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Latticepathology and symmetric functions

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Abstract. In this article, we revisit and extend a list of formulas based on lattice path surgery: cut-and-paste methods, rotations, factorizations, kernel method... We focus on the natural model of directed lattice paths (also called generalized Dyck paths) and the related constrained objects (excursions, meanders, bridges). We show that each of the fundamental families of symmetric polynomials are offering remarkable expressions for lattice path generating functions. We also give some natural decompositions for the associated context-free grammars, and bijective proofs of bivariate versions of Spitzer/Sparre Andersen/Wiener–Hopf formulas, paving the way for many joint distribution studies.

Résumé. Cet article revisite et étend diverses identités liées à des décompositions des chemins: méthodes de copier-coller, permutations cycliques, factorisations, méthode du noyau... Nous considérons les chemins ayant un jeu fini de sauts et dirigés sur le réseau $\mathbb{N}^2$ (également appelés chemins de Dyck généralisés) et, par analogie avec le mouvement brownien, leurs variantes classiques: excursions, méandres, ponts. Nous montrons que chacune des familles fondamentales e_k, p_k, m_k de polynômes symétriques énumère une classe remarquable de tels chemins. Nous donnons aussi des décompositions naturelles pour les grammaires algébriques associées, et une preuve bijective des versions bivariées des identités de Spitzer, de Sparre Andersen et de Wiener–Hopf.

Zusammenfassung. In diesem Artikel behandeln und erweitern wir Formeln, die auf bestimmten Operationen auf Gitterwegen basieren: Zerlegungen, Rotationen, Faktorisierungen, Andwendungen der Kernmethode, usw. Wir konzentrieren uns hiebei auf das natürliche Modell von gerichteten Gitterwegen, welche auch als verallgemeinerte Dyck Pfade bekannt sind. Eines unserer Hauptresultate zeigt, dass drei fundamentale Familien symmetrischer Polynome natürliche Interpretationen im Sinne von erzeugenden Funktionen bestimmter Gitterwegsklassen besitzen. Weiters präsentieren wir eine natürliche Zerlegung der zugehörigen kontextfreien Grammatik, sowie bijektive Beweise bivariater Versionen der Formeln von Spitzer, Sparre Andersen und Wiener–Hopf.

Keywords: *Lattice path, generating function, algebraic function, kernel method, context-free grammar, Sparre Andersen formula, Spitzer's identity, Wiener–Hopf factorization.*

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1 Introduction and definitions

The recursive nature of lattice paths makes them amenable to context-free grammar techniques, their geometric nature makes them amenable to cut-and-paste bijections, their step-by-step nature makes them amenable to functional equations solvable by the kernel method (see e.g. [2–8, 11, 20, 22, 24] for many applications of these ideas). We present in a unified way some consequences of these observations in Section 2 on context-free grammars and in Section 3 on Spitzer and Wiener–Hopf identities. This helps to state our main resuts: this article gives new connections with symmetric functions in Section 4, see Table 2. This greatly extends the enumerative formulas given in [3].

Definition 1.1 (Jumps and lattice paths)**.** A *step set* $\mathcal{S} \subset \mathbb{Z}$ is a finite set $\{s_1, \ldots, s_m\}$. The elements of $\mathcal{S}$ are called *steps* or *jumps*. An n-step *lattice path* or *walk* ω is a sequence $(v_1, \ldots, v_n) \in \mathcal{S}^n$. The *length* $|\omega|$ of a lattice path is its number n of jumps.

Such sequences are one-dimensional objects. Geometrically, they can be interpreted as two-dimensional objects which justifies the name *lattice path*. Indeed, $(v_1, \ldots, v_n)$ may be seen as a sequence of points $(\omega_0, \omega_1, \ldots, \omega_n)$, where ω_0 is the starting point and $\omega_i - \omega_{i-1} = (1, v_i)$ for $i = 1, \ldots, n$. Except when mentioned differently, the starting point ω_0 of these lattice paths is $(0,0)$.

Let $s_k := \sum_{i=1}^{k} v_i$ be the partial sum of the first k steps of the walk ω. We define the *height* or *maximum of* ω as $\max_k s_k$, and the *final altitude of* ω as s_n. For example, the first walk in Table 1 has height 3 and final altitude 1. Table 1 is also illustrating the four following classical types of paths:

Definition 1.2 (Excursions, arches, meanders, bridges)**.**

- *Excursions* are paths never going below the x-axis and ending on the x-axis;
- *Arches* are excursions that only touch the x-axis twice: at the beginning and at the end;
- *Meanders* are prefixes of excursions, i.e., paths never going below the x-axis;
- *Bridges* are paths ending on the x-axis (allowed to cross the x-axis any number of times).

Let $c := -\min \mathcal{S}$ be the maximal negative step, and let $d := \max \mathcal{S}$ be the maximal positive step. To avoid trivial cases we assume $\min \mathcal{S} < 0 < \max \mathcal{S}$. Then we associate to this set of steps the following *step polynomial*:

$$P(u) = \sum_{i=-c}^{d} p_i u^i. \tag{1.1}$$

The weight of a lattice path is the product of the weights of its steps. (In this article, we will not need to assume that the weights p_j are real or non-negative: they are allowed to be complex numbers, or members of any field of characteristic 0.)

The generating functions of directed lattice paths can be expressed in terms of the roots of the *kernel equation*

$$1 - zP(u) = 0. \tag{1.2}$$

More precisely, this equation has $c + d$ solutions in u. The *small roots* $u_i(z)$, for $i = 1, \ldots, c$, are the c solutions with the property $u_i(z) \sim 0$ for $z \sim 0$. The remaining d solutions are called *large roots* as they satisfy $|v_i(z)| \sim +\infty$ for $z \sim 0$. The generating functions of the four classical types of lattice paths introduced above are shown in Table 1.

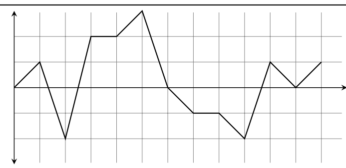
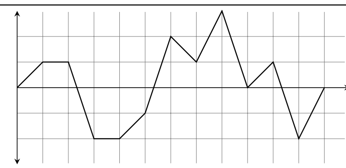
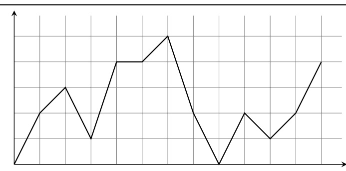
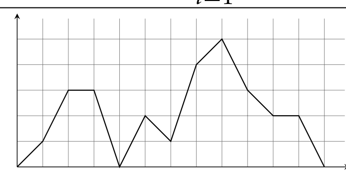
	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path ($\mathcal{W}$) $W(z) = \frac{1}{1-zP(1)}$	 bridge ($\mathcal{B}$) $B(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
constrained (on $\mathbb{Z}$)	 meander ($\mathcal{M}$) $M(z) = \frac{1}{1-zP(1)} \prod_{i=1}^{c}(1 - u_i(z))$	 excursion ($\mathcal{E}$) $E(z) = \frac{(-1)^{c-1}}{p_{-c}z} \prod_{i=1}^{c} u_i(z)$

Table 1: The four types of paths: walks, bridges, meanders and excursions, and the corresponding generating functions for directed lattice paths.

These results follow from the expression for the bivariate generating function $M(z, u)$ of meanders. Indeed, let $m_{n,k}$ be the number of meanders of length n going from altitude 0 to altitude k, then we have

$$M(z, u) = \sum_k M_k(z)u^k = \sum_{n,k \geqslant 0} m_{n,k} z^n u^k = \frac{\prod_{i=1}^{c}(u - u_i(z))}{u^c(1 - zP(u))}. \tag{1.3}$$

This formula is obtained by the kernel method: it consists of substituting $u = u_i(z)$ (the small roots of (1.2)) in the functional equation which mimics the recursive definition of a meander. This results in new and simpler equations which lead to the closed form (1.3). The generating function of excursions is $E(z) := M(z, 0)$, see [3,7] for more details.

2 Context-free grammars for directed lattice paths

Context-free grammars are a powerful tool to tackle problems related to directed lattice paths (we refer to [18] for a detailed presentation of grammar techniques). In this section, we introduce some key families of lattice paths (arches, prime walks) that we will also use in the next section. En passant, we also give grammars generating the most fundamental classes of lattice paths (excursions, bridges, meanders); this generalizes and unifies results from [8,11,22,24]. Illustrating the philosophy of "latticepathology", we give short concise visual proofs based on lattice path surgery. What is more, the following decompositions hold for any set of jumps: it is straightforward to extend them to multiplicities (jumps with different colours) or even to an infinite set of jumps.

All our grammars are non-ambiguous: there is only one way to generate each lattice path. They require the introduction of the following class of lattice paths:

Definition 2.1. An *arch from i to j* is a walk starting at altitude i ending at altitude j and staying always strictly above altitude $\max(i,j)$ except for its first and final position.

An important consequence of this definition is that arches cannot have an excursion as left or right factor. Note that an arch from i to j can be considered as an arch from 0 to $j-i$. This justifies that we now focus on arches starting at 0. Let A_k be the class of arches from 0 to k. For example, one has $(2,-1) \in A_1$ and $(1) \in A_1$, but $(1,1,-1) \notin A_1$. Following the tradition of several authors, we refer to *arches* (omitting the start and end point) as arches from 0 to 0, see e.g. [3]. Thus, an excursion is clearly a sequence of arches. More generally we have the following decompositions:

Theorem 2.2 (Grammars)**.** *Excursions are generated by the following grammar:*

$$E \;\to\; \varepsilon + \sum_{k=1}^{d} A_k\, E\, A_{-k}\, E \qquad \textit{(excursions)},$$

where one also adds the rule $E \to 0E$ if $0 \in \mathcal{S}$. The arches A_k from 0 to k are generated by

$$A_k \;\to\; \sum_{j=k+1}^{d} A_j\, E\, A_{k-j} \qquad \textit{(arches for } k \geqslant 0),$$

$$A_k \;\to\; \sum_{j=-c}^{k-1} A_{k-j}\, E\, A_j \qquad \textit{(arches for } k < 0),$$

and one adds to these two rules the rule $A_k \to k$ whenever $k \in \mathcal{S}$. Meanders are generated by

$$M \;\to\; E + \sum_{k=1}^{d} E\, A_k\, M \qquad \textit{(meanders)},$$

which can be rephrased as "meanders are sequences of prime walks": $M = \mathrm{Seq}\left(\sum_{k=0}^{d} A_k\right)$.

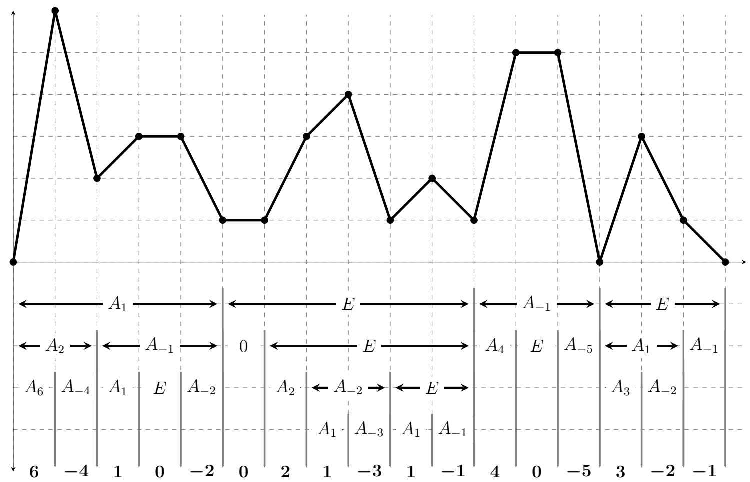


Figure 1: Example of our non-ambiguous decomposition of an excursion

Proof. Let us start with arches A_k from 0 to $k \geqslant 0$. (The results for A_{-k} follow analogously.) For such arches of length > 1, we cut them at the first and the last time their minimal altitude (not taking end points into account) is attained. The first factor goes from 0 to j and stays in-between always strictly above j, and therefore is given by A_j. The second factor is a (possibly empty) excursion. The last factor is an arch from j to k given by A_{k-j}. This gives $A_k = A_j E A_{k-j}$. From this, it is immediate to get the grammar for excursions, as they are a sequence of arches A_0, thus $E = \varepsilon + A_0 E$.

Now take any meander and cut it at the last time it touches altitude 0. The first part is a (possibly empty) sequence of arches. We cut the second part at the first point where its minimal altitude > 0 is attained. The remaining part is again a meander. This gives the factorization $E A_k M$. The set $\sum_{k=0}^{d} A_k$ can thus be considered as *prime walks*: any positive meander is a sequence of such "prime" factors, in a unique way.

All these decompositions are clearly 1-to-1 correspondences, as shown in Figure 1. □

We end this section with the grammar of bridges. It uses another class of walks: the negative arches from 0 to k, denoted by $\bar{A}_k$. These stay always strictly below $\min(0, k)$. Their grammar is just the mirror of the one for A_k given in Theorem 2.2.

Theorem 2.3. *Bridges $B = B_0$ are generated by the following grammar:*

$$B_0 \quad \to \quad \varepsilon + \sum_{k \in \mathcal{S}} k B_k,$$

where B_k stands for the "bridges starting at k", i.e. walks on $\mathbb{Z}$ from k to 0, given by

$$B_k \quad \to \quad \sum_{j=-c}^{0} (A_j \, B_{k+j}) \qquad (\textit{if } k > 0),$$

$$B_k \quad \to \quad \sum_{j=0}^{d} (\bar{A}_j \, B_{k+j}) \qquad (\textit{if } k < 0).$$

3 Latticepathology and surgery of paths

The decompositions of lattice paths mentioned in the previous section find application in the bivariate versions of the Sparre Andersen/Spitzer/Wiener–Hopf formulas [1,16,17,23,26]. It gives for free elegant short proofs for these fundamental results which were definitively missing in [3], again illustrating the latticepathology philosophy!

Theorem 3.1 (Bivariate version of Spitzer/Sparre Andersen's identities)**.** *The generating function* $W^+(z,u) = \sum_n w_n^+(u) z^n$ *of walks on* $\mathbb{Z}$ *ending at a positive altitude and the generating function* $M(z,u) = \sum_n m_n(u) z^n$ *of meanders (where* u *encodes the final altitude and* z *encodes the length of the lattice path) are related by the formulas*

$$W^+(z,u) = 1 + z\frac{M'(z,u)}{M(z,u)} \qquad \text{or, equivalently,} \tag{3.1}$$

$$M(z,u) = \exp\left(\int_0^z \frac{W^+(t,u)-1}{t}\,dt\right) = \exp\left(\sum_{n\geqslant 1} \frac{w_n^+(u)}{n} t^n\right). \tag{3.2}$$

Proof. We give a bijective proof. It consists in splitting any non-empty walk ω ending at a positive altitude into 3 factors: $\omega = \phi_1.M.\phi_2$ where M is the longest meander starting at the first minimum of the walk and such that $\phi_2.\phi_1$ is a prime walk (pointed, in order to remember where to split). The fact that it exists and is unique follows from the positivity of ω and from the non-ambiguous grammar decomposition from Section 2. This decomposition directly keeps track of the last altitude of each of its factors, so

$$W^+(z,u) - 1 = M(z,u) z \frac{\partial}{\partial z}\left(1 - \frac{1}{M(z,u)}\right). \qquad \square$$

A similar surgery of lattice paths (details in the long version of this article) leads to:

Theorem 3.2 (Bivariate version of Wiener–Hopf formula)**.** *The bivariate generating functions* $W_{\text{pos. height}}(z,u)$ *and* $W_{\text{neg. height}}(z,u)$ *of walks on* $\mathbb{Z}$ *with* u *marking the positive and negative height (not the altitude!) are related to the bivariate generating functions* $M^+(z,u)$ *of positive meanders and* $M^-(z,u)$ *of reversed meanders (with* u *marking the final altitude, see Figure 2):*

$$W_{\text{pos. height}}(z,u) = M^-(z)E(z)M^+(z,u) = -\frac{1}{p_d z}\left(\prod_{j=1}^{c} \frac{1}{1-u_j(z)}\right)\left(\prod_{\ell=1}^{d} \frac{1}{u - v_\ell(z)}\right), \tag{3.3}$$

$$W_{\text{neg. height}}(z,u) = M^-(z,u)E(z)M^+(z) = -\frac{1}{p_d z}\left(\prod_{j=1}^{c} \frac{1}{1-u_j(z)/u}\right)\left(\prod_{\ell=1}^{d} \frac{1}{1 - v_\ell(z)}\right). \tag{3.4}$$

This Wiener–Hopf factorization $W = M^- E M^+$ *thus gives:*

$$M^-(z) = \frac{W(z)}{M(z)} = \prod_{j=1}^{c} \frac{1}{1-u_j(z)} \qquad \text{and} \qquad M^+(z) = \frac{M(z)}{E(z)} = \prod_{\ell=1}^{d} \frac{1}{1-1/v_\ell(z)}.$$

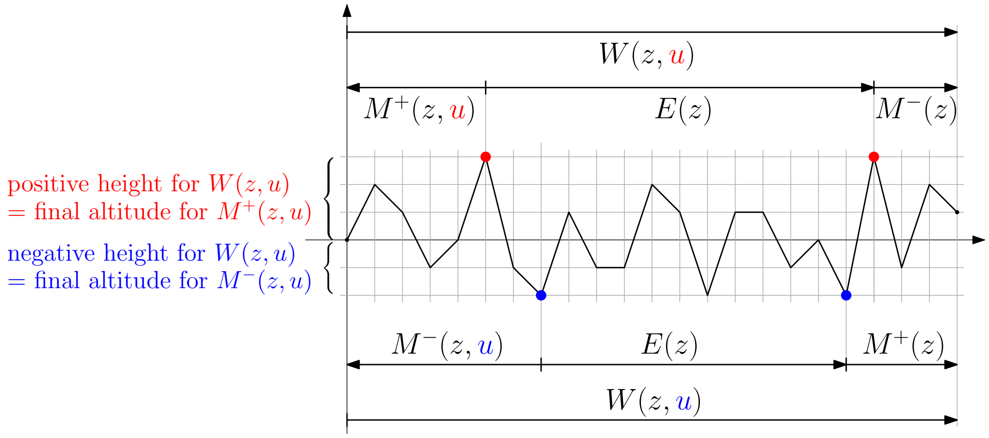


Figure 2: The Wiener–Hopf decomposition of a walk: $W = M^-EM^+$, a product of a reversed-meander, an excursion, and a positive meander. See e.g. [16] for the importance of this factorization for lattice path enumeration. It offers a link between two important parameters (height and final altitude): the proof uses a 180^o rotation of the first factor. The above picture crystalizes the key idea behind the theorems given by Feller in his nice introduction to the Wiener–Hopf factorization [13, Chapter XVIII.3 and XVIII.4]. It also explains why this decomposition holds for Lévy processes, which are the continuous time and space version of lattice paths, see [21].

4 Lattice paths and symmetric functions

In this section we show that three fundamental classes of symmetric polynomials evaluated at the small branches of the kernel have a natural combinatorial interpretation in terms of directed lattice paths. For our main results see Table 2. We first recall the definitions of these symmetric polynomials (see e.g. [27] for more on these objects).

Definition 4.1. The *complete homogeneous symmetric polynomials* h_k of degree k in the d variables $x_1, \dots, x_d$ are defined as

$$\sum_{k\geqslant 0} h_k(x_1,\dots,x_d)u^k = \prod_{i=1}^{d} \frac{1}{1-ux_i}. \tag{4.1}$$

The *elementary homogeneous symmetric polynomials* e_k of degree k in the d variables $x_1, \dots, x_d$ are defined as

$$\sum_{k=0}^{c} e_k(x_1,\dots,x_d)u^k = \prod_{i=1}^{d} (1+ux_i). \tag{4.2}$$

The *power sum homogeneous symmetric polynomials* p_k of degree k in the d variables $x_1, \dots, x_d$ are defined as[1]

$$\sum_{k\geqslant 0} p_k(x_1,\dots,x_d)u^k = \sum_{i=1}^{d} \frac{1}{1-ux_i}. \tag{4.3}$$

[1] To avoid confusion with the coefficients of the step polynomial $P(u)$, namely $p_i = [u^i]P(u)$, we will always write the power sum symmetric polynomials with their variables: $p_k(x_1, \dots, x_d)$.

Many variants of directed lattice paths satisfy functional equations which are solvable via the kernel method and lead to formulas involving a quotient of Vandermonde-like determinants, see e.g. [3]. It is thus natural that Schur polynomials intervene, they e.g. play an important role for lattice paths in a strip, see [4,6]. It is nice that the other symmetric polynomials also have a combinatorial interpretation, as presented in the following table.

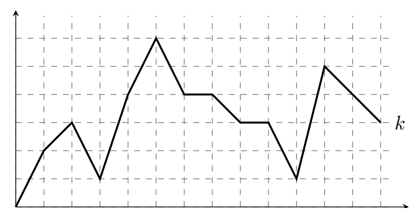
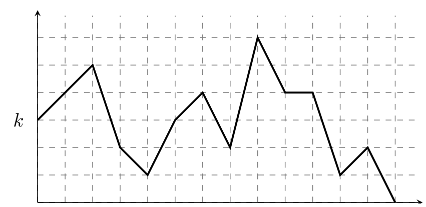
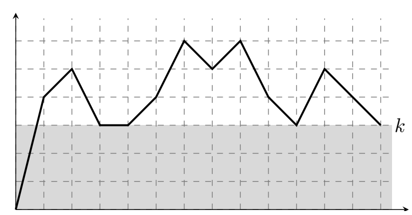
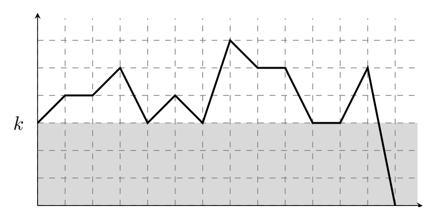
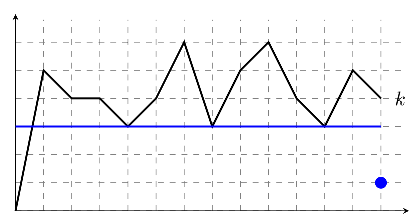
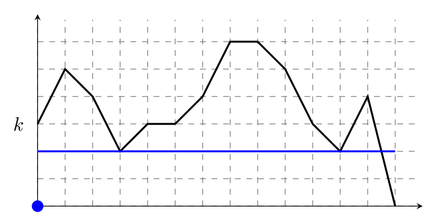
	from 0 to k	from k to 0
positive meander	 $M^{+}_{0,k}(z) = h_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{+}_{k,0}(z) = h_k\left(u_1(z), \dots, u_c(z)\right)$
positive meander avoiding $(0,k)$	 $M^{\geqslant}_{0,k}(z) = (-1)^{k-1} e_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{\geqslant}_{k,0}(z) = (-1)^{k-1} e_k\left(u_1(z), \dots, u_c(z)\right)$
positive meander marked below the minimum	 $M^{\bullet}_{0,k}(z) = p_k\left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right)$	 $M^{\bullet}_{k,0}(z) = p_k\left(u_1(z), \dots, u_c(z)\right)$

Table 2: In this article, we show that the fundamental symmetric polynomials (the complete homogeneous, elementary, and power sum type) are counting families of positive meanders (walks touching the x-axis only at one of the end points and staying always above the x-axis). The functions $v_j(z)$ for $j = 1, \dots, d$ are the branches of the kernel equation $1 - zP(u) = 0$ with $\lim_{z=0} |v_j(z)| = +\infty$, whereas the functions $u_i(z)$ for $i = 1, \dots, c$ are the branches such that $\lim_{z=0} u_i(z) = 0$.

Let us now give a more formal definition of the corresponding objects and a proof of the formulas for the associated generating functions.

Definition 4.2. A *positive meander* is a path from $\ell \geqslant 0$ to $k \geqslant 0$ staying strictly above the x-axis (except possibly at the end points). The generating function is denoted by $M^{+}_{\ell,k}(z)$.

In Table 2, we focus on positive meanders from 0 to k and from k to 0. Note that it suffices to consider the paths from 0 to k as by time-reversion they are mapped to each other. In particular, let u_i and v_j be the small and large branches of the initial model. Then, after time-reversion the small branches are $\frac{1}{v_j}$ and the large branches are $\frac{1}{u_i}$. More details are given in the long version.

Theorem 4.3 (Generating function of positive meanders)**.**

$$M^{+}_{0,k}(z) = h_k\left(\frac{1}{v_1(z)}, \ldots, \frac{1}{v_d(z)}\right).$$

Proof (Sketch). We observe that positive meanders are classical meanders factored by excursions. By [3, Theorem 2] their generating function is the coefficient of u^k in $\prod_{j=1}^{d} \frac{1}{1-u/v_j(z)}$. Consequently, by Equation (4.1) this is the generating function of the complete homogeneous symmetric polynomials $h_k(1/v_1(z), \ldots, 1/v_d(z))$. □

This theorem gives a shorter proof of [3, Corollary 3]:

Corollary 4.4. *The generating function $M_k(z)$ of meanders ending at altitude k are given by*

$$M_k(z) = E(z) h_k\left(\frac{1}{v_1(z)}, \ldots, \frac{1}{v_d(z)}\right) = \frac{1}{p_d z} \sum_{\ell=1}^{d} \left(\prod_{j \neq \ell} \frac{1}{v_j - v_\ell} \right) \frac{1}{v_\ell^{k+1}}.$$

Proof. As in the proof of Theorem 4.3, we use that a meander ending at altitude k can be uniquely decomposed into an initial excursion followed by a positive meander from 0 to k. Using a partial fraction decomposition of (4.1), one gets the claimed form. □

Definition 4.5. A *positive meander avoiding a strip of width k* is a positive meander from 0 to k that always stays above any point of altitude $j < k$ except for its start point. The generating function is denoted by $M^{\geqslant}_{0,k}(z)$.

Theorem 4.6 (Positive meanders avoiding the strip $[0,k]$)**.**

$$M^{\geqslant}_{0,k}(z) = (-1)^{k-1} e_k\left(\frac{1}{v_1(z)}, \ldots, \frac{1}{v_d(z)}\right). \tag{4.4}$$

Proof (sketch). We proceed by induction on k. The base case $k = 1$ holds due to $M^{\geqslant}_{0,1}(z) = M^{+}_{0,1}(z) = 1/v_1(z) + \cdots + 1/v_d(z)$. Next assume the claim holds for $M^{\geqslant}_{0,1}(z), \ldots, M^{\geqslant}_{0,k-1}(z)$.

Take an arbitrary positive meander from 0 to k. Either it is a positive meander avoiding the strip of width k, or at least one of its lattice points has an altitude smaller than k.

Let $0 < i < k$ be the altitude of the last step below altitude k. Then the path can be uniquely decomposed into an initial part from altitude 0 to this altitude i and a part from this point to the end. Note that by the construction the initial part starts at altitude 0 and then always stays above the x-axis, whereas the last part avoids a strip of width $k-i$. In terms of generating functions this gives

$$M^{\geqslant}_{0,k}(z) = M^{+}_{0,k}(z) - \sum_{i=1}^{k-1} M^{+}_{0,i}(z) M^{\geqslant}_{0,k-i}(z) .$$

Inserting the known expressions we get

$$M^{\geqslant}_{0,k}(z) = \sum_{i=1}^{k} (-1)^{k-i} e_{k-i}\left(\frac{1}{v_1}, \dots, \frac{1}{v_d}\right) h_i \left(\frac{1}{v_1}, \dots, \frac{1}{v_d}\right) = (-1)^{k-1} e_k \left(\frac{1}{v_1}, \dots, \frac{1}{v_d}\right),$$

thanks to the fundamental involution relation [27, Equation (7.13)] between elementary symmetric polynomials and complete homogeneous symmetric polynomials. □

We end our discussion with a third class of positive meanders.

Definition 4.7. A *positive meanders marked below the minimum* is a positive meander with an additional marker in $\{1, \dots, m\}$ where m is its minimal positive altitude. The generating function for such paths from 0 to k is denoted by $M^{\bullet}_{0,k}(z)$.

For example it is immediate that $M^{\bullet}_{0,1}(z) = M^{\geqslant}_{0,1}(z) = M^{+}_{0,1}(z)$ as the only restriction is to avoid the x-axis. Furthermore, $M^{\bullet}_{0,0}(z) = 0$ while $M^{\geqslant}_{0,0}(z) = M^{+}_{0,0}(z) = 1$.

Theorem 4.8 (Positive meanders marked below the minimum)**.**

$$M^{\bullet}_{0,k}(z) = p_k \left(\frac{1}{v_1(z)}, \dots, \frac{1}{v_d(z)}\right).$$

Proof (sketch). Every path from 0 to k has to touch at least one of the altitudes $1, \dots, d$, as the largest possible up step is $+d$. We decompose any positive meander from 0 to k into two parts by cutting at the unique last positive minimum m. The first part is an arch avoiding the strip of width m, whereas the second part is a positive meander from m to k. Translating this decomposition into generating functions, we get

$$M^{\bullet}_{0,k}(z) = \sum_{m=1}^{d} m M^{\geqslant}_{0,m}(z) M^{+}_{0,k-m}(z),$$

where the factor m encodes the m possible ways to put a mark below the minimum, see Definition 4.7. Note that $M^{\geqslant}_{0,k}(z) = 0$ for $k > d$. Thus, by Theorems 4.3 and 4.6 we get

$$\sum_{k\geqslant 1} M^{\bullet}_{0,k}(z) u^k = \left(u \frac{\partial}{\partial u} \sum_{j\geqslant 0} M^{\geqslant}_{0,j}(z) u^j \right) \left(\sum_{i \geqslant 0} M^{+}_{0,i}(z) u^i \right) = \sum_{i=1}^{d} \frac{u/v_i(z)}{1-u/v_i(z)}.$$

By Equation (4.3) this proves the claim. □

In the long version of this article, we show that this approach also allows us to derive simple expressions in terms of Schur polynomials for walks and meanders from i to j.

5 Conclusion

In this article, we gave several identities and decompositions which are a consequence of lattice path surgery. It is interesting that most of these decompositions extend to lattice paths in which one forbids (or counts the number of occurrences of) a given pattern, see [2]. The fact that many of these decompositions keep track of some additional parameter also gives access to many joint distribution studies (see e.g. [9]).

Our work also offers new links with symmetric polynomials, adding to previous fundamental connections with algebraic combinatorics via Vandermonde determinants, the Jacobi–Trudi identity, and Schur functions (see [4,6]). En passant, we illustrate the old Schützenberger philosophy: most of the identities in the commutative world are images of structural identities in the non-commutative world. It is natural to ask how far we can extend the link between lattice paths and the non-commutative symmetric world; note that further non-commutative points of view are developed in [12,14,15].

It is striking that astonishingly powerful formulas can be obtained by astonishingly simple tools from symbolic combinatorics. Such formulas, e.g. the Spitzer formula for bridges have some unexpected avatars. Indeed, bridges of length n can be seen as $[u^0]P(u)^n$ for some Laurent polynomial $P(u)$, the same holds with multivariate polynomials; this leads to some interesting connections between the non-commutative world, the Laurent phenomenon, and lattice paths, as studied by Kontsevich (see e.g. [10,19,25]). In one sense, this paves the way for an extension of the Weil conjectures on the rationality of zeta functions and trace formulas to the algebraic world.

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