Intersection Types for Implicit Computational Complexity

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3. Polynomial time strongly normalizable terms: a static characterization
4. Beyond PSN: real challenge

Preliminary

- Λ: the set of untyped lambda terms.
- $W = \{0,1\}^*$.
- Fix a Church-coding of binary words in W, eg.

$$\overline{01101} \equiv \lambda f_0 f_1 z. f_0(f_1(f_1(f_0(f_1 z)))).$$

- Any $M \in \Lambda$ represents a partial function $f_M : W \longrightarrow W$:

$$\begin{array}{rcll} f_M(w) & = & w' & \text{if } M\overline{w} \longrightarrow^* \overline{w}' \\ & = & \uparrow & \text{otherwise.} \end{array}$$

- A type system $\mathbf{L}$ (such as System F) determines a set $\mathcal{T} \subseteq \Lambda$ of typable terms.
- Usually, such $\mathcal{T}$ is r.e.

Lambda-characterizations of FP

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Examples: $\Lambda\mathbf{BC}$ (Bellantoni-Cook 92) translated into Λ, $\mathbf{1}\lambda^p(\mathbf{W})$ (Leivant-Marion 94), $\mathbf{LLL}$ (Girard 98), $\mathbf{SLL}$ (Lafont 04), $\mathbf{DLAL}$ (Baillot-Terui 04)

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- $\mathcal{T}$ is intensionally P-complete if in addition

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  - (restricted) iteration scheme
- But so what?
- Intensional P-completeness is desired...

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- Proof: Reduction of Hilbert's 10th problem.
- For any polynomial $P(X_1, \ldots, X_n)$ with integer coefficients, there is M_P that works on unary integers as follows:

$$\begin{array}{rcll} M_P(0) & = & 1 & \\ M_P(x+1) & = & 1 & \text{if } \exists z_1, \ldots, z_n.(-x \leq z_1, \ldots, z_n \leq x \\ & & & \quad \wedge P(z_1, \ldots, z_n) = 0) \\ & = & 2 \cdot M_P(x) & \text{otherwise.} \end{array}$$

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& && \quad \wedge P(z_1, \ldots, z_n) = 0) \\
&= 2 \cdot M_P(x) && \text{otherwise.}
\end{aligned}
$$

- M_P can be considered as a program on binary words by:

$$
\text{u2b}(n) = \underbrace{11 \cdots 111}_{n\ times} \qquad \text{b2u}(\underbrace{011 \cdots 100}_{n\ times}) = n.
$$

- $M_P \in \Lambda\mathbf{P}$ iff $P(X_1, \ldots, X_n) = 0$ admits an integer solution.
- Hence $\Lambda\mathbf{P}$ is not r.e (nor co-r.e).

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- Cf. Given $M \in \Lambda_F$, it is decidable in Ptime whether M is typable in **DLAL** (Atassi-Baillot-Terui 2006).

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- Cf. Given $M \in \Lambda_F$, it is decidable in Ptime whether M is typable in **DLAL** (Atassi-Baillot-Terui 2006).
- We are looking for a better approximation of $\Lambda\mathbf{P}$.

Subclasses of ΛP

- $\mathbf{\Lambda P}_{SN}$: the class of Ptime strongly normalizable lambda terms:
  $M \in \mathbf{\Lambda P}_{SN} \Longleftrightarrow$
  $\exists d \in N. \forall w \in W.$ for any reduction sequence

$$M\overline{w} \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_k$$

  $k = O(|w|^d)$; $\quad |M_i| = O(|w|^d)$ for any $1 \leq i \leq k$.

Subclasses of ΛP

- R: feasible reduction strategy (such as leftmost, innermost): given M, R picks up a redex of M, if any, in Ptime.
- $\mathbf{\Lambda P}_R$: the class of Ptime R-normalizable lambda terms M: $\exists d \in N. \forall w \in W.$ for the R-reduction sequence

$$M\overline{w} \longrightarrow_R M_1 \longrightarrow_R \cdots \longrightarrow M_k$$

with M_k in normal form

$k = O(|w|^d)$; $\quad |M_i| = O(|w|^d)$ for any $1 \leq i \leq k$.

- Finally,

$$\mathbf{\Lambda P}_{WN} := \bigcup_{R:\text{feasible strategy}} \mathbf{\Lambda P}_R.$$

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- $\mathbf{\Lambda BC} \not\subset \mathbf{\Lambda P}_{SN}$, $\mathbf{\Lambda BC} \subset \mathbf{\Lambda P}_{innermost}$.

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- $\mathbf{DLAL} \subset \mathbf{\Lambda P}_{SN}$
- $\mathbf{\Lambda BC} \not\subset \mathbf{\Lambda P}_{SN}$, $\mathbf{\Lambda BC} \subset \mathbf{\Lambda P}_{innermost}$.
- Intersection types provide static (i.e. reduction-free) characterization of Λ_{PSN} (though not r.e.).

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- Most systems satisfy (Pottinger 80, CDV 81):
  M is typable without using $\omega \iff M$ is strongly normalizable.
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- As a consequence, type inference is as difficult as normalization.

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- Affine intersection types satisfy Weakening but not Idempotency/Contraction. Typability=SN.
- Relationship with linear logic is suggested by (Regnier 92, Mairson-Møller 04, Carlier-Wells 04).

Explicit vs Implicit typings

- Explicit typing: introduces constructor/destructor to the subject.
- Implicit typing: does not change the subject.
- Most logical connectives admit both interpretations.
- Explicit Quantifier:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \Lambda\alpha.M : \forall\alpha.A} \qquad \frac{\Gamma \vdash M : \forall\alpha.A}{\Gamma \vdash MB : A[B/\alpha]}$$

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Explicit vs Implicit typings

- Explicit Plus:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{inl}(M) : A \oplus B}$$

$$\frac{x : A, \Gamma \vdash M : C \quad y : B, \Gamma \vdash N : C}{z : A \oplus B, \Gamma \vdash \mathtt{case}\ z\ \mathtt{of}\ \mathtt{inl}(x) \Rightarrow M |\ \mathtt{inr}(y) \Rightarrow N : C}$$

- Implicit Plus (Union Type):

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M : A \oplus B} \qquad \frac{x : A, \Gamma \vdash M : C \quad x : B, \Gamma \vdash M : C}{x : A \oplus B, \Gamma \vdash M : C}$$

Explicit vs Implicit typings

- Explicit Tensor:

$$\frac{\Gamma \vdash M : A \quad \Delta \vdash N : B}{\Gamma, \Delta \vdash M \otimes N : A \otimes B} \qquad \frac{x : A, y : B, \Gamma \vdash M : C}{z : A \otimes B, \Gamma \vdash \mathtt{let}\ z = x \otimes y\ \mathtt{in}\ M : C}$$

- Implicit Tensor (Non-idempotent intersection type):

$$\frac{\Gamma \vdash M : A \quad \Delta \vdash M : B}{\Gamma, \Delta \vdash M : A \otimes B} \qquad \frac{x : A, x : B, \Gamma \vdash M : C}{x : A \otimes B, \Gamma \vdash M : C}$$

Explicit vs Implicit typings

- Explicit One:

$$\frac{}{\vdash * : \mathbf{1}} \qquad \frac{\Gamma \vdash M : C}{x : \mathbf{1}, \Gamma \vdash \texttt{let } z = * \texttt{ in } M : C}$$

- Implicit One (Omega):

$$\frac{M \text{ arbitrary}}{\vdash M : \mathbf{1}} \qquad \frac{\Gamma \vdash M : C}{x : \mathbf{1}, \Gamma \vdash M : C}$$

- Contraction, Weakening. . .

Explication

- For each system $\mathbf{L}$, $e\mathbf{L}$ (explicit), $i\mathbf{L}$ (implicit)
- Explication map: $(\bullet)^e : i\mathbf{L} \longrightarrow e\mathbf{L}$

$$\frac{\Gamma \vdash M : A \quad \Delta \vdash M : B}{\Gamma, \Delta \vdash M : A \otimes B} \quad \Longrightarrow \quad \frac{\tilde{\Gamma} \vdash M^e : A \quad \tilde{\Delta} \vdash M^e : B}{\tilde{\Gamma}, \tilde{\Delta} \vdash M^e \otimes M^e : A \otimes B}$$

$$\frac{x : A, x : B, \Gamma \vdash M : C}{x : A \otimes B, \Gamma \vdash M : C} \quad \Longrightarrow \quad \frac{x_1 : A, x_2 : B, \tilde{\Gamma} \vdash M^e : C}{x_0 : A \otimes B, \tilde{\Gamma} \vdash \texttt{let}\ x_0 = x_1 \otimes x_2\ \texttt{in}\ M^e}$$

- We only consider strict types: No $A \multimap B \otimes C$, $A \multimap B \oplus C$, $A \multimap \mathbf{1}$, etc.

Explication

- Theorem: For any $\mathbf{L}$,

$$\begin{array}{ccc} i\mathbf{L} & \longrightarrow & e\mathbf{L} \\ \hline \Gamma \vdash M : A & \Longleftrightarrow & \tilde{\Gamma} \vdash M^e : A \end{array}$$

$$\begin{array}{ccc} M & & M^e \\ \downarrow * & \Longleftrightarrow & \downarrow * \\ N & & N^e \end{array}$$

- Corollary: Subject reduction/Normalization for $e\mathbf{L} \Longrightarrow$ the same for $i\mathbf{L}$.
- $e\mathbf{L}$ term-nonuniform: $M \otimes N$, $M \otimes N \otimes K$, ...
- $i\mathbf{L}$ type-nonuniform: $M : A$, $M : A \otimes B$, $M : A \oplus B$,...

Implicit implication

- Is implicit implication useful?

$$\frac{x : A, \Gamma \vdash M : B \quad x \notin FV(M)}{\Gamma \vdash M : A \multimap B} \qquad \frac{\Gamma \vdash M : A \multimap B \quad x : A \vdash x : A}{x : A, \Gamma \vdash M : B}$$

- Yes, when implicating **LFPL** (Hofmann):

$$\frac{\dfrac{\overline{\vdash cons : \Diamond \multimap A \multimap L(A) \multimap L(A)}}{x : \Diamond \vdash cons : A \multimap L(A) \multimap L(A)} \quad \overline{a : A \vdash a : A} \quad \overline{l : L(A) \vdash l : L(A)}}{x : \Diamond, a : A, l : L(A) \vdash cons(a, l) : L(A)}$$

$$\frac{\dfrac{x : \Diamond, a : A, l : L(A) \vdash cons(a, l) : L(A)}{x : \Diamond, l : L(A) \vdash \lambda a.cons(a, l) : A \multimap L(A)}}{l : L(A) \vdash \lambda a.cons(a, l) : \Diamond \multimap A \multimap L(A)}$$

- Explication: $i\mathbf{LFPL} \longrightarrow e\mathbf{LFPL}$ makes it the original system.

Linearization

1. $M \in SN \Longleftrightarrow M \in i\mathbf{MAL}\ (= i\mathbf{MLL} + Weak)$.
2. $M \in WN \Longleftrightarrow M \in i\mathbf{MLL}+$ Negative **1**

- We will sharpen 1. (The following results are also found by Ronchi Della Rocca and Gaboardi independently)
- 2. will be sharped by Daniel de Calvalho's talk.

Strength of logic

- What is the strength of a logic?
- Explicit typing: How many terms it types.
  - $\mathbf{MAL}_{\multimap} \subsetneq \mathbf{SimTyp}_{\multimap} \subsetneq \mathbf{SysF}_{\multimap\forall}$
- Implicit typing: How short typing derivations are.
  - In $i\mathbf{MAL}$, normalization cost $\approx$ derivation size
  - In $i\mathbf{SimTyp}$, non-elementary terms have very short proofs.
  - In $i\mathbf{SysF}$, ...
- Connection with propositional proof systems?

Type system iMAL

- Variables: α, β
- Intersections: $D, E ::= A_1 \otimes \ldots \otimes A_n \quad (n \geq 1)$
- Types: $A, B, C ::= \alpha \mid D \multimap A$
- Environments: $\Gamma, \Delta, \Sigma ::= \{x_1 : A_1, \ldots, x_n : A_n\}$

(Multiset. $x_1, \ldots, x_n$ not necessarily distinct.)

- Type inference rules:

$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \qquad \frac{\Gamma, x : A_1, \ldots, x : A_n \vdash M : B \quad x \notin Var(\Gamma)}{\Gamma \vdash \lambda x.M : A_1 \otimes \cdots \otimes A_n \multimap B}\ (\multimap I)$$

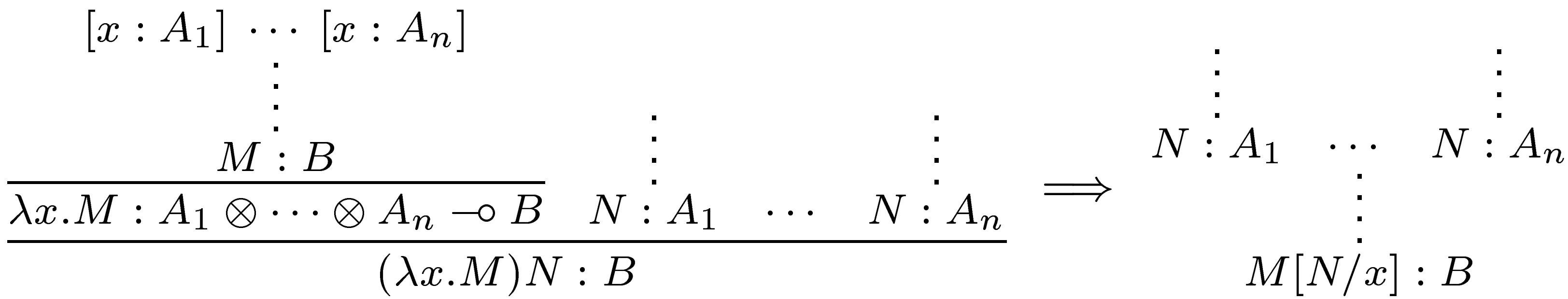
$$\frac{\Gamma_0 \vdash M : A_1 \otimes \cdots \otimes A_n \multimap B \quad \Gamma_1 \vdash N : A_1 \quad \cdots \quad \Gamma_n \vdash N : A_n}{\Gamma_0, \Gamma_1, \ldots, \Gamma_n \vdash M N : B}\ (\multimap E)$$

Type system iMAL

- $\mathcal{D} \rhd \Gamma \vdash M : A \iff \mathcal{D}$ is a derivation for $\Gamma \vdash M : A$.
- $\mathcal{D} \rhd M \iff$ there are Γ, A such that $\mathcal{D} \rhd \Gamma \vdash M : A$.
- $|M|$: the number of λ and applications in the term M.
- $|\mathcal{D}|$: the number of $(\multimap I)$ and $(\multimap E)$ in the derivation $\mathcal{D}$.
- Lemma: $\mathcal{D} \rhd M \Longrightarrow |\mathcal{D}| \geq |M|$.
- Lemma: M is in nf $\Longrightarrow$ there is $\mathcal{D}$ such that $\mathcal{D} \rhd M$ and $|\mathcal{D}| = |M|$.

Derivation size bounds normalization cost

- Subject Reduction Theorem: $\mathcal{D} \rhd \Gamma \vdash M : B$ and $M \longrightarrow N \Longrightarrow \mathcal{D}' \rhd \Gamma \vdash N : B$ and $|\mathcal{D}| > |\mathcal{D}'|$.



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$$\dfrac{\dfrac{\begin{matrix}[x : A_1] \cdots [x : A_n] \\ \vdots \\ M : B\end{matrix}}{\lambda x.M : A_1 \otimes \cdots \otimes A_n \multimap B} \quad \begin{matrix}\vdots \\ N : A_1\end{matrix} \quad \cdots \quad \begin{matrix}\vdots \\ N : A_n\end{matrix}}{(\lambda x.M)N : B} \Longrightarrow \begin{matrix}\vdots & & \vdots \\ N : A_1 & \cdots & N : A_n \\ & \vdots & \\ & M[N/x] : B & \end{matrix}$$

- Corollary: If $\mathcal{D} \triangleright M$, then

Derivation size bounds normalization cost

- Subject Reduction Theorem: $\mathcal{D} \triangleright \Gamma \vdash M : B$ and $M \longrightarrow N \Longrightarrow \mathcal{D}' \triangleright \Gamma \vdash N : B$ and $|\mathcal{D}| > |\mathcal{D}'|$.

$$\dfrac{\dfrac{\begin{matrix}[x:A_1] \cdots [x:A_n]\\ \vdots \\ M:B\end{matrix}}{\lambda x.M : A_1 \otimes \cdots \otimes A_n \multimap B} \quad \begin{matrix}\vdots\\ N:A_1\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\\ N:A_n\end{matrix}}{(\lambda x.M)N : B} \Longrightarrow \begin{matrix}\vdots & & \vdots\\ N:A_1 & \cdots & N:A_n\\ & \vdots & \\ & M[N/x]:B & \end{matrix}$$

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- Corollary: If $\mathcal{D} \rhd M$, then
  1. M strongly normalizes in $|\mathcal{D}|$ steps.
  2. $M \longrightarrow^* N \Longrightarrow |N| \leq |\mathcal{D}|$.

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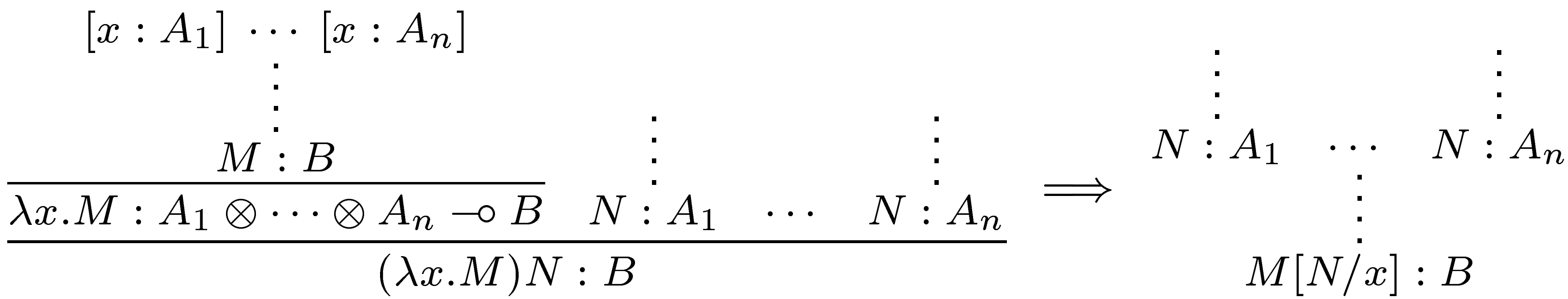
- Lemma 1: If $\mathcal{D} \triangleright \Gamma \vdash M[N/x] : B$ and $x \in FV(M)$, then there is $\mathcal{D}'$ such that $\mathcal{D}' \triangleright \Gamma \vdash (\lambda x.M)N : B$ and $|\mathcal{D}'| = |\mathcal{D}| + 2$.



Normalization cost bounds derivation size

- Lemma 2: If $\mathcal{D} \triangleright \Gamma \vdash M : B$, $x \notin FV(M)$ and N is in nf, then there are $\mathcal{D}', \Delta$ such that $\mathcal{D}' \triangleright \Gamma, \Delta \vdash (\lambda x.M)N : B$ and $|\mathcal{D}'| = |\mathcal{D}| + |N| + 1$.

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ M : B\end{matrix}}{\lambda x.M : A \multimap B} \quad \begin{matrix}\Delta\\ \vdots\\ N : A\end{matrix}}{(\lambda x.M)N : B} \Longrightarrow \begin{matrix}\Gamma\\ \vdots\\ M : B\end{matrix}$$

- Consider the following perpetual reduction strategy:

$$\frac{x \in FV(M)}{(\lambda x.M)N\vec{L} \longrightarrow_P M[N/x]\vec{L}} \qquad \frac{x \notin FV(M) \quad N_1 \longrightarrow_P N_2}{(\lambda x.M)N_1\vec{L} \longrightarrow_P (\lambda x.M)N_2\vec{L}}$$

$$\frac{x \notin FV(M) \quad N \text{ in nf}}{(\lambda x.M)N\vec{L} \longrightarrow_P M\vec{L}} \qquad \frac{M_1 \longrightarrow_P M_2}{\lambda x.M_1 \longrightarrow_P \lambda x.M_2} \qquad \frac{N_1 \longrightarrow_P N_2}{x\vec{M}N_1\vec{K} \longrightarrow_P x\vec{M}N_2\vec{K}}$$

Normalization cost bounds derivation size

- Weak Subject Expansion Theorem: If $\mathcal{D} \rhd N$ (with $\mathcal{D}$ "canonical") and $M \longrightarrow_P N$, then there is a "canonical" $\mathcal{D}'$ such that $\mathcal{D}' \rhd M$ and $|\mathcal{D}'| \leq |\mathcal{D}| + |M| + 1$.
- Two delicate points: In so far as expanding along $\longrightarrow_P$,
  1. the previous lemmas are sufficient to obtain the type for M (Møller);
  2. P-redex is not located above intersections, so that the increase can be bounded by $|M| + 1$:

$$\dfrac{\begin{array}{c} R : B_1 \\ \vdots \\ L : A_1 \end{array} \quad \cdots \quad \begin{array}{c} R : B_1 \\ \vdots \\ L : A_n \end{array}}{\begin{array}{c} L : A_1 \otimes \cdots \otimes A_n \\ \vdots \end{array}} \quad \not\Longrightarrow \quad \dfrac{\begin{array}{c} R' : B_1 \\ \vdots \\ L : A_1 \end{array} \quad \cdots \quad \begin{array}{c} R' : B_n \\ \vdots \\ L : A_n \end{array}}{\begin{array}{c} L : A_1 \otimes \cdots \otimes A_n \\ \vdots \end{array}}$$

Characterization of $\Lambda\mathrm{P}_{SN}$

- Theorem 1: $\mathcal{D} \triangleright M \implies M$ strongly normalizes in length $|\mathcal{D}|$ and size $|\mathcal{D}|$.

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- Static Characterization of Λ_{PSN}: $M \in \Lambda_{PSN} \Longleftrightarrow$ $\exists d \in N. \forall w \in W.$ there is $\mathcal{D}_w$ such that

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- In short: $M \in \Lambda\mathbf{P}_{SN} \iff \{M\overline{w}\}_{w \in W}$ have polynomial size derivations.

Degree of intensional expressivity

- Let $M \in \Lambda\mathbf{P}_{SN}$. How complex are the derivations $\{\mathcal{D}_w \rhd M\overline{w}\}_{w \in W}$?
- $w \mapsto \mathcal{D}_w$ is at least Ptime computable.
- Let $M : \mathbf{W}^n \multimap \mathbf{B} \ \in \mathbf{SLL}$.
- $\{(\mathcal{D}_n)\mathcal{E}_w \rhd M\overline{w}\}_{w \in W, |w|=n}$
- The main part $\{\mathcal{D}_n\}_{n \in N}$ only depends on the input length. Moreover, $n \mapsto \mathcal{D}_n$ is Logspace computable.
- Proposal: Degree of intensional expressivity = complexity of derivations.
- Open Problem: What about other systems such as DLAL?

Parallelization

- $M \in SN \Longrightarrow M \in i\mathbf{MAL} \Longrightarrow M^e \in e\mathbf{MAL}$
- In (Terui 05), we have shown:

$e\mathbf{MLL}$ Proof nets	$\approx$	Boolean circuits
Size	$\approx$	Size
Logical depth of cut-formulas	$\approx$	Circuit depth
Poly-size, Polylog-depth PNs	$\approx$	NC circuits

- If the logical depth is shallow, $M \mapsto M^e$ leads to effective parallelization.

Beyond $\Lambda\mathrm{P}_{SN}$

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- $\Lambda\mathbf{P}_{SN}$ is too small to be of practical interest.
- Iteration of if-then-else not allowed in $\Lambda\mathbf{P}_{SN}$.
- Let $M \equiv \lambda x.(\text{if } x = 0 \text{ then } x \text{ else } x)$. Then

$$M^n\overline{0} \longrightarrow^*_{CBV} M^{n-1}\overline{0} \longrightarrow^*_{CBV} \cdots \longrightarrow^*_{CBV} \overline{0}$$

takes $O(n)$ steps.

$$M\overline{0} \longrightarrow^* \text{if } M^{n-1}\overline{0} = \overline{0} \text{ then } M^{n-1}\overline{0} \text{ else } M^{n-1}\overline{0}$$
$$\longrightarrow^* \cdots \longrightarrow^* 0$$

takes $O(2^n)$ steps.

Beyond $\Lambda\mathbf{P}_{SN}$

- $\Lambda\mathbf{P}_{SN}$ is too small to be of practical interest.
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  takes $O(2^n)$ steps.
- In particular, $\Lambda\mathbf{BC} \not\subset \Lambda\mathbf{P}_{SN}$.

Beyond $\Lambda\mathbf{P}_{SN}$

- Let R be a feasible reduction strategy.
- An (abstract) size function s : Derivations $\longrightarrow \mathbb{N}$ is admissible for (M, R) if for any M_1 such that $M\overline{w} \longrightarrow_R^* M_1$,
  1. $\mathcal{D} \rhd M_1 \implies s(\mathcal{D}) \geq |M_1|$
  2. $(\mathcal{D}_1 \rhd M_1 \longrightarrow_R \mathcal{D}_2 \rhd M_2) \implies s(\mathcal{D}_1) > s(\mathcal{D}_2)$
- Theorem: $M \in \Lambda\mathbf{P}_R \iff$
  1. there is a size function s admissible for (M, R)
  2. $\{M\overline{w}\}_{w\in\mathbf{W}}$ has polynomial size derivations w.r.t. s.
- In practice, we have to find suitable size function s for each term M to be analyzed.

Case Study: a ramified Ptime system

- Recall (Leivant-Marion 94) characterizes $\mathbf{FP}$ based on lambda-calculus with pairing and some constants, where
- Lower tier: word algebra terms ϵ, $0(\epsilon)$, $1(0(\epsilon))$, $\ldots$ of base type o
- Higher tier: Church words: $\lambda fx.f0(f1(f1(x)))$, $\ldots$
- $f \in \mathbf{FP} \iff f$ represented by

$$M_f : ((A \to A) \to A \to A) \to o$$

where $A \equiv o \times \cdots \times o$.

Case Study: a ramified Ptime system

- We consider a variant characterization based on pure lambda-calculus with linear polymorphism and linear recursive types, where
- Lower tier: Scott words of type $\forall\alpha.\mu\beta.(\mathbf{B} \multimap \beta \multimap \alpha) \multimap \alpha \multimap \alpha$
- Higher tier: Church words of type $\forall\alpha.(\mathbf{B} \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha$ (with $A \Rightarrow B \equiv !A \multimap B$)

The system $\mathbf{AL}_{!\multimap\forall\mu}$

- $\mathbf{AL}_{!\multimap\forall\mu}$: Intuitionistic affine linear logic with

$$\frac{\Gamma \vdash M : A \quad \alpha \notin FV(\Gamma)}{\Gamma \vdash M : \forall\alpha.A} \qquad \frac{\Gamma \vdash M : \forall\alpha.A}{\Gamma \vdash M : A[L/\alpha]}$$

$$\frac{\Gamma \vdash M : L[\mu\alpha.L/\alpha]}{\Gamma \vdash M : \mu\alpha.L} \qquad \frac{\Gamma \vdash M : \mu\alpha.L}{\Gamma \vdash M : L[\mu\alpha.L/\alpha]}$$

- where L is purely linear, i.e., without !.

Scott numerals

- For simplicity, we consider unary numerals rather than words.
- Church numerals of type $\mathbf{N}_C \equiv \forall\alpha.(\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha$
  nonlinear, support iteration
- Scott numerals of type $\mathbf{N}_S \equiv \forall\alpha.\mu\beta.(\beta \multimap \alpha) \multimap \alpha \multimap \alpha$
  linear, support basic functions

$$
\begin{array}{rcll}
\overline{0} & \equiv & \lambda xy.y & : \mathbf{N}_S \\
\overline{n+1} & \equiv & \lambda xy.x\overline{n} & : \mathbf{N}_S \\
suc & \equiv & \lambda z.\lambda xy.xz & : \mathbf{N}_S \multimap \mathbf{N}_S \\
prd & \equiv & \lambda z.z(\lambda x.x)\overline{0} & : \mathbf{N}_S \multimap \mathbf{N}_S \\
cond & \equiv & \lambda z_1 z_2 z_3.z_1(\lambda w.z_2)z_3 & : \mathbf{N}_S^3 \multimap \mathbf{N}_S
\end{array}
$$

Representation of FP

- Theorem: Any $f \in \mathbf{FP}$ can be represented by a closed term $M_f : \mathbf{W}_C \Rightarrow \mathbf{W}_S$.
- Proof: Any $M'_f : ((A \to A) \to A \to A) \to o$ with $A \equiv o \times \cdots \times o$ in (LM94) can be easily translated into $\mathbf{AL}_{!\multimap\forall\mu}$.
- Exercise: Give a characterization of PSPACE in $\mathbf{AL}_{!\multimap\forall\mu}$.

Pruned size of derivations

- Pruned size $s(\mathcal{D})$ defined by:
- When $\mathcal{D}$ is (var), $s(\mathcal{D}) = 0$.
- $s(\mathcal{D}) = s(\mathcal{D}_0) + 1$ when $\mathcal{D}$ is

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_0\\ \Gamma, x : A_1, \ldots, x : A_n \vdash M : B\end{array}}{\Gamma \vdash \lambda x.M : A_1 \otimes \cdots \otimes A_n \multimap B}\ (\multimap I)$$

- $s(\mathcal{D}) = s(\mathcal{D}_0) + min(s(\mathcal{D}_1), \ldots, s(\mathcal{D}_n)) + |\Gamma_1, A_1, \ldots, \Gamma_n, A_n|$, when $\mathcal{D}$ is

$$\dfrac{\begin{array}{c}\vdots\ \mathcal{D}_0\\ \Delta_0, \Gamma_0 \vdash M : A_1 \otimes \cdots \otimes A_n \multimap B\end{array} \quad \begin{array}{c}\vdots\ \mathcal{D}_1\\ \Delta_1, \Gamma_1 \vdash N : A_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\ \mathcal{D}_n\\ \Delta_n, \Gamma_n \vdash N : A_n\end{array}}{\Delta_0, \Gamma_0, \Delta_1, \Gamma_1, \ldots, \Delta_n, \Gamma_n \vdash MN : B}\ (\multimap E)$$

where Δ_i consists of redundant types (to be explained later).

Normalization bound

- Given $M : \mathbf{N}_C \Rightarrow \mathbf{N}_S$ and $\overline{n} : \mathbf{N}_C$:
- Linearize $\mathcal{D} \rhd M\overline{n}$ in $\mathbf{AL}_{!\multimap\forall\mu}$ into $\mathcal{D}_0 \rhd M\overline{n} : \mathcal{L}^k \multimap \mathbf{N}_C$ in $i\mathbf{MAL}_{\forall\mu}$:

$$\begin{array}{lcl} L & \mapsto & L \\ !L \multimap L & \mapsto & \mathbf{L} \equiv L \otimes \cdots \otimes L \multimap L \\ !(!L \multimap L) \multimap !L \multimap L & \mapsto & \mathcal{L} \equiv \mathbf{L}_1 \otimes \cdots \otimes \mathbf{L}_k \multimap \mathbf{L}_{k+1} \end{array}$$

- Call the images of !-types redundant types.

Normalization bound

- Strategy R: First reduce all redices of type $\mathcal{L}$ and $\mathbf{L}$ to obtain M_1, then reduce the rightmost $(\lambda x.M)N$ with N in nf.
- After M_1, the pruned size s is admissible for R:
  - $\mathcal{D}_i \rhd M_i \implies s(\mathcal{D}_i) \geq |M_i|$
  - $(\mathcal{D}_i \rhd M_i \longrightarrow_R \mathcal{D}_{i+1} \rhd M_{i+1}) \implies s(\mathcal{D}_i) > s(\mathcal{D}_{i+1})$
- the latter because of
  - If $(\lambda x.M)N$ is the redex to be reduced, N is in nf and does not contain a free variable of redundant type.

Normalization bound

- If $\mathcal{D}_1 \triangleright M_1$, $s(M_1)$ is polynomial in N.
- By finite dispatching, the size function s' admissible for (M, R) can be obtained.
- Theorem: $M : \mathbf{W}_C \Rightarrow \mathbf{W}_S$ in $\mathbf{AL}_{!\multimap\forall\mu}$
  $\Longrightarrow M \in \Lambda\mathbf{P}_R \subseteq \Lambda\mathbf{P}_{WN}$.

Logspace

- Let $\mathbf{W}'_C \equiv \forall\alpha.(\alpha \multimap \alpha) \Rightarrow \alpha \multimap \alpha$.
- Any $f \in \mathbf{BC}^{--}$ can be represented by $M_f : \mathbf{W}'_C \multimap \mathbf{W}_S$.
- $\mathbf{BC}$: safe recursion

$$\begin{aligned} f(0, x; y) &= g(x; y) \\ f(n+1, x; y) &= h(n, x; f(n), y) \end{aligned}$$

- $\mathbf{BC}^-$: safe recursion with linear safe arguments

$$\begin{aligned} f(0, x : y) &= g(x : y) \\ f(n+1, x : y) &= h(n, x : f(n, x : y)) \end{aligned}$$

- A slight extension of $\mathbf{BC}^-$ captures LOGSPACE (Mairson and Møller-Neergaard)

Logspace

- $\mathbf{BC}^{--}$: safe recursion with linear safe arguments and without regressive arguments

$$\begin{array}{rcl} f(0, x : y) & = & g(x : y) \\ f(n+1, x : y) & = & h(x : f(n)) \end{array}$$

- Given $M : \mathbf{W}'_C \multimap \mathbf{W}_S$ and $\overline{w} : \mathbf{W}'_C$ in $\mathbf{AL}_{! \multimap \forall \mu}$,
- $M\overline{w}$ can be linearized into $i\mathbf{MAL}$ and explicated into $e\mathbf{MAL}$ and η-expanded as $(M\overline{w})_\eta$.
- $(M\overline{w})_\eta$ has size polynomial in $|w|$.
- Logspace normalization procedure for η-expanded $\mathbf{MLL}$ (Terui 2002, Mairson 2004) could be applied...
- Seems to suggest a connection between two systems associated to Logspace.

Summary

- $\Lambda\mathbf{P}$, $\Lambda\mathbf{P}_{SN}$, $\Lambda\mathbf{P}_{WN}$ not r.e.
- Implicit vs Explicit typings:
  - Explication: $i\mathbf{L} \longrightarrow e\mathbf{L}$
  - Linearization: $SN = i\mathbf{MAL}$
- $\Lambda\mathbf{P}_{SN}$ = terms having polysize derivations in $i\mathbf{MAL}$.
- Degree of intensional expressivity = Complexity of derivations ?
- $\mathbf{W}_C \Rightarrow \mathbf{W}_S$ in $\mathbf{AL}_{!\multimap\forall_\mu}$ captures $\mathbf{FP}$
- Showed $M : \mathbf{W}_C \Rightarrow \mathbf{W}_S$ belongs to $\Lambda\mathbf{P}_{WN}$ by using an abstract size function.
- Suggested a connection between recursion theoretic and logical characterizations of LOGSPACE.