A Measure of Space for Computing over the Reals

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Plan of the Talk

- The BSS Model of Computation over the Reals
- Michaux's Result - Computing in Constant Space
- Koiran's Weak Model
- A Weak Measure of Space

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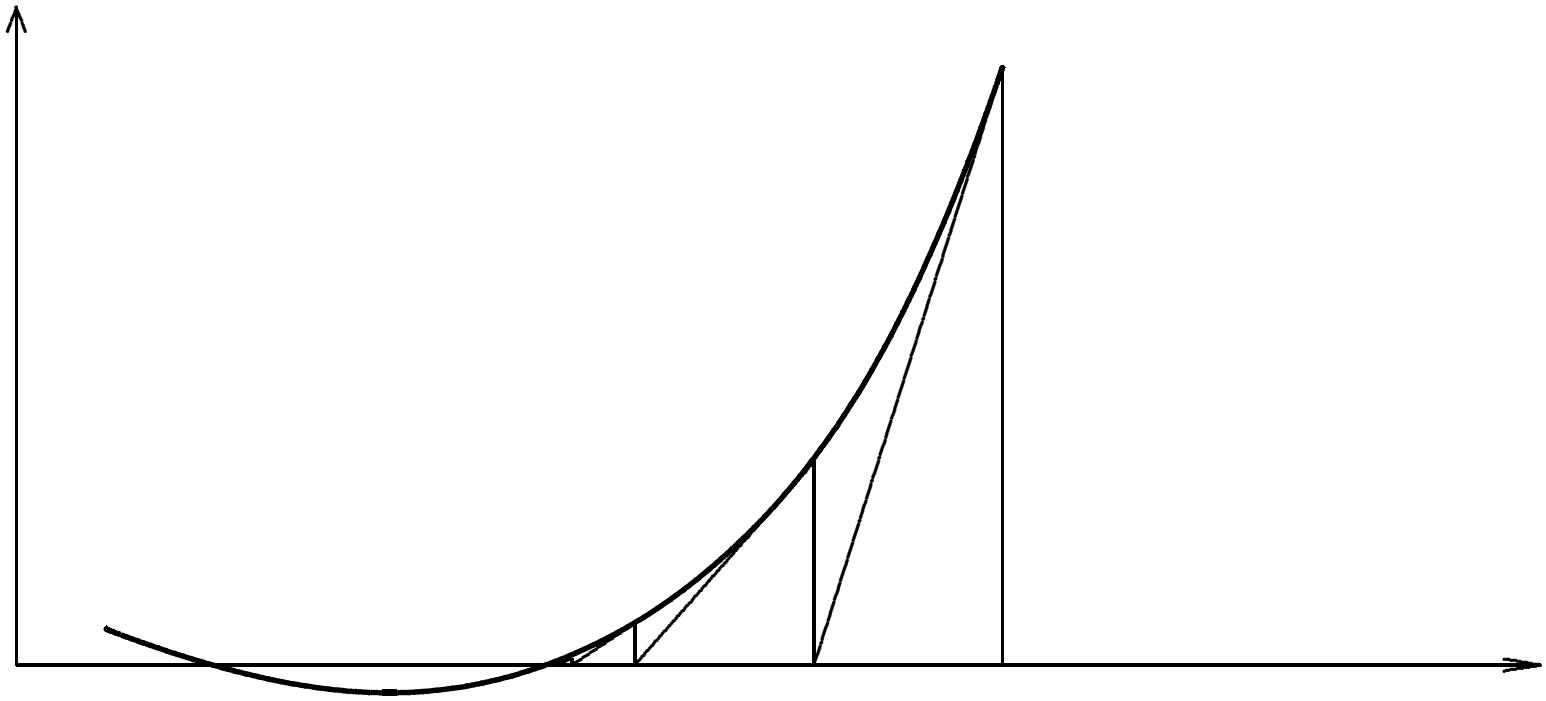

Newton's Method for finding a zero of a function.

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- some shift nodes move the scanning head on the tape,
- some copy nodes duplicate the content of some cells.

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Convention: One Input Tape, one Work Tape, one Output Tape.

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- Decision Problems - or Languages - are subsets of $\mathbb{R}^*$.

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- The set of points that converge under Newton's algorithm is undecidable.

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- $\mathrm{EXP}_{\mathbb{R}}$: subsets of $\mathbb{R}^*$ decided in exponential time.

Complexity (2)

There exist natural $\mathrm{NP}_{\mathbb{R}}$ and $\mathrm{coNP}_{\mathbb{R}}$-complete problems. Ex: $4\mathrm{FEAS}_{\mathbb{R}}$ (existence of a zero for a real polynomial of degree 4) is $\mathrm{NP}_{\mathbb{R}}$-complete (reductions in $\mathrm{P}_{\mathbb{R}}$).

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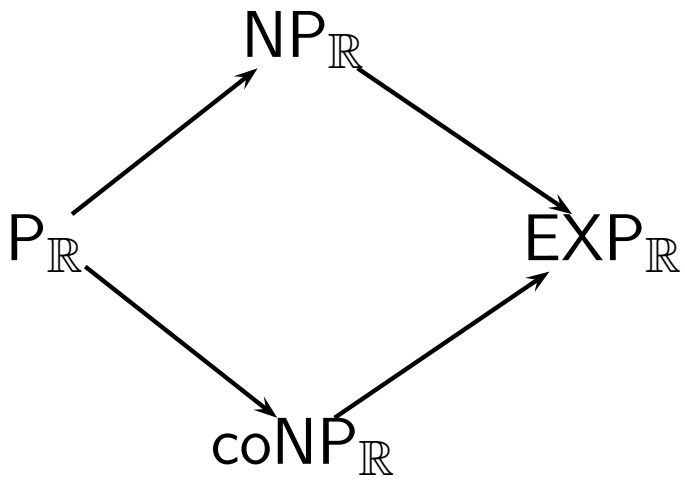
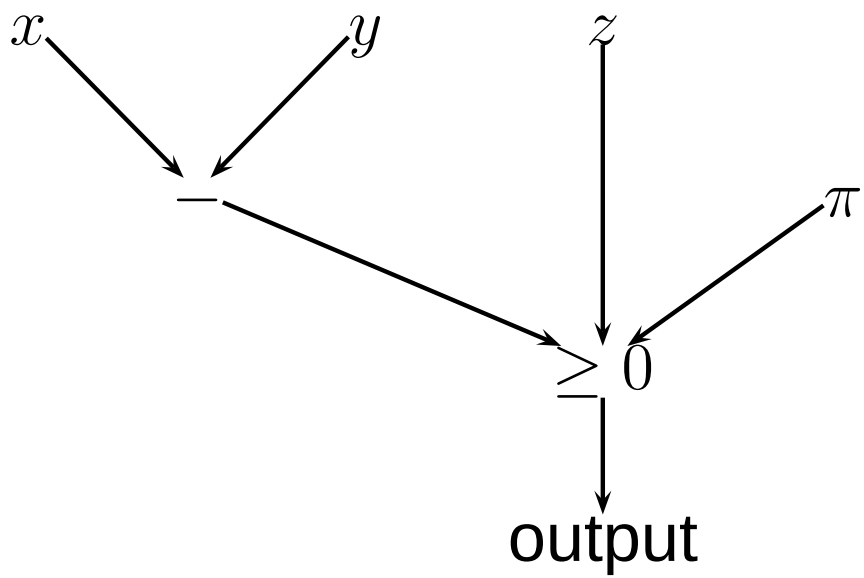


Question: $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$?

Algebraic Circuits

An algebraic circuit $\mathscr{C}$ computes $F_{\mathscr{C}} : \mathbb{R}^n \to \mathbb{R}^m$.



if $x - y \geq 0$ then z else π, $n = 3, m = 1$

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- $\mathsf{NC}_{\mathbb{R}} = \bigcup_{k\in\mathbb{N}} \mathsf{NC}^k_{\mathbb{R}}$.
- $\mathsf{PAR}_{\mathbb{R}}$: subsets of $\mathbb{R}^*$ decided by a uniform family of circuits of polynomial depth.

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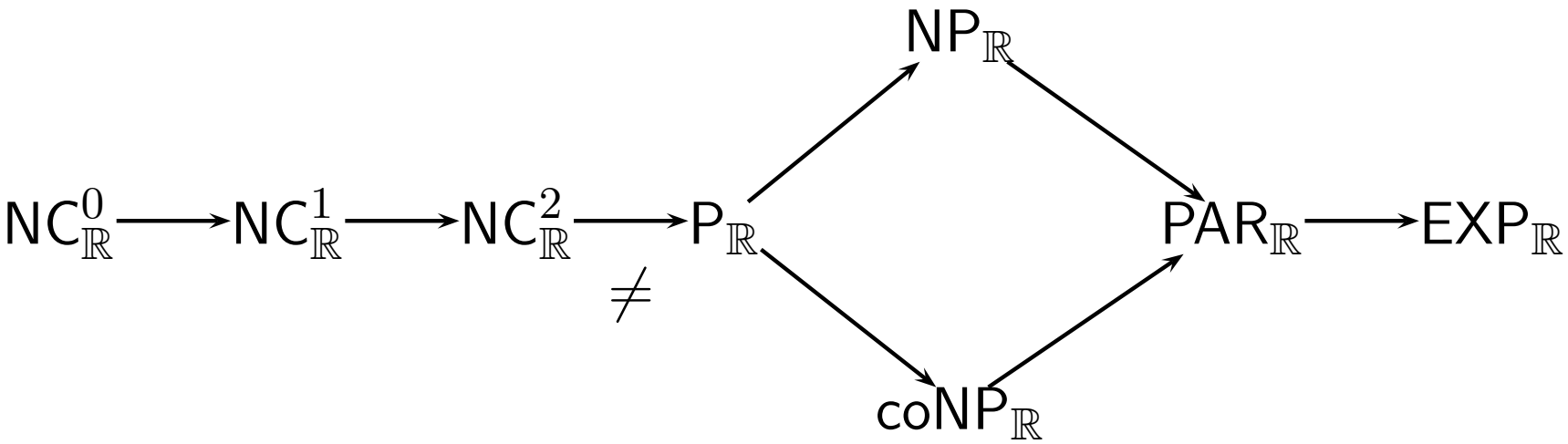

Questions of Space

Unit measure of space: # of tape cells used.

Michaux's Result
Let $L \subseteq \mathbb{R}^*$ be a language decided in bounded time by a machine M. There exists $k \in \mathbb{N}$ and a machine M' deciding L in bounded time and working space less than k.

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$M' \in \mathsf{NC}^2_{\mathbb{R}}$. Existence of a more natural class?

Michaux's Result: Computed Values

Let M be a machine, with

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Lemma:
On any input $x_1, \ldots, x_n \in \mathbb{R}^n$, at any computation step k, any non-empty cell e_l on the work tape holds the evaluation of a rational fraction $f_{l,k} \in \mathbb{Z}(X_1, \ldots, X_{n+m})$ on $(x_1, \ldots, x_n, A_1, \ldots, A_m)$.

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Simulation of one arithmetical computation step: symbolic binary computation.

Simulation of one branch step: numerical evaluation of the arguments, and comparison.

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Weak cost of $y * z$: max of the degrees and of the coefficient heights of f_y and f_z.

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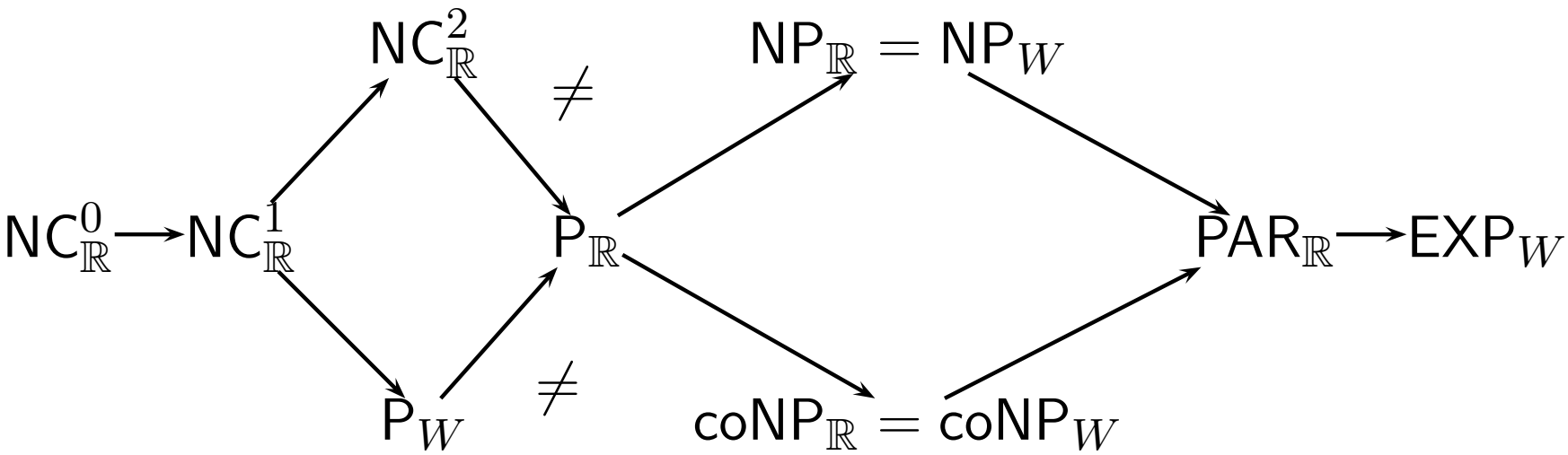

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$M' \in \mathsf{NC}^2_{\mathbb{R}} \cap \mathsf{P}_W$
$M' \notin \mathsf{NC}^1_{\mathbb{R}}$?

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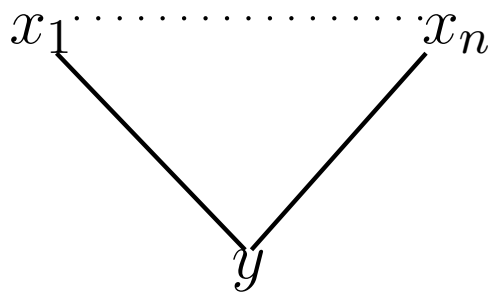

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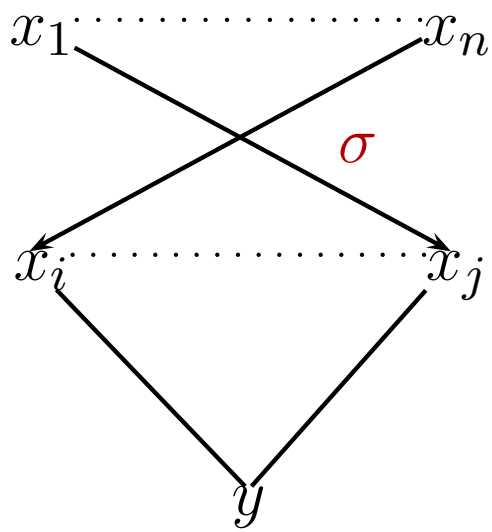

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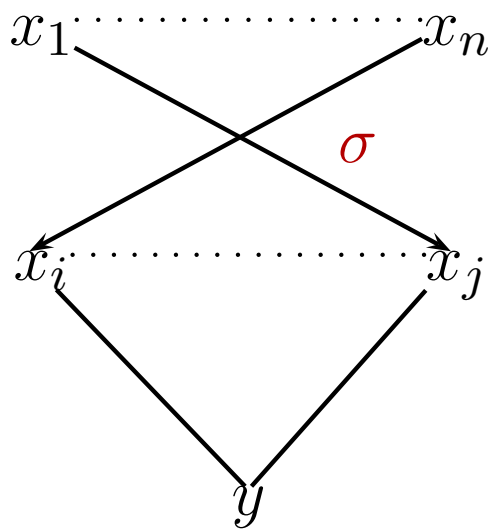

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PB 2: computation of a minimal tree $\to$ factorization problem.

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$|y|_W$ is the size of a explicit (sequence of monomials) boolean description of f_y, modulo the permutation.

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- The algorithm M' is in $\mathsf{LOGSPACE}_W$.
- Michaux's simulation of M' is in $\mathsf{LOGSPACE}_W$.
- There exists some problems decidable in bounded time, not decidable in constant weak space.

$\mathsf{P}_{\mathbb{R}}$-completeness

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Proof: P-completeness of the Boolean Circuit Decision Problem under $\mathsf{LOGSPACE}$-reductions.

Structural Complexity

Theorem:

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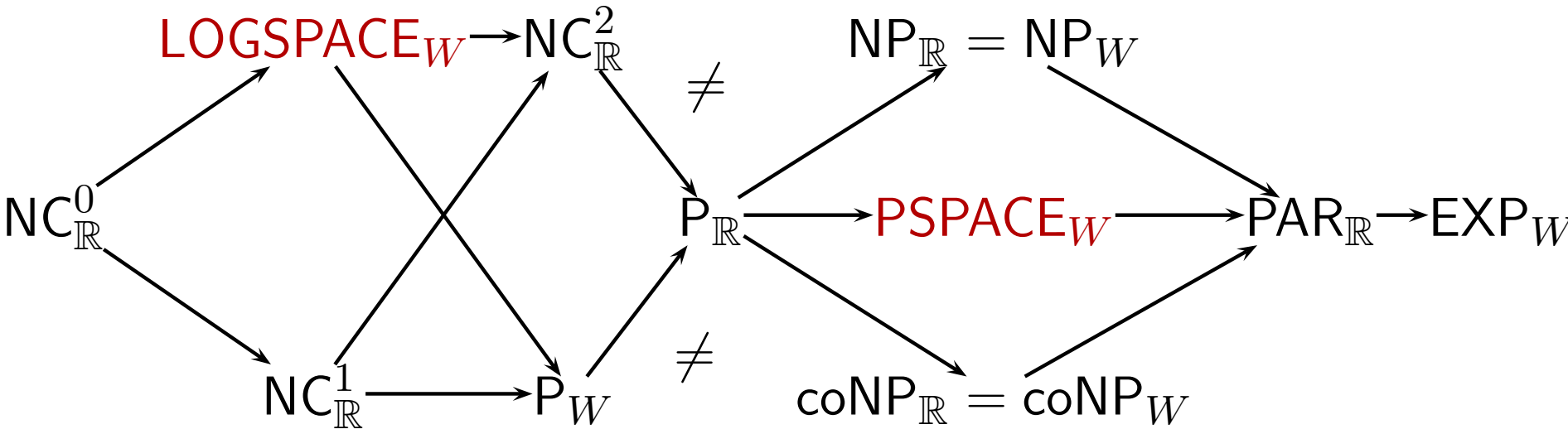
Proof:

- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_{\mathbb{R}}$: enumeration of all configurations.
- $\mathsf{LOGSPACE}_W \subset \mathsf{P}_W$: Koiran's Lemma.
- $\mathsf{LOGSPACE}_W \subset \mathsf{NC}^2_{\mathbb{R}}$: $\mathsf{P}_{\mathbb{R}}$-uniform construction of the configuration graph of a $\mathsf{LOGSPACE}_W$ machine, and graph reachability in NC^2.
- $\mathsf{PSPACE}_W \subset \mathsf{PAR}_{\mathbb{R}}$: Corollary.

Summary and Open Questions

Inclusions:



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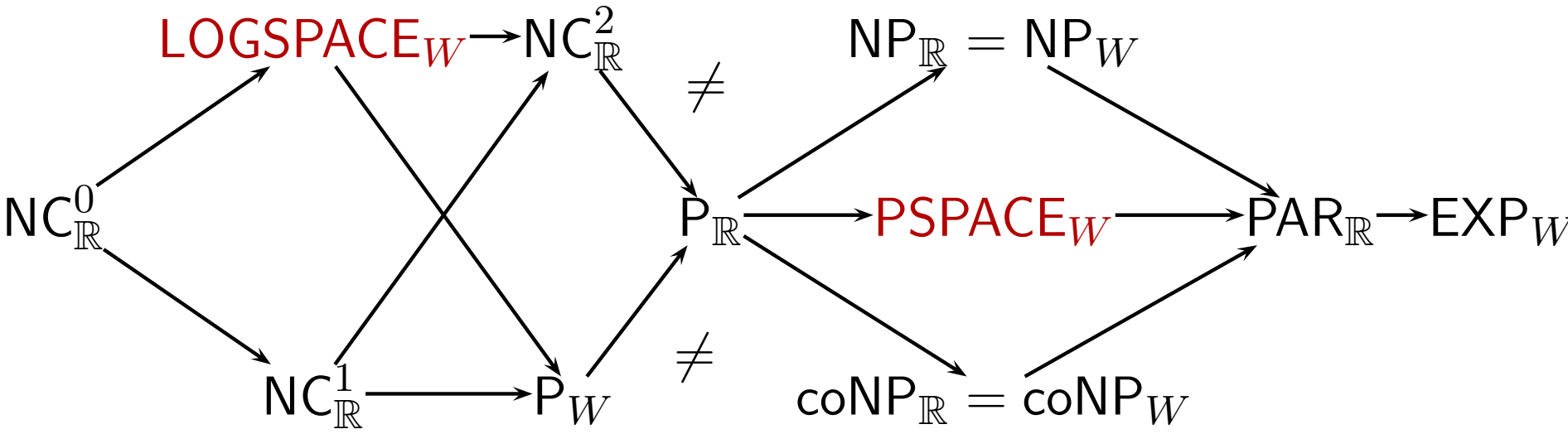

Conjectures:

- $\mathsf{NC}^1_{\mathbb{R}} \not\subset \mathsf{LOGSPACE}_W$, $\mathsf{LOGSPACE}_W \not\subset \mathsf{NC}^1_{\mathbb{R}}$.
- $\mathsf{PSPACE}_W = \mathsf{PAR}_{\mathbb{R}}$