A phase transition in block-weighted random maps

Séminaire au LIPN
13 septembre 2022

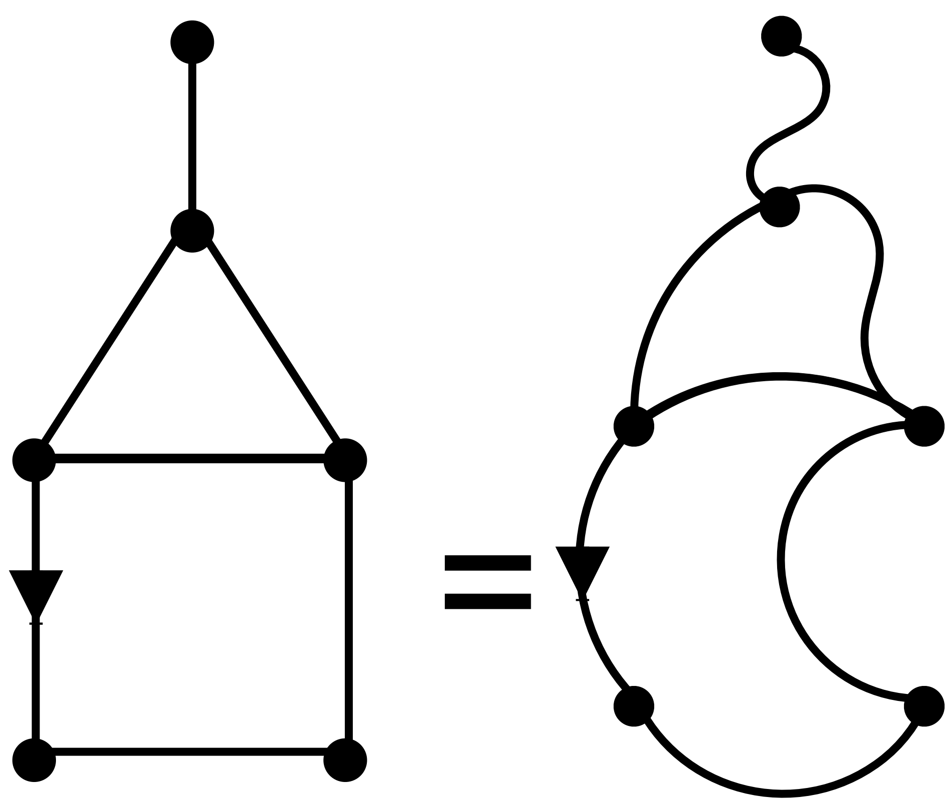
Zéphyr Salvy

William Fleurat

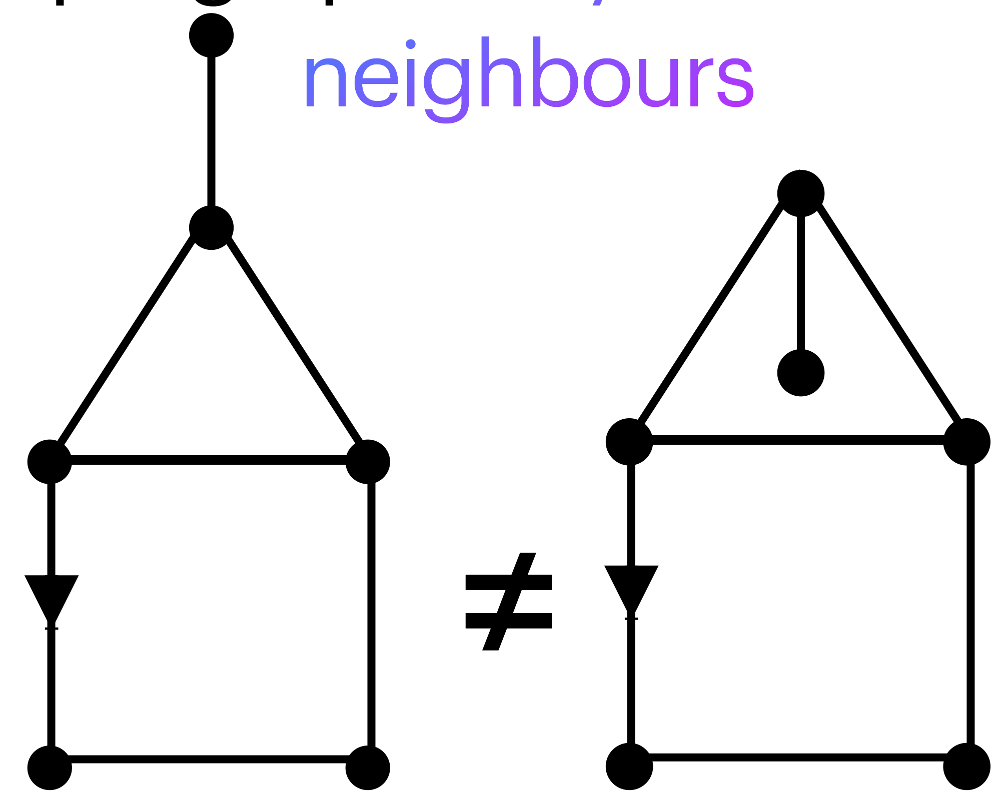
LIGM, Université Gustave Eiffel

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms

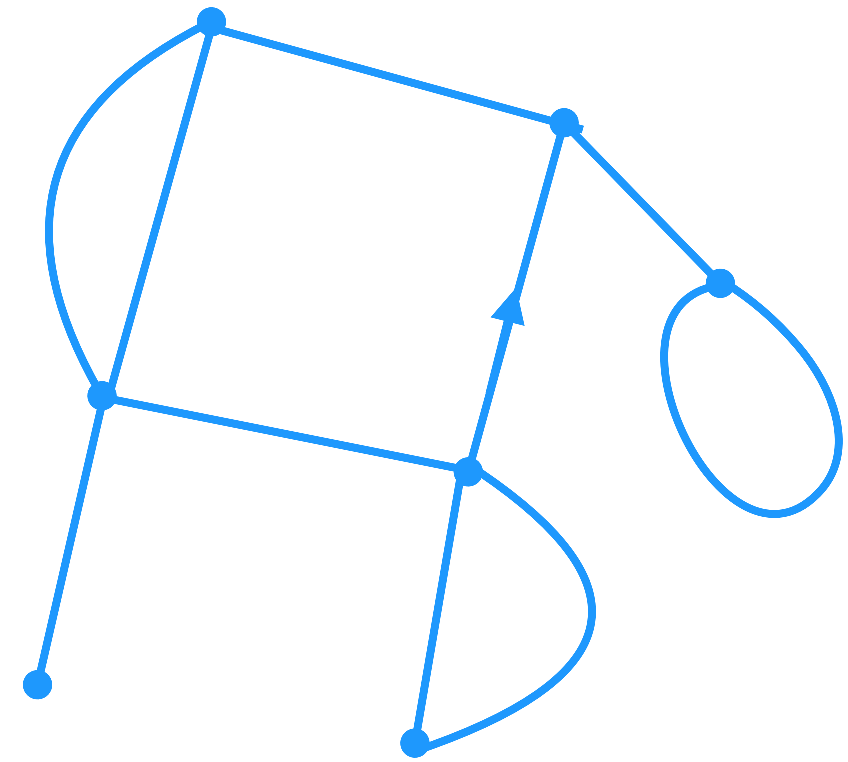


Map = graph + cyclic order on neighbours

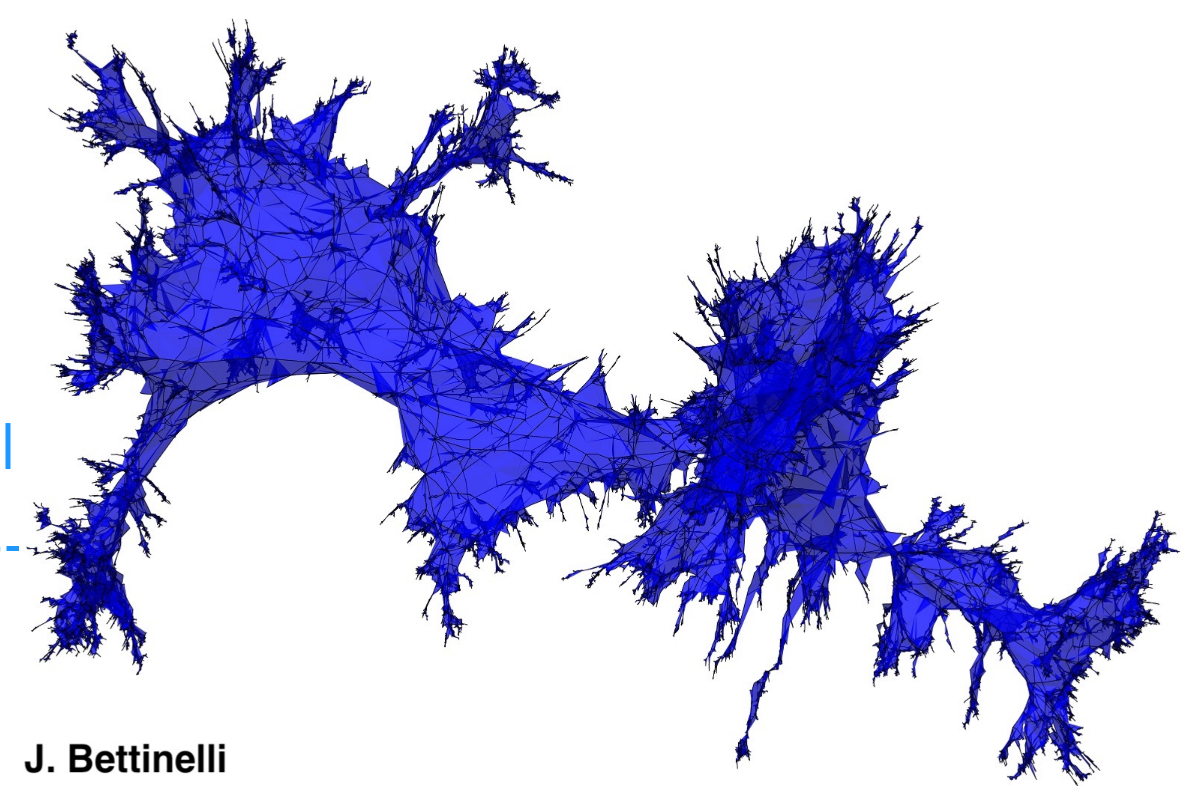


- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Universality results for planar maps



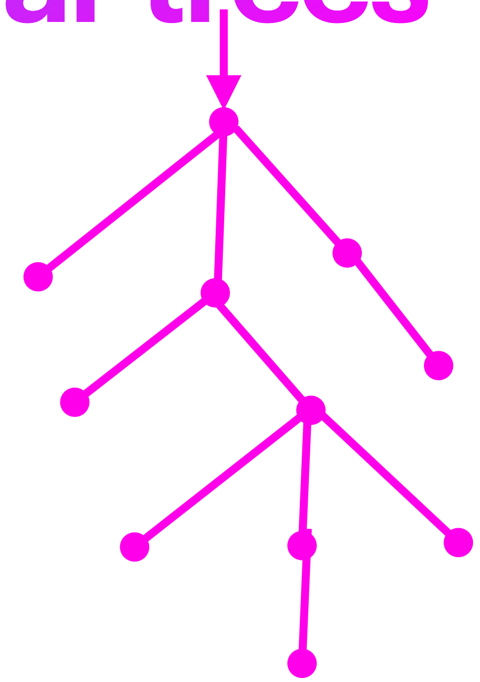
- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963, Drmota, Noy, Yu 2020];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for arbitrary maps [Bettinelli, Jacob, Miermont 2014];
- Universality:
  - Same enumeration;
  - Same scaling limit, e.g. for quadrangulations [Miermont 2013], triangulations & 2q-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].



J. Bettinelli

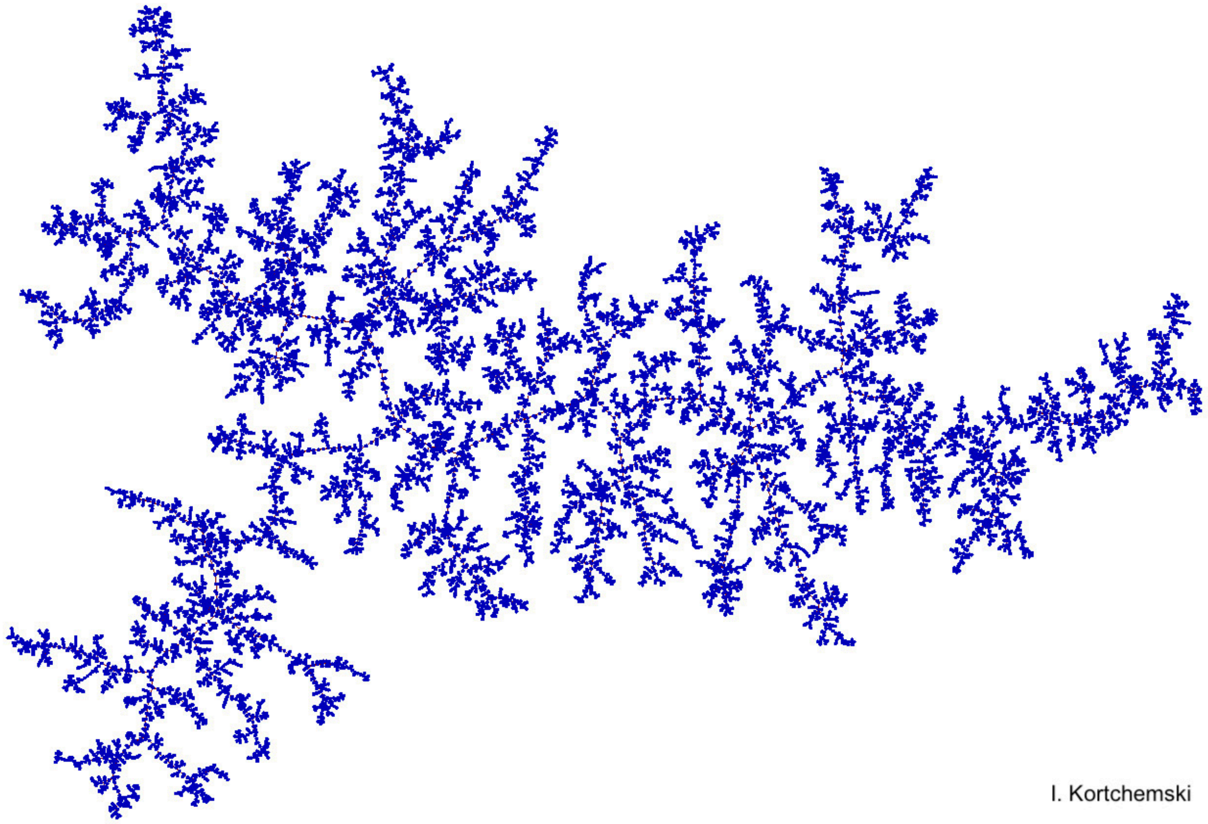
Brownian sphere $\mathcal{S}_e$

Universality results for planar trees



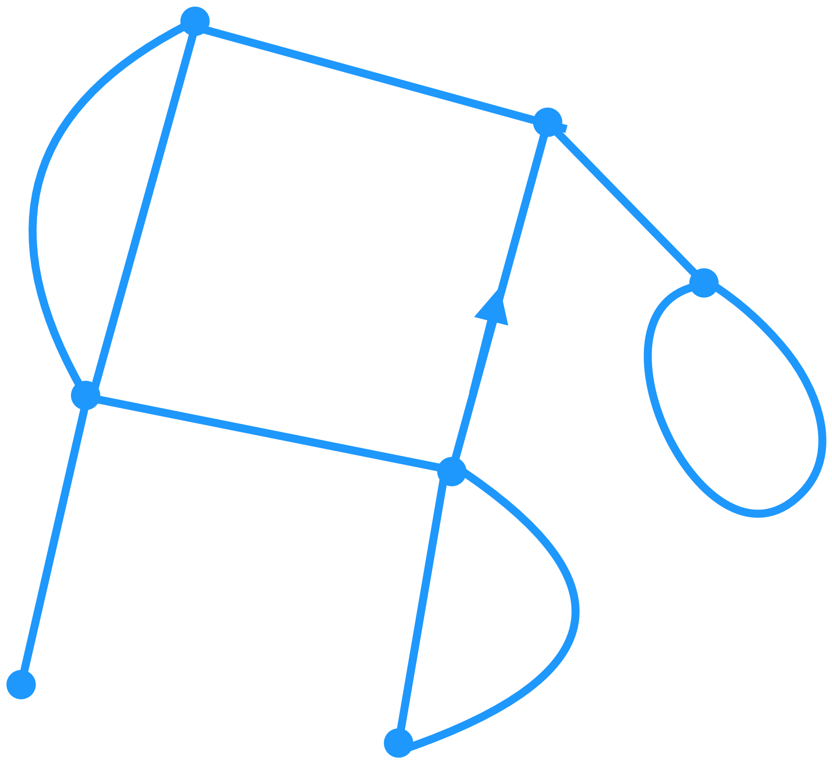
- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

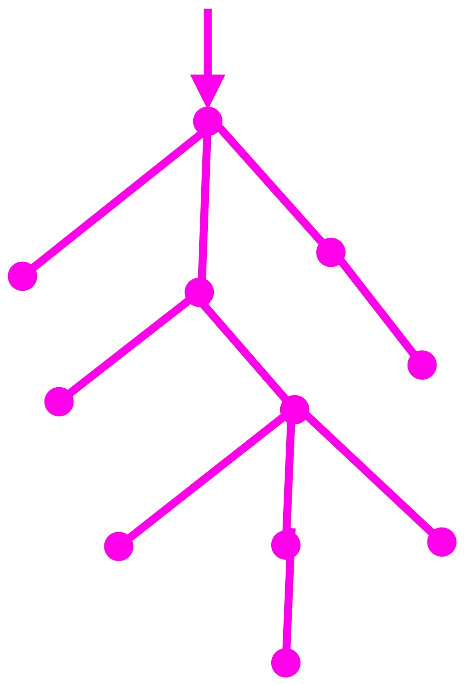
Models with (very) constrained boundaries



Brownian tree $\mathscr{T}_e$

Motivation





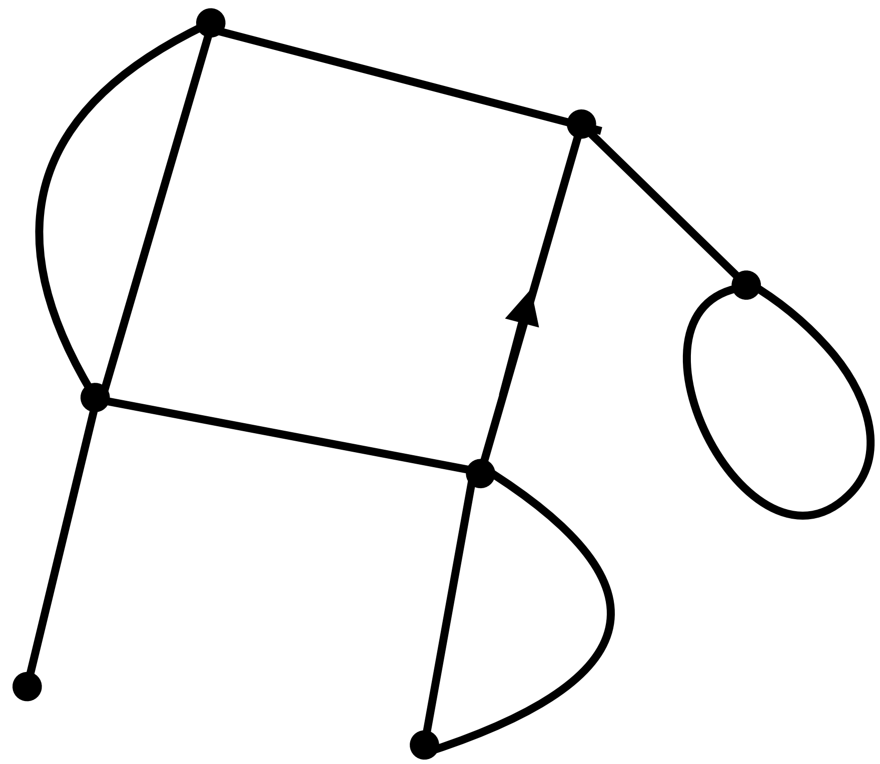
Interpolating model?

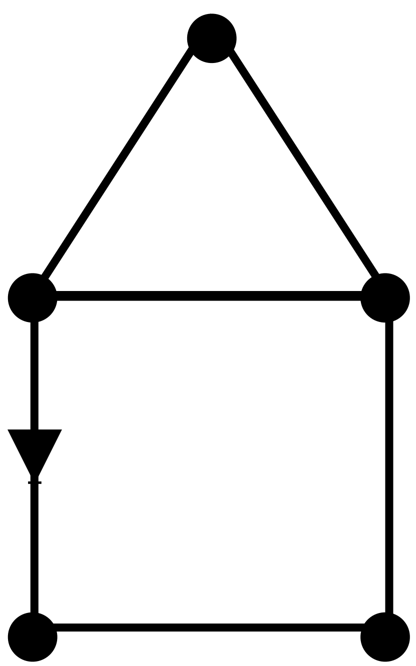
2-connectivity

Cut vertex: vertex that when removed disconnects the map

2-connected: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

Block = maximal (for inclusion) 2-connected submap



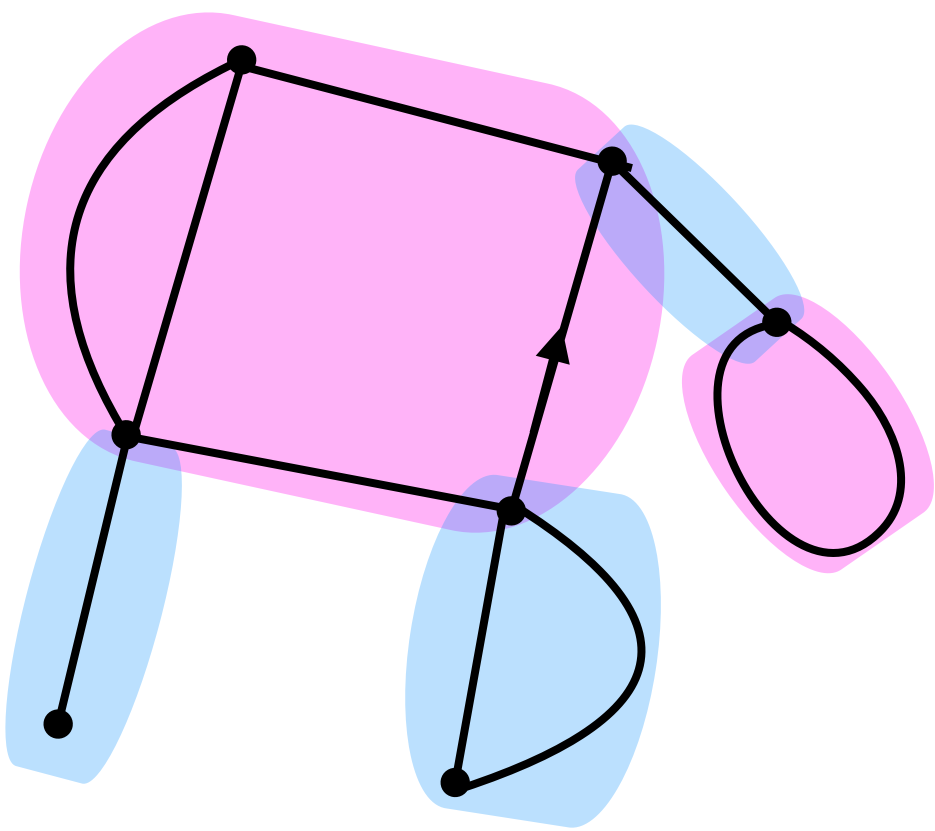


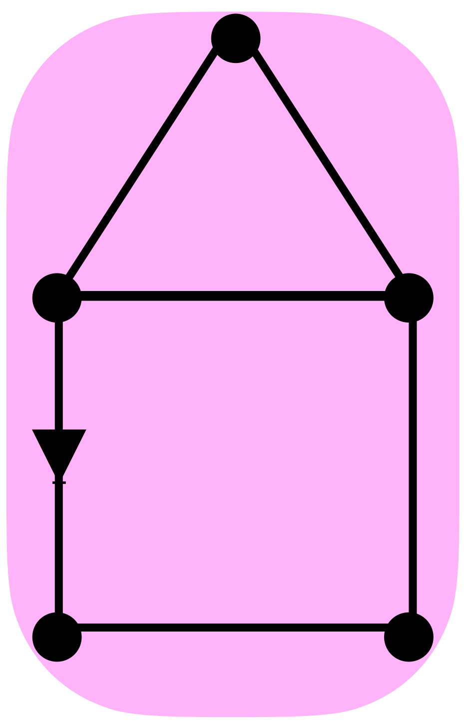
2-connectivity

Cut vertex: vertex that when removed disconnects the map

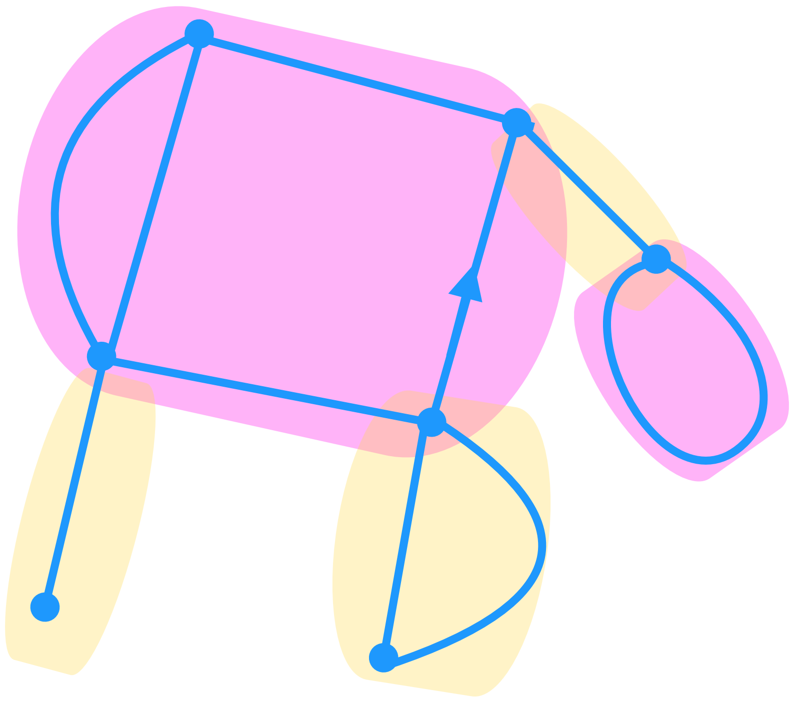
2-connected: no cut vertex (=to be able to disconnect, at least two vertices must be removed)

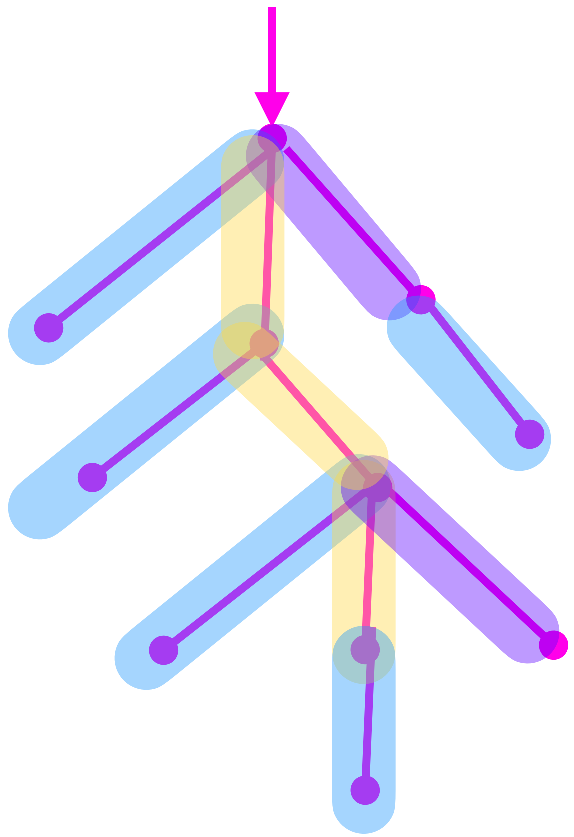
Block = maximal (for inclusion) 2-connected submap





Motivation





Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

Only small blocks.

Interpolating model?

Outline of the talk

A phase transition in block-weighted random maps

I. Approach

II. Largest blocks

III. Similar model: quadrangulations

IV. Scaling limits

V. Perspectives

I. Approach

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

$\mathscr{M}_n$ = {maps of size n},

$\mathfrak{m} \in \mathscr{M}_n$,

$Z_{n,u}$ = normalisation.

Inspired by [Bonzom 2016].

- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

Given u, asymptotic behaviour when $n \to \infty$?

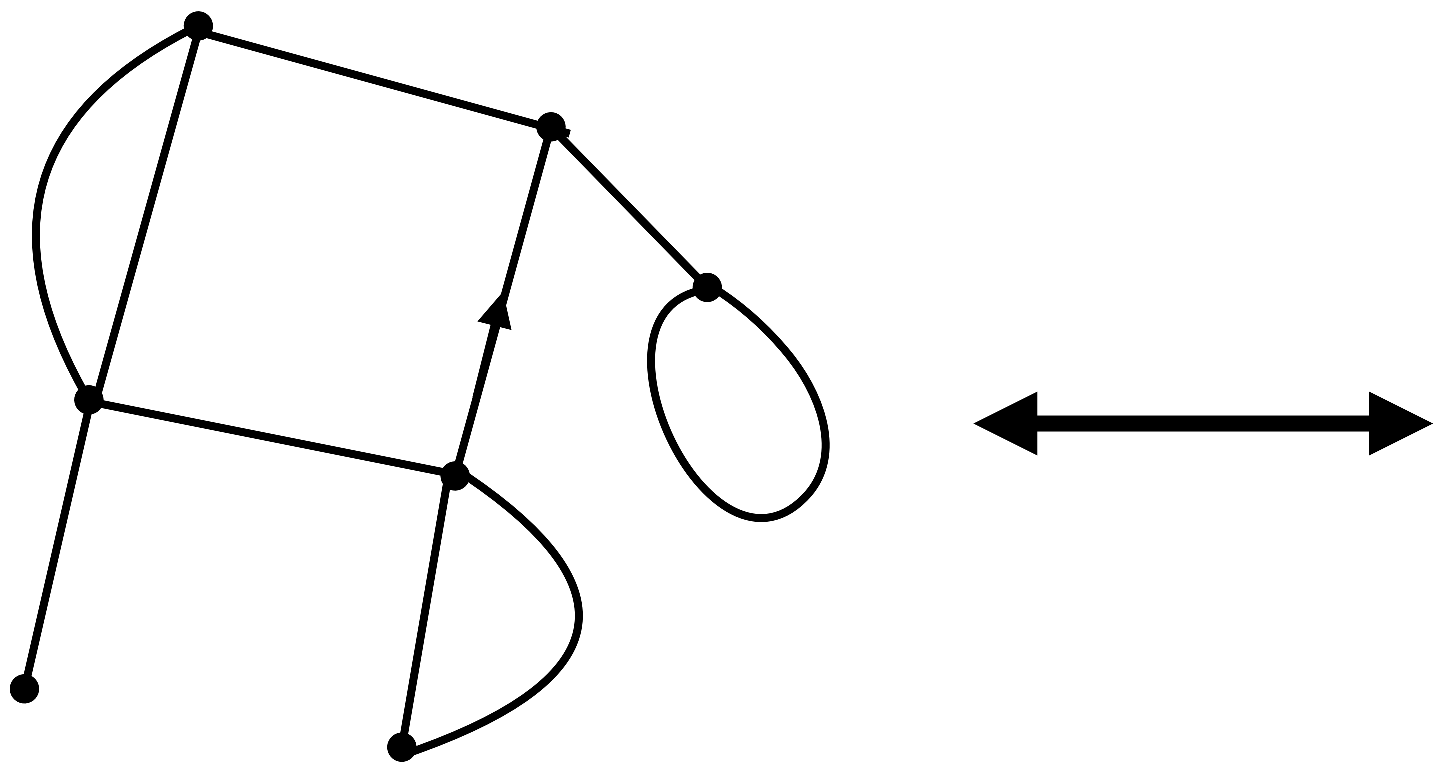
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

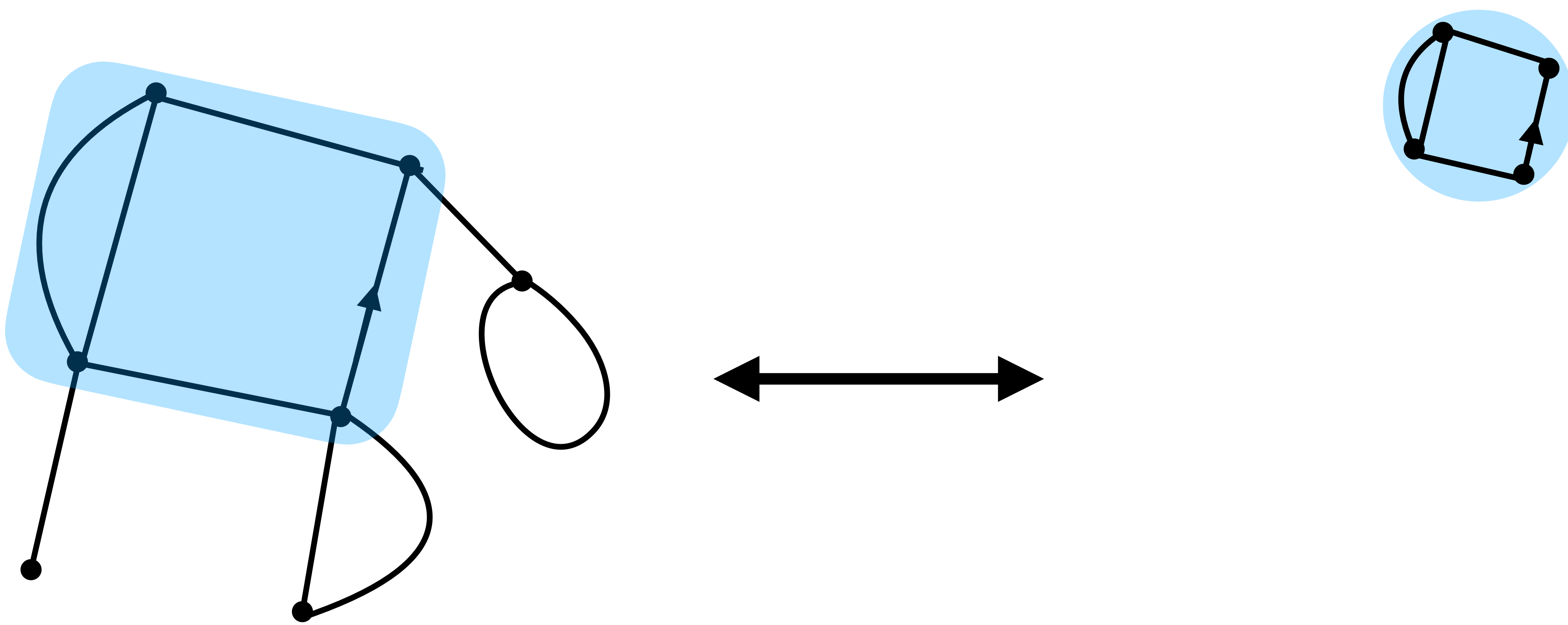
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

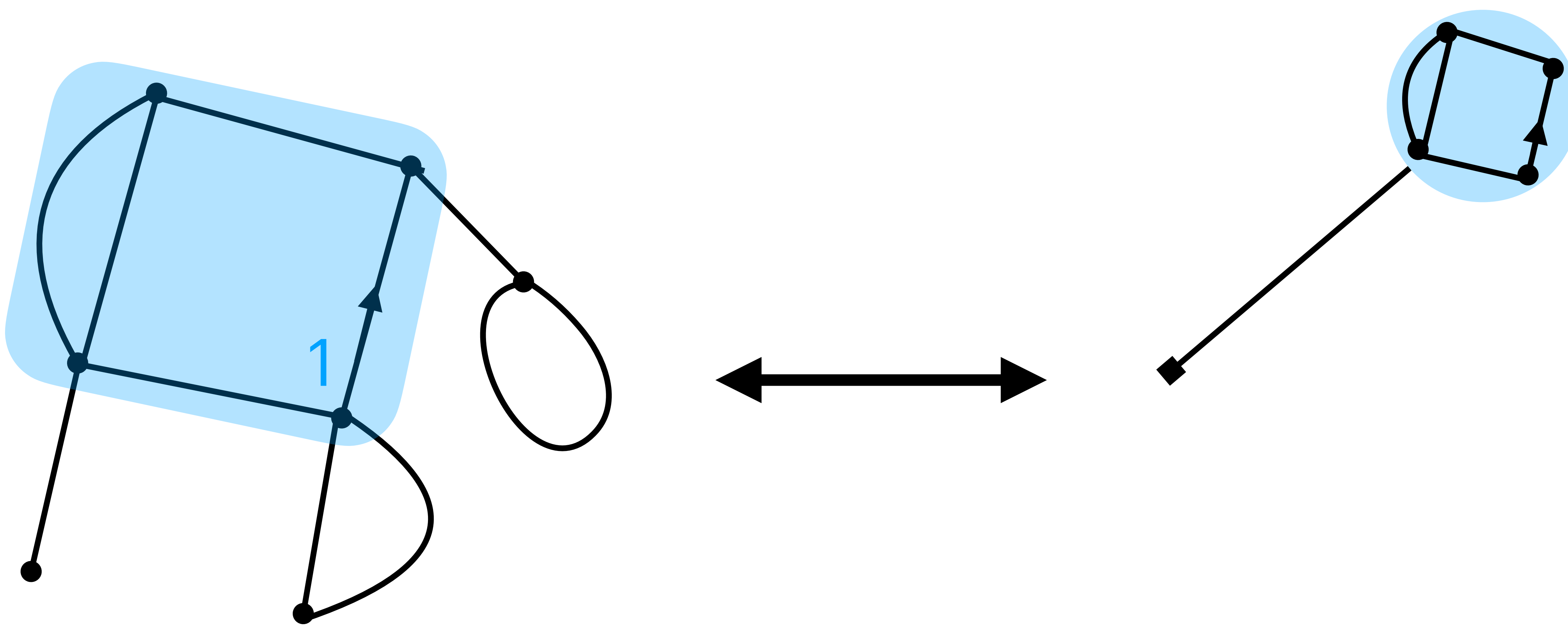
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

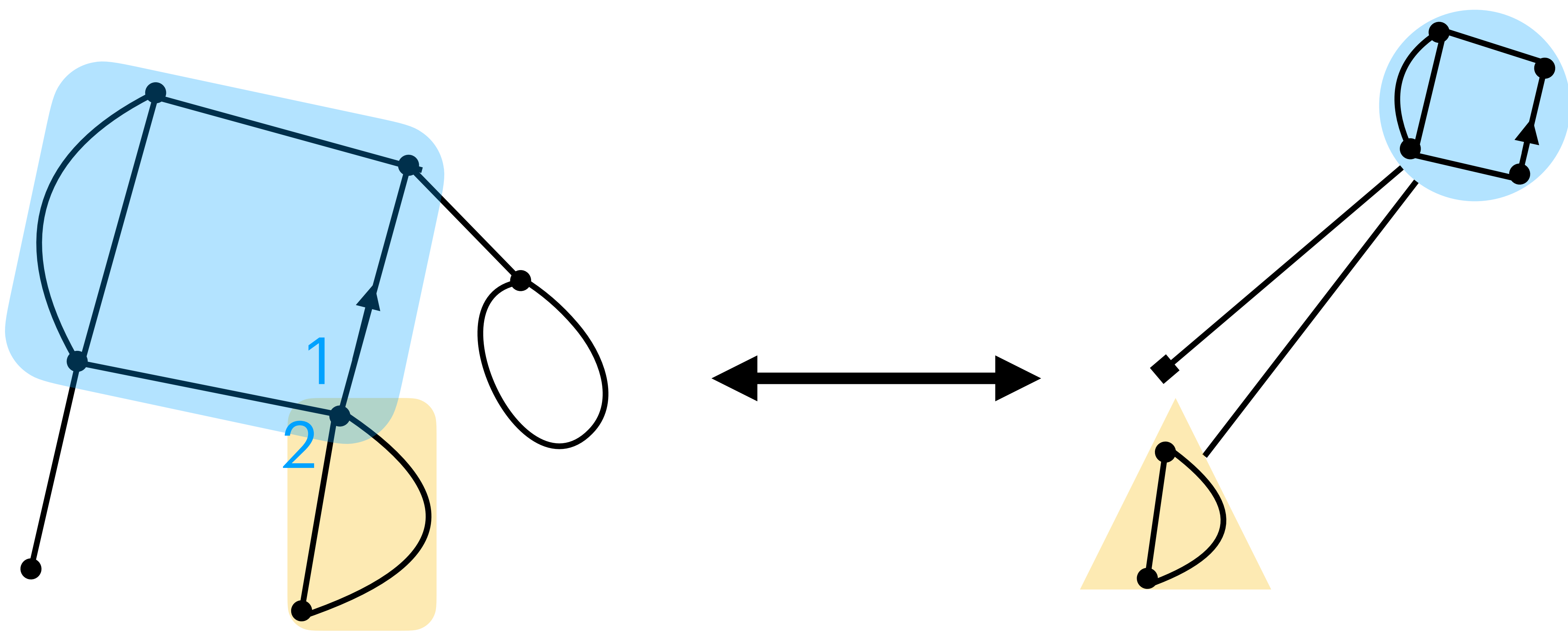
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

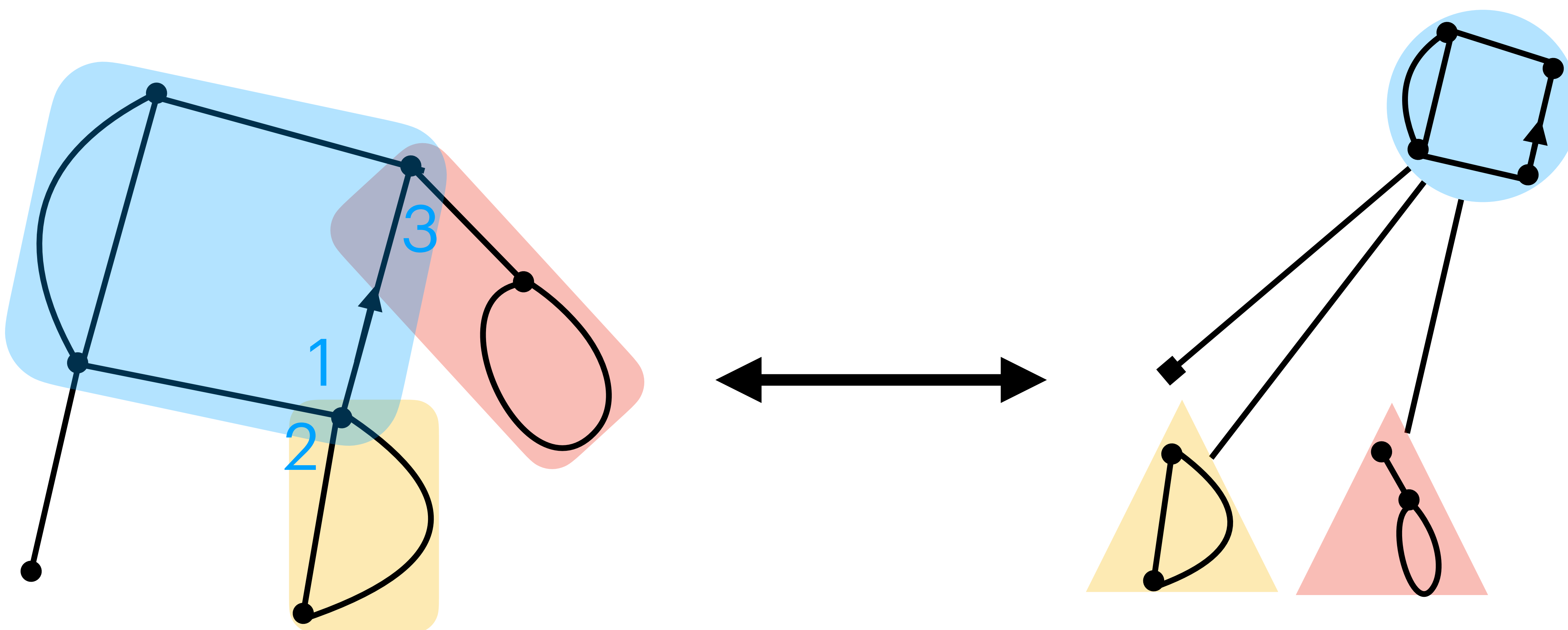
$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

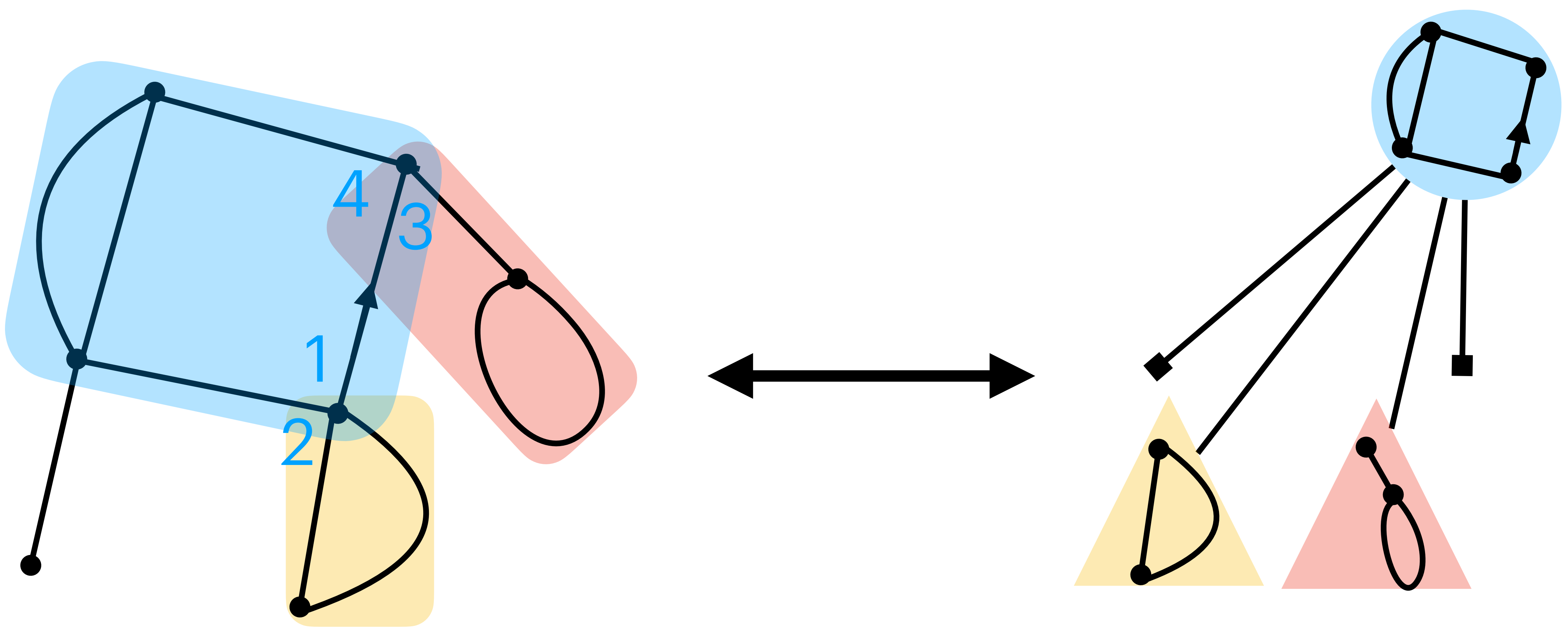
$$M(z,u) = \sum_{\mathfrak{m}\in\mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

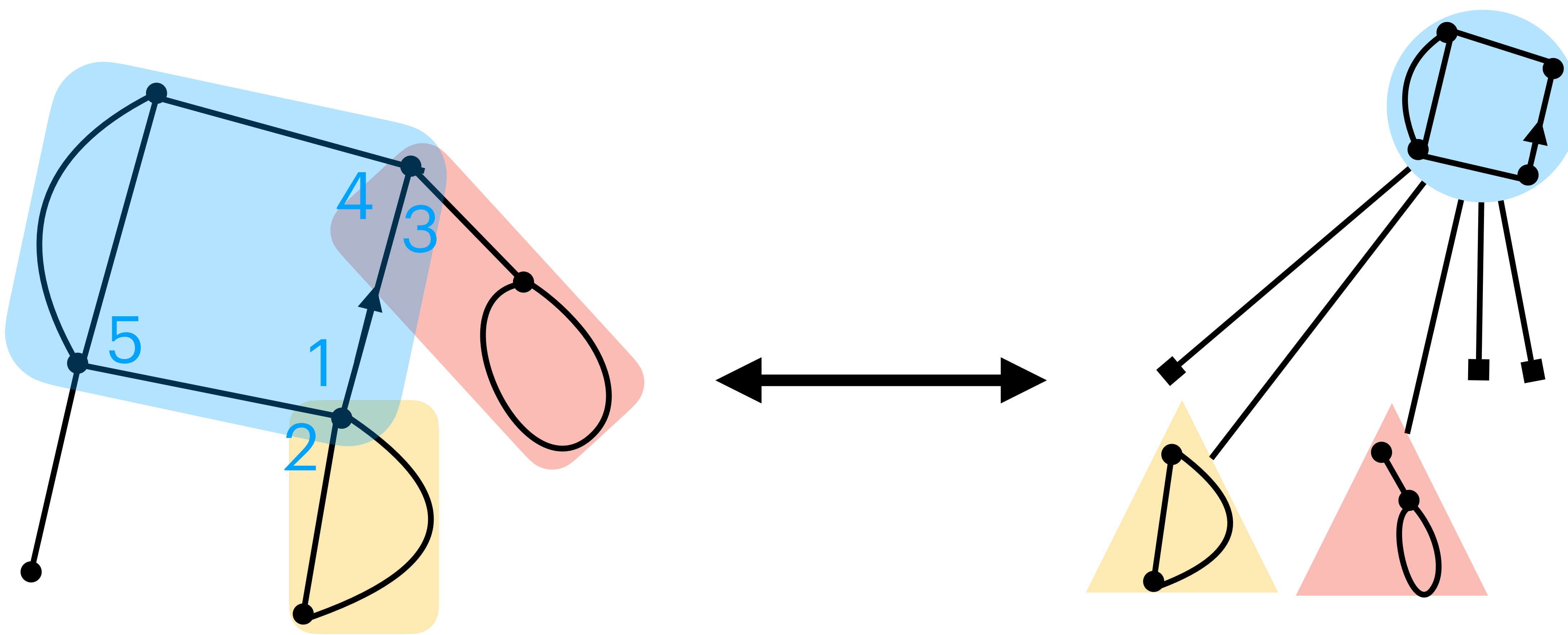
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

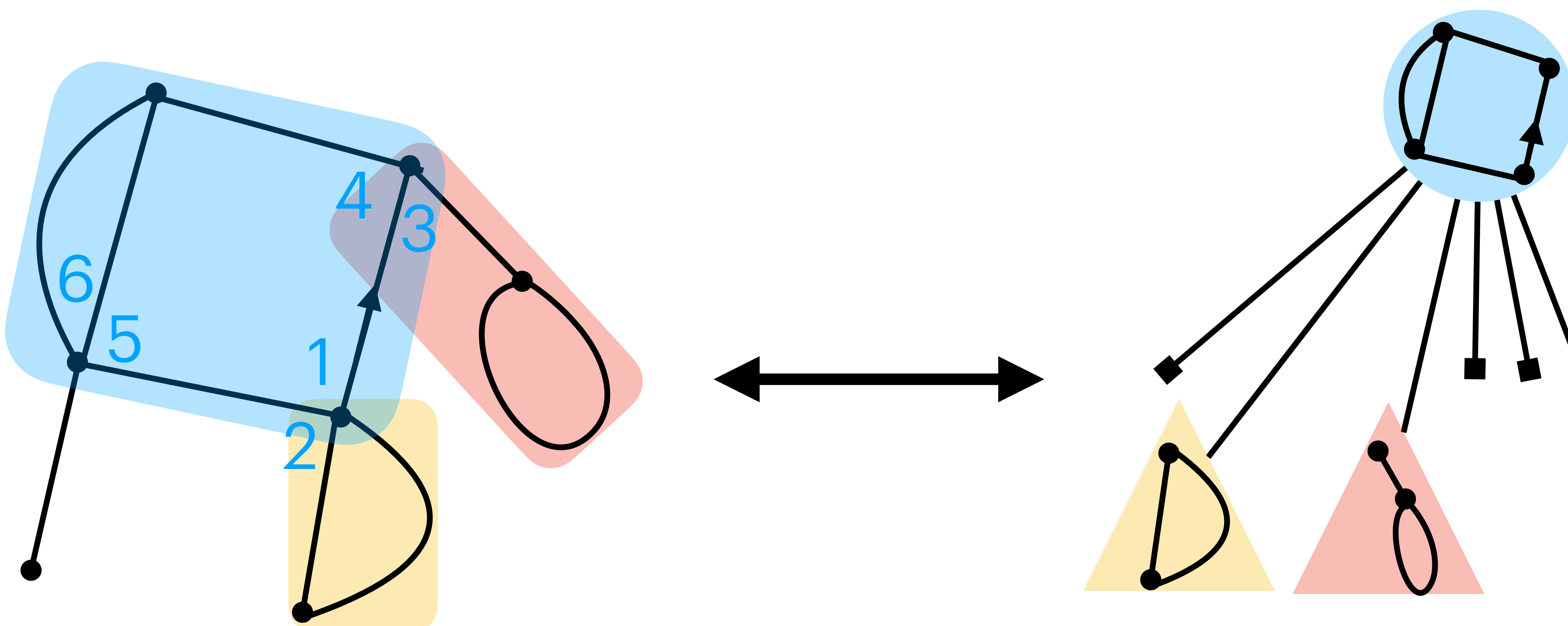
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

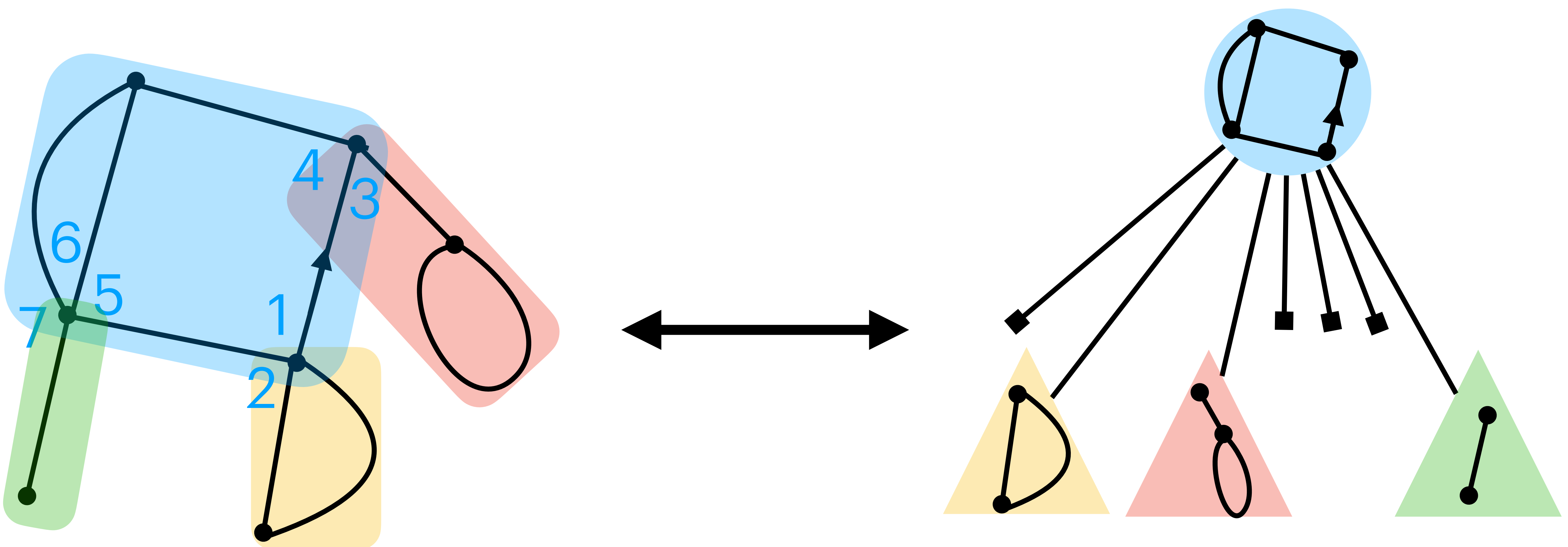
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

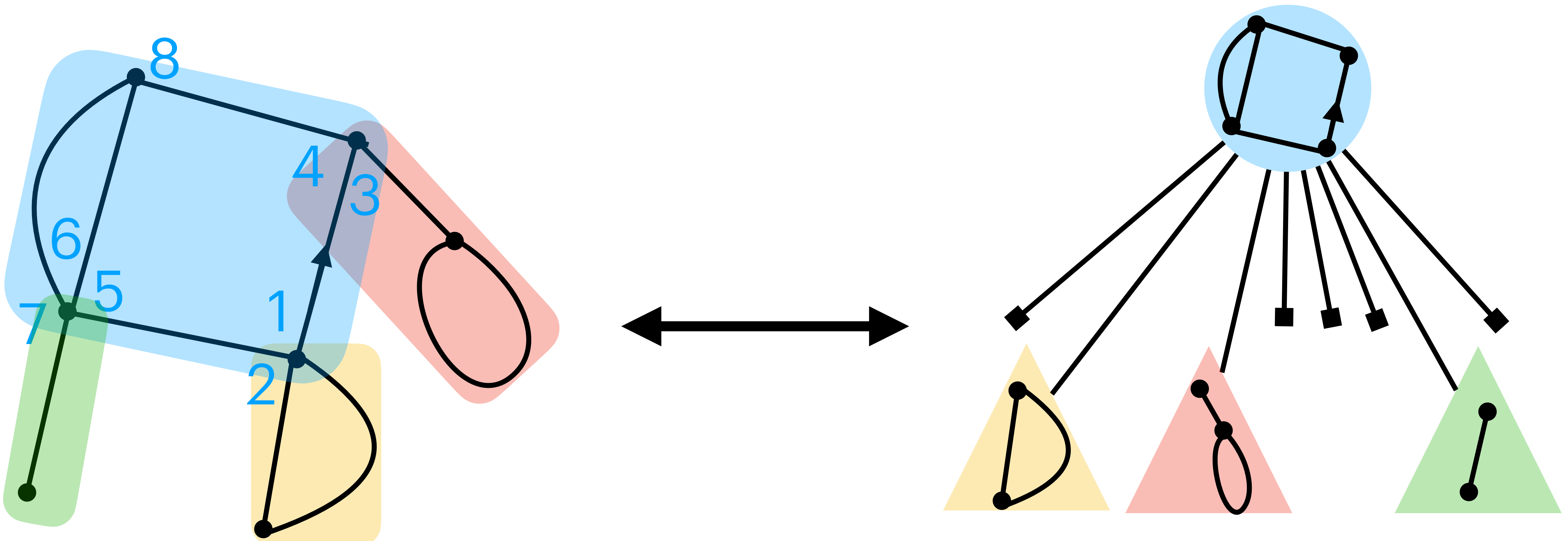
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

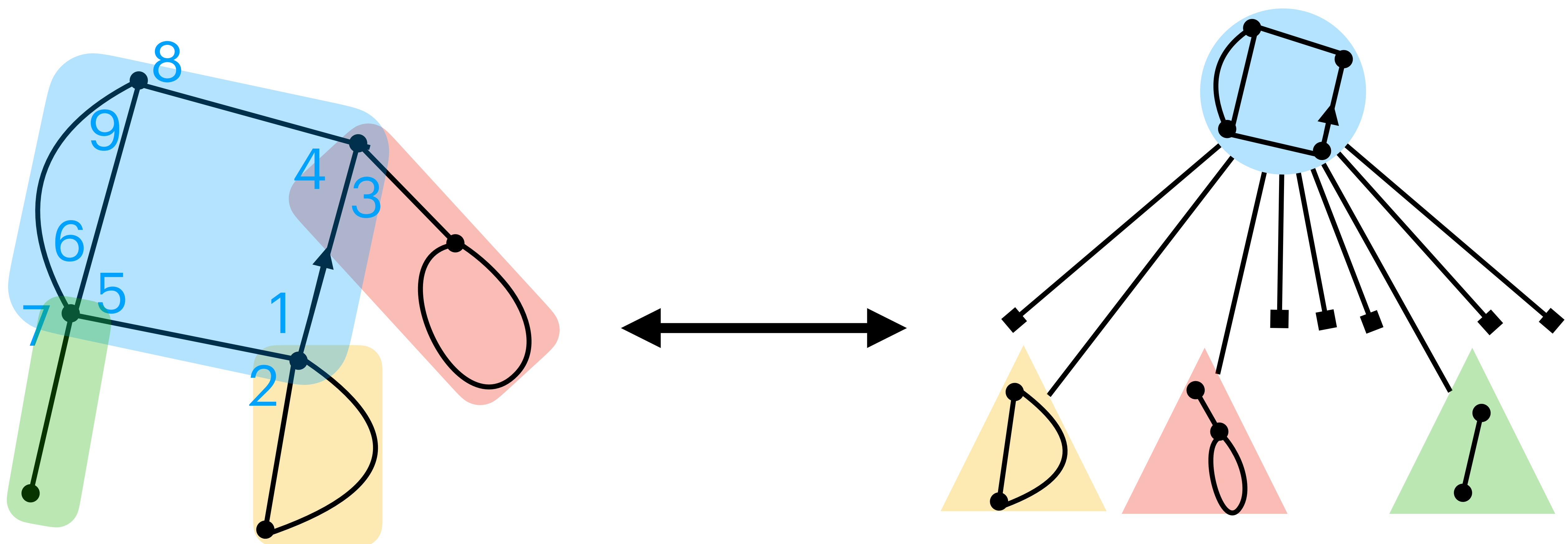
$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

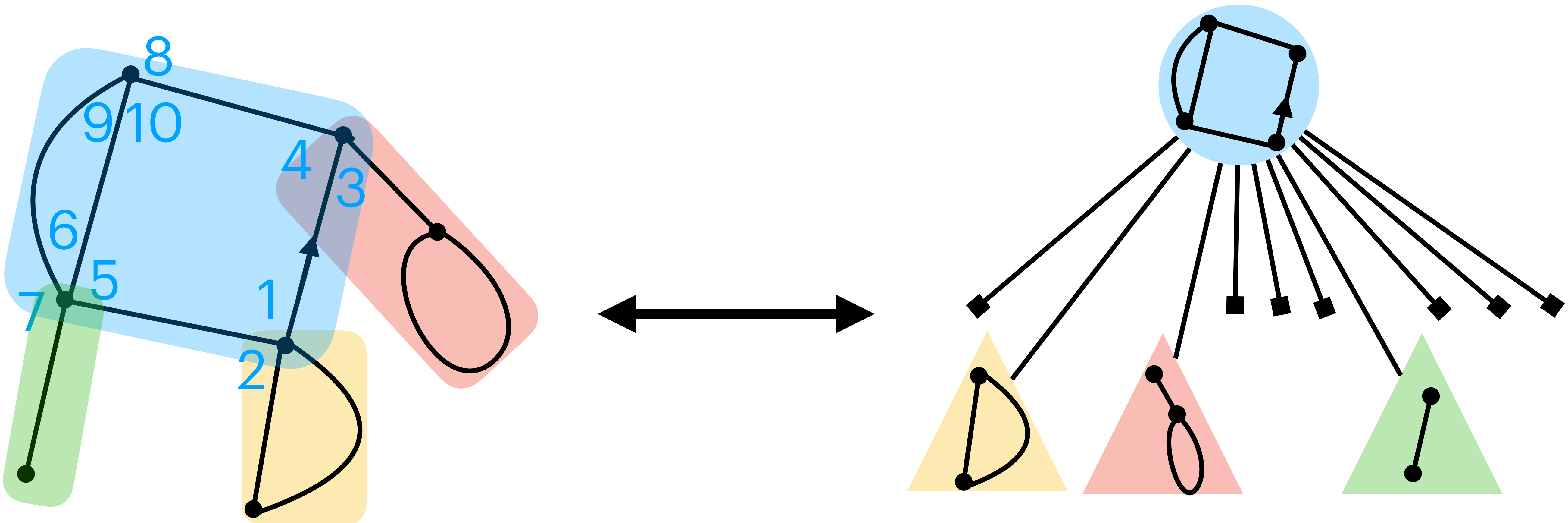
$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

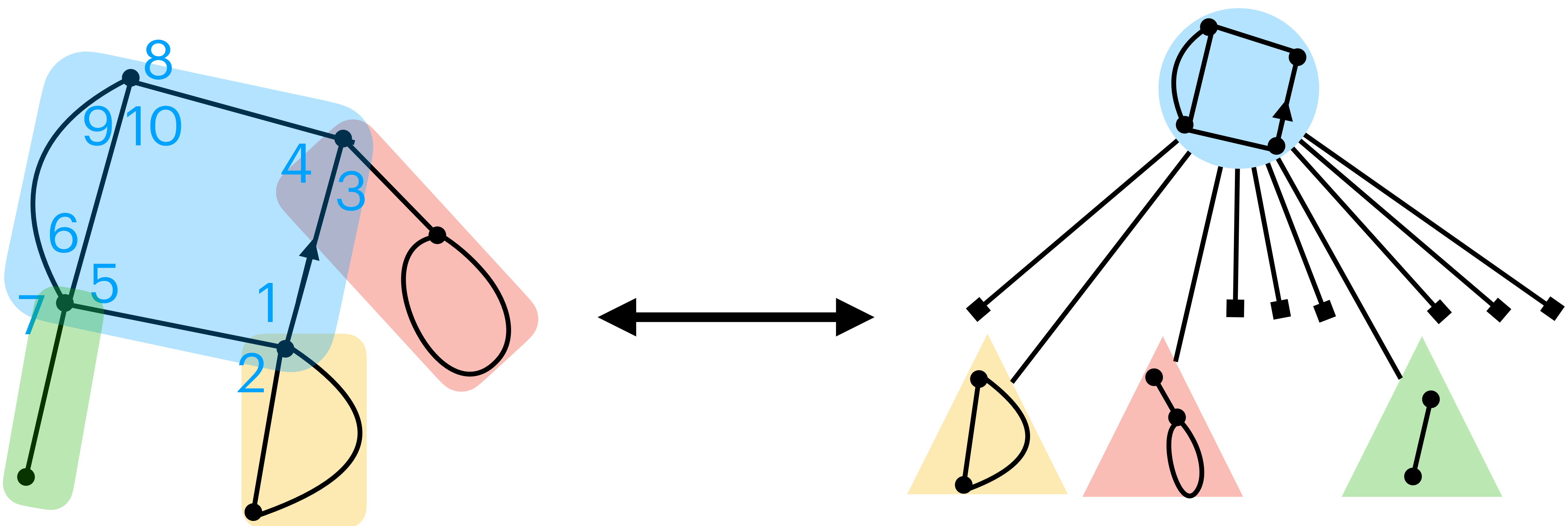
$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



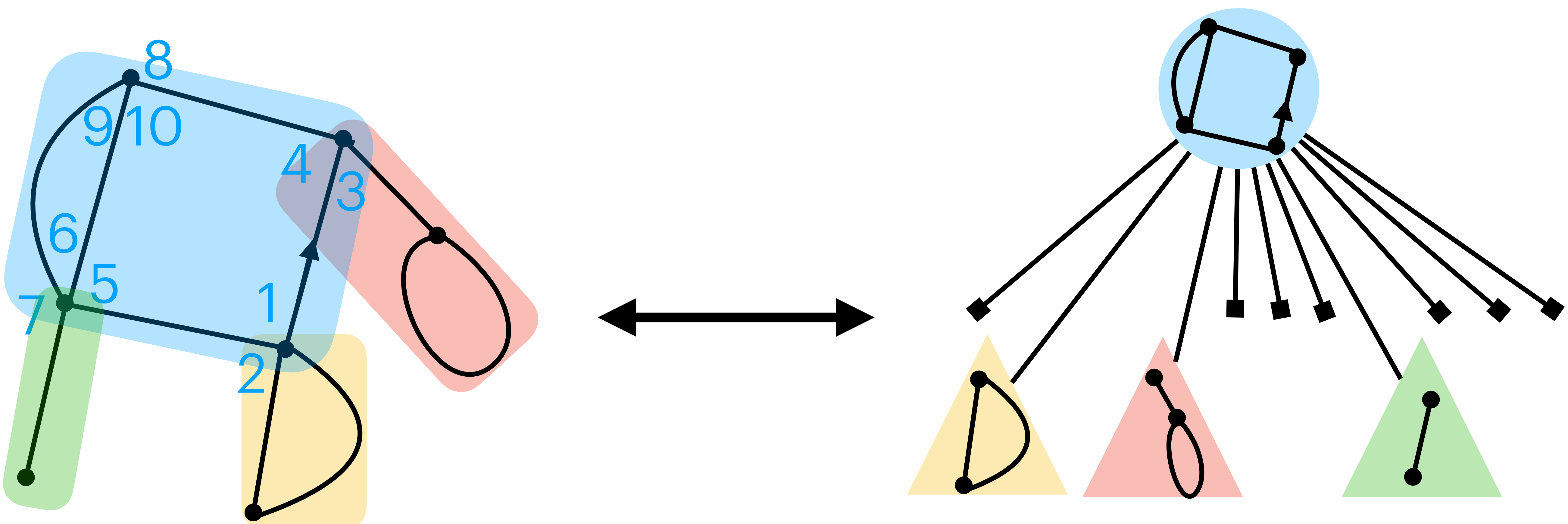
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

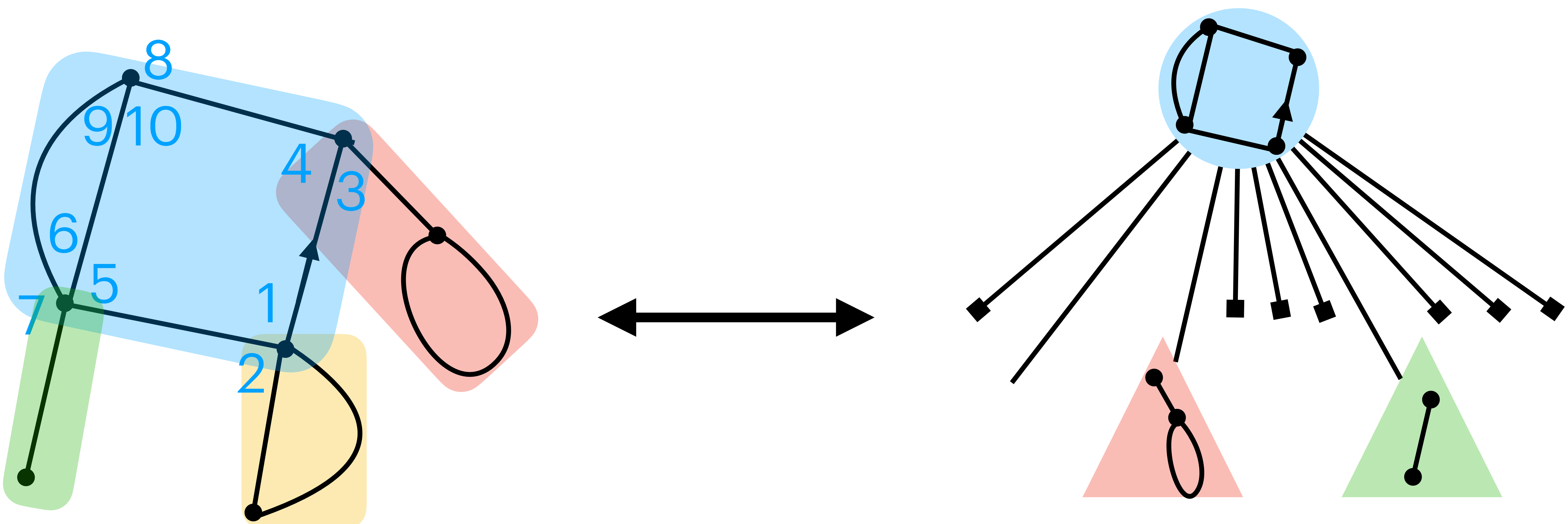
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

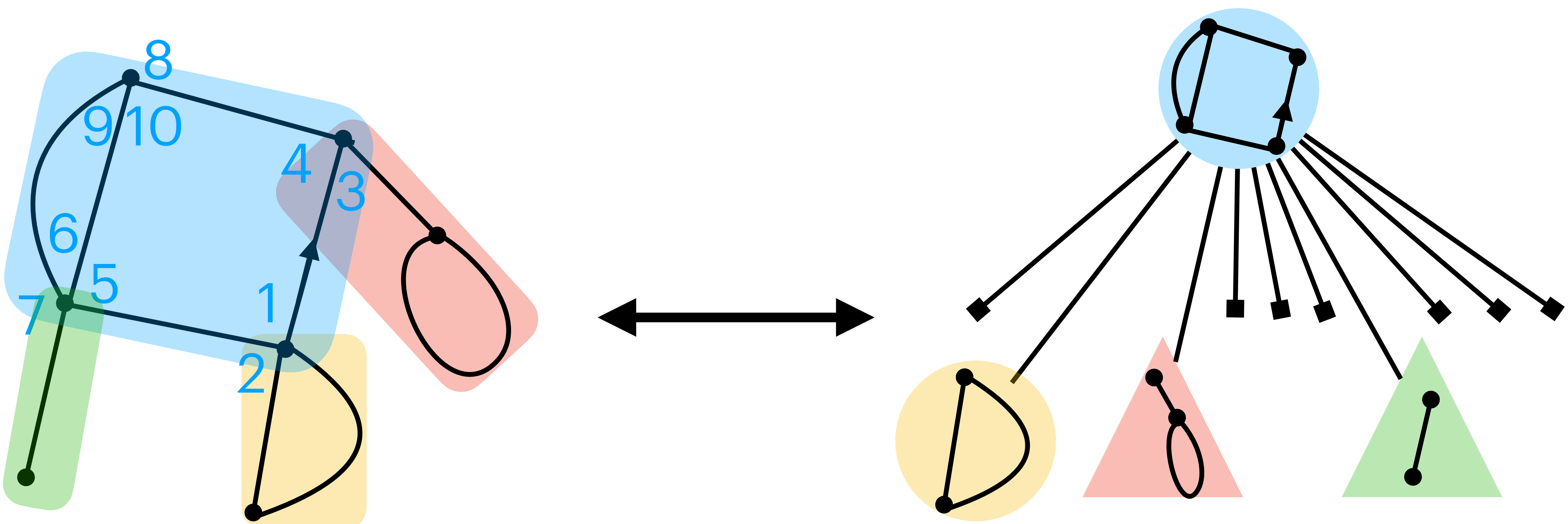
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

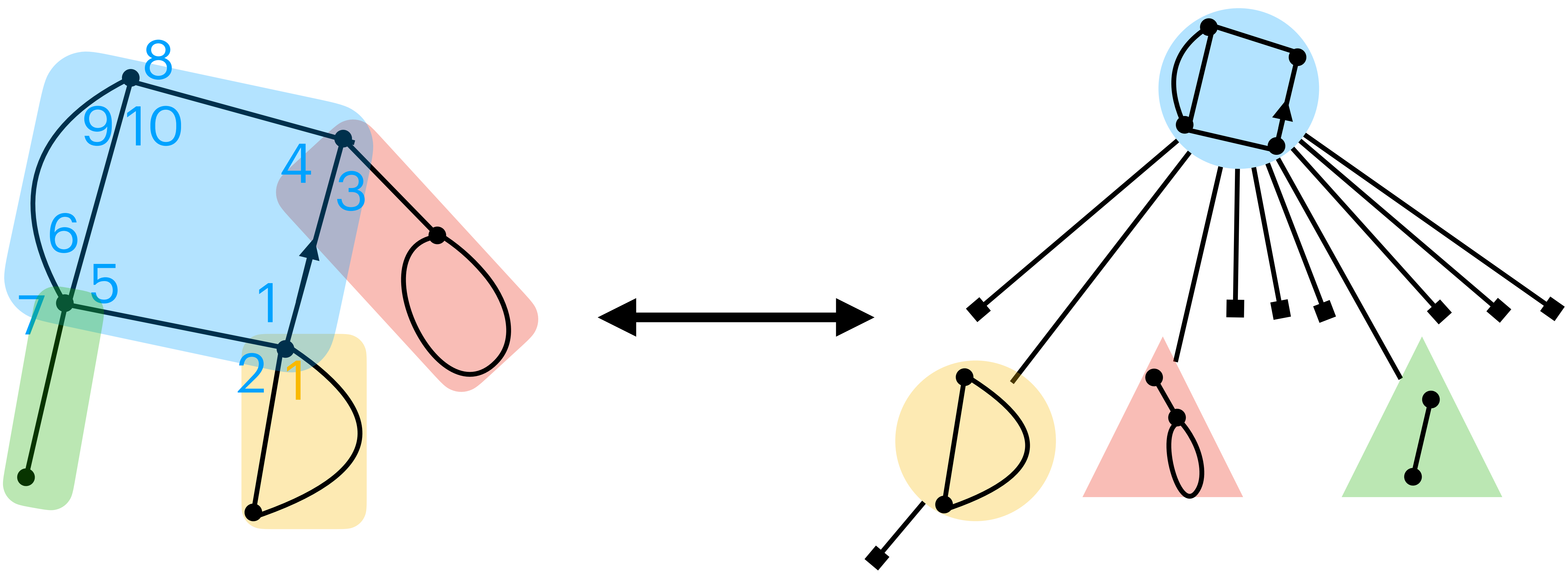
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

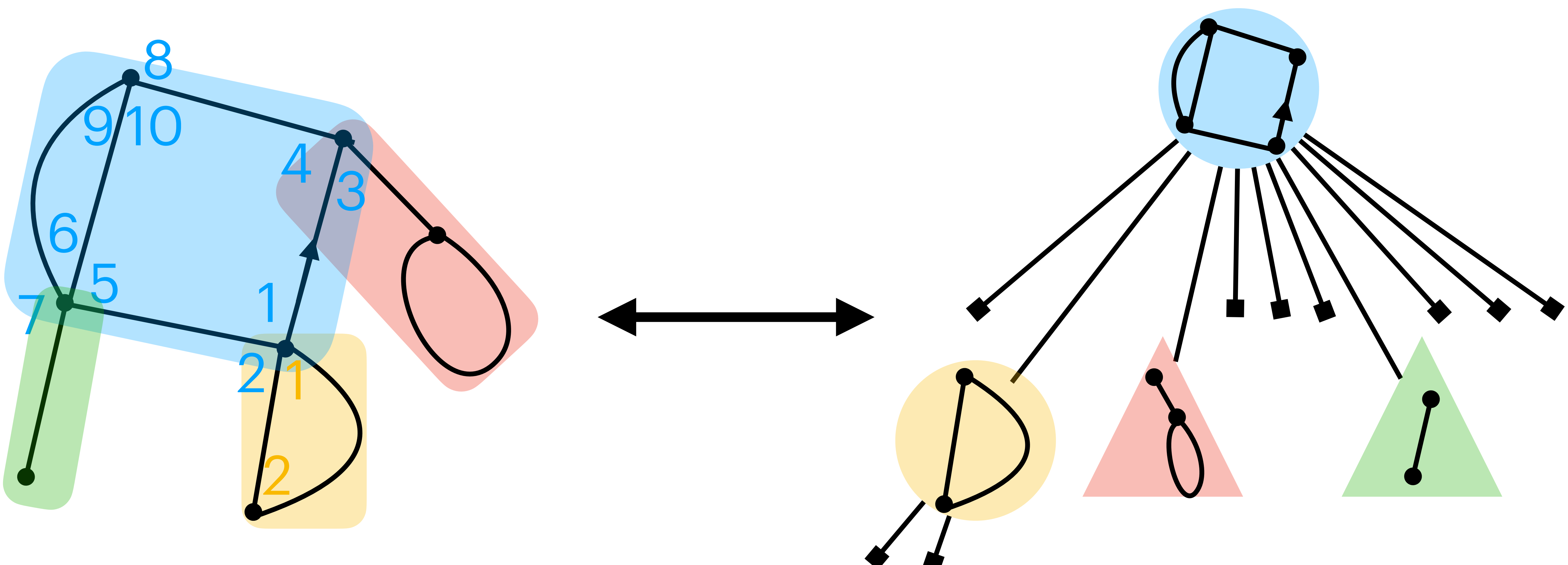
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

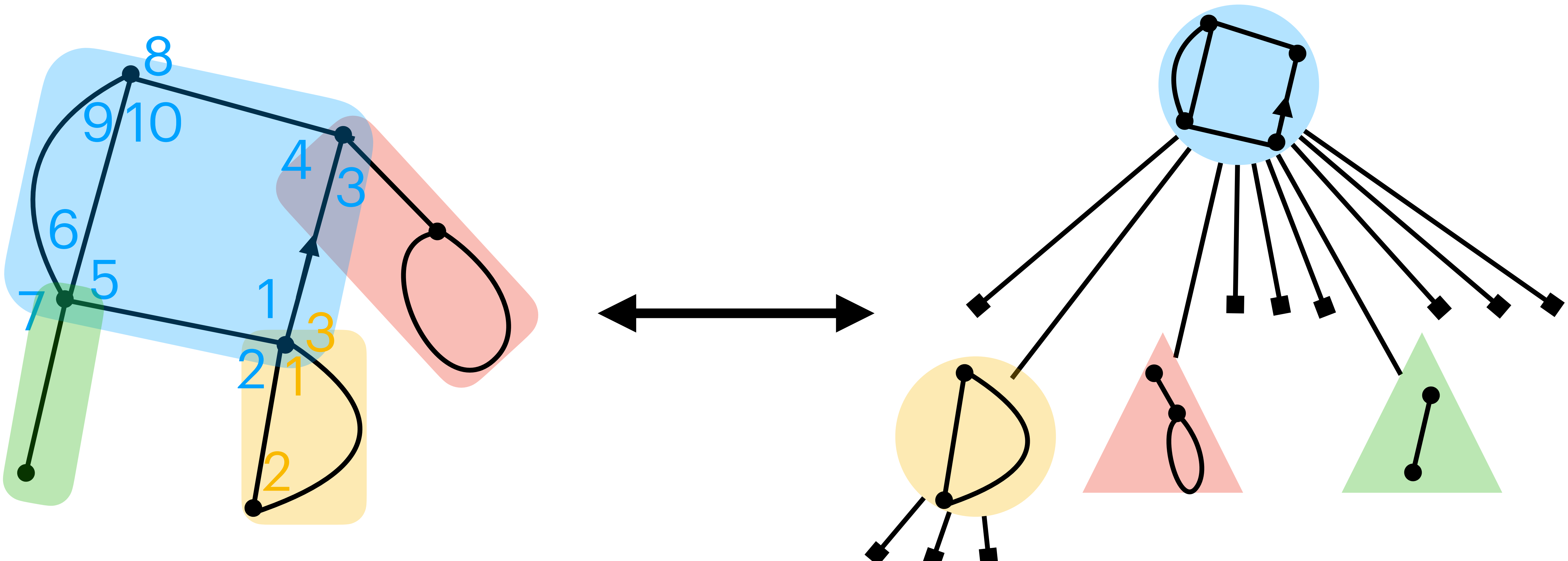
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

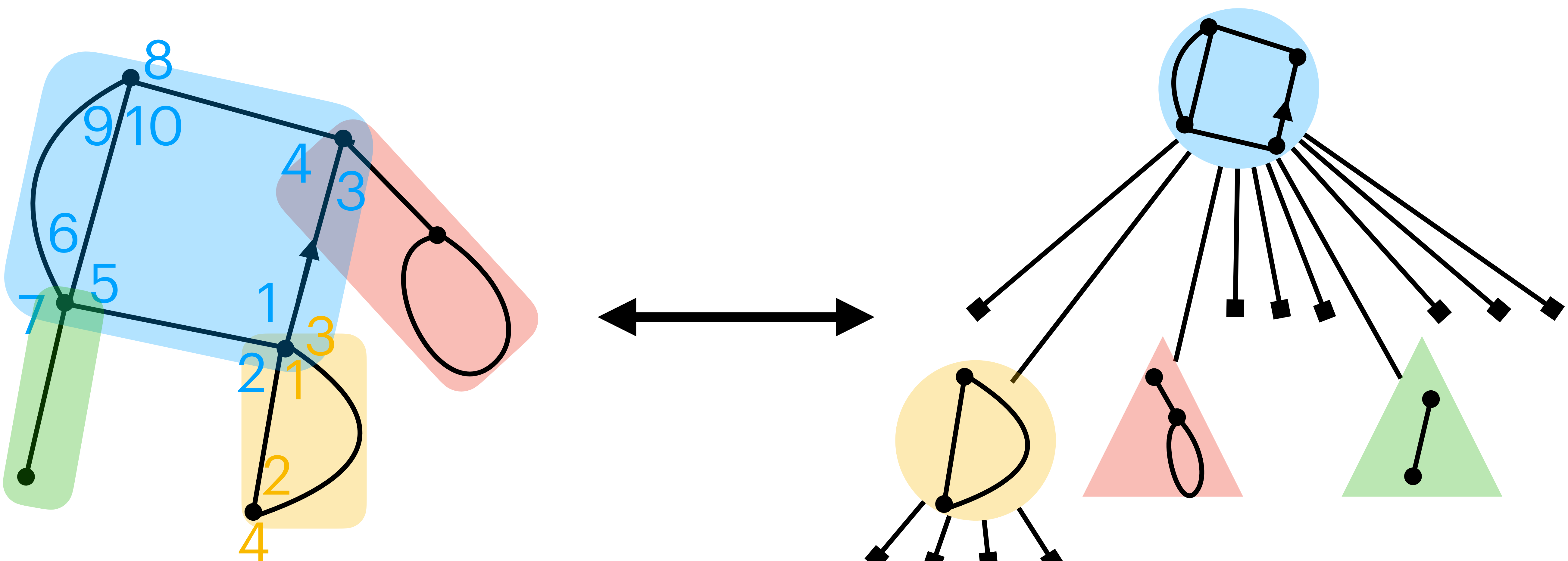
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

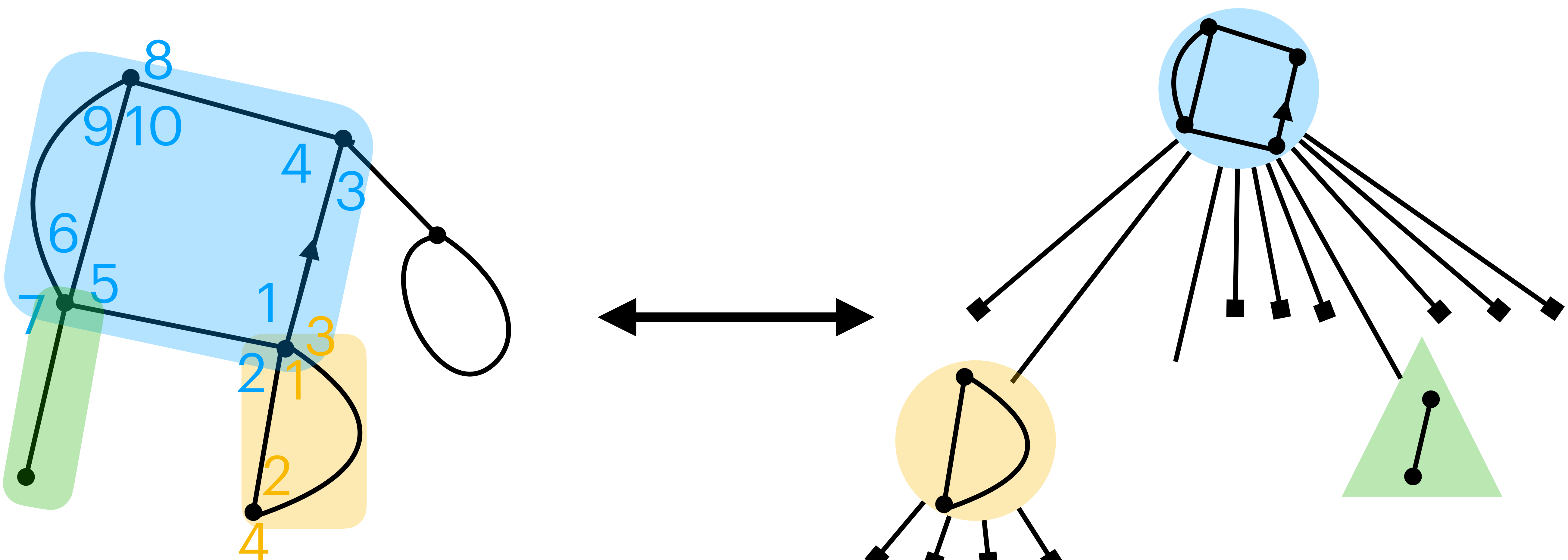
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

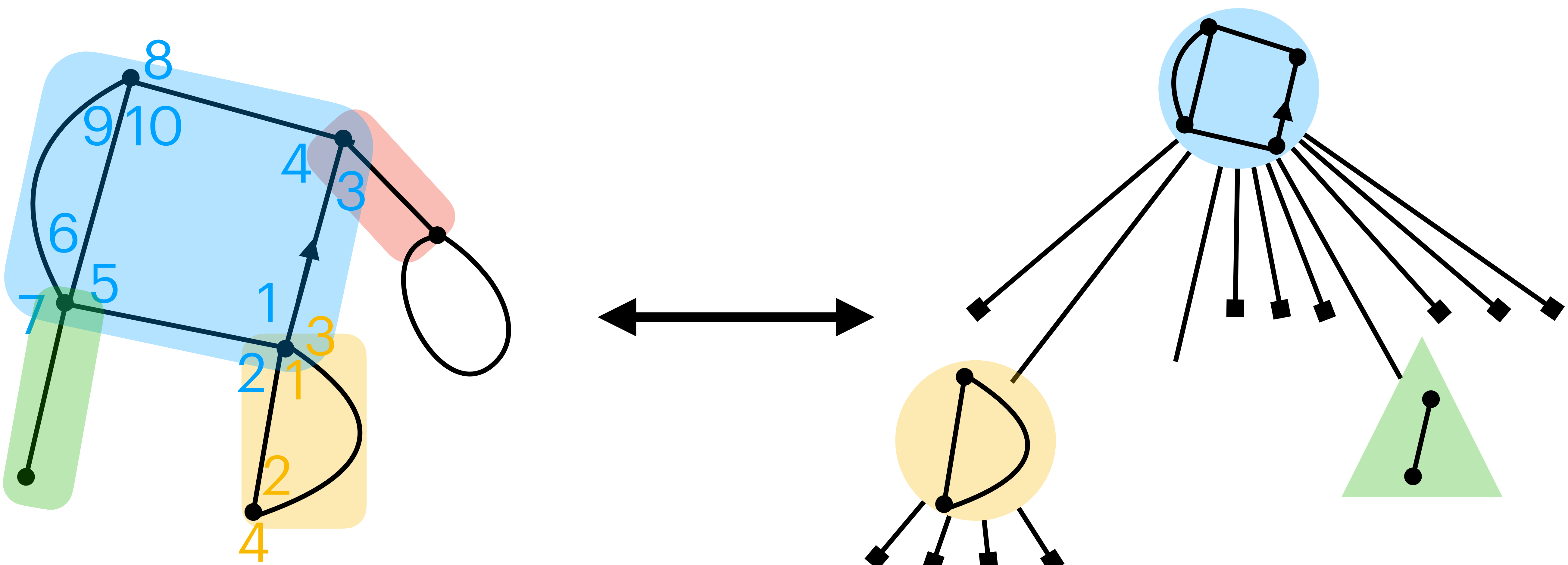
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

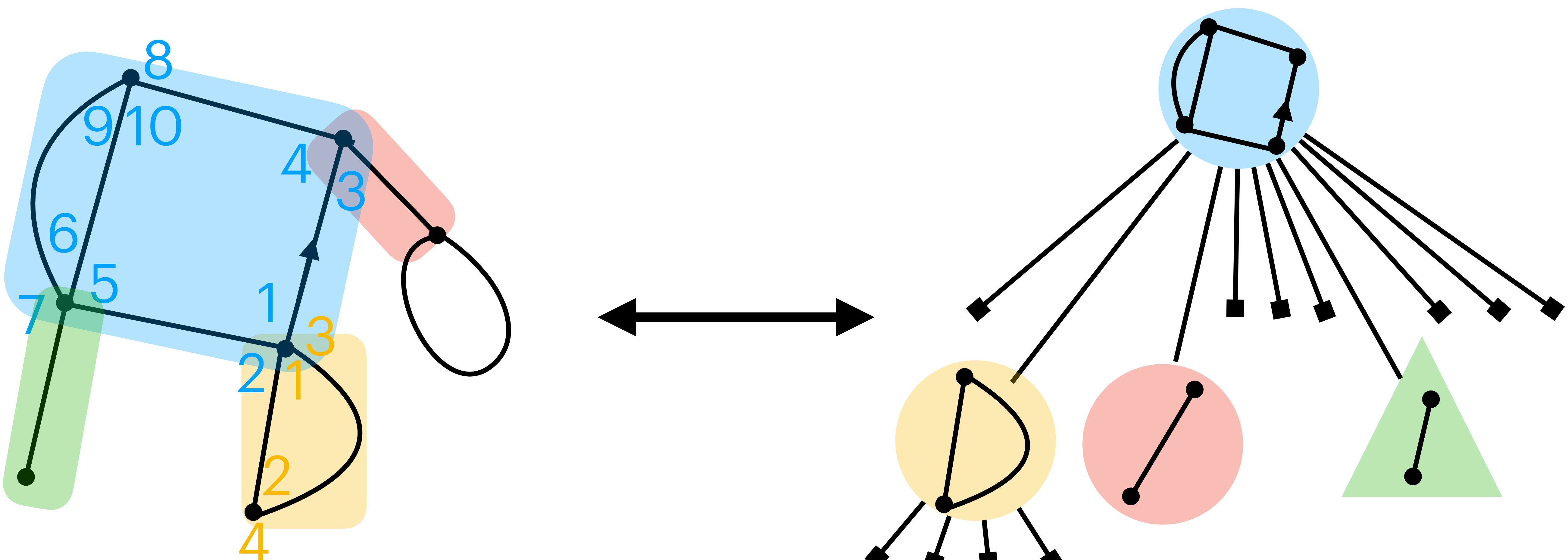
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

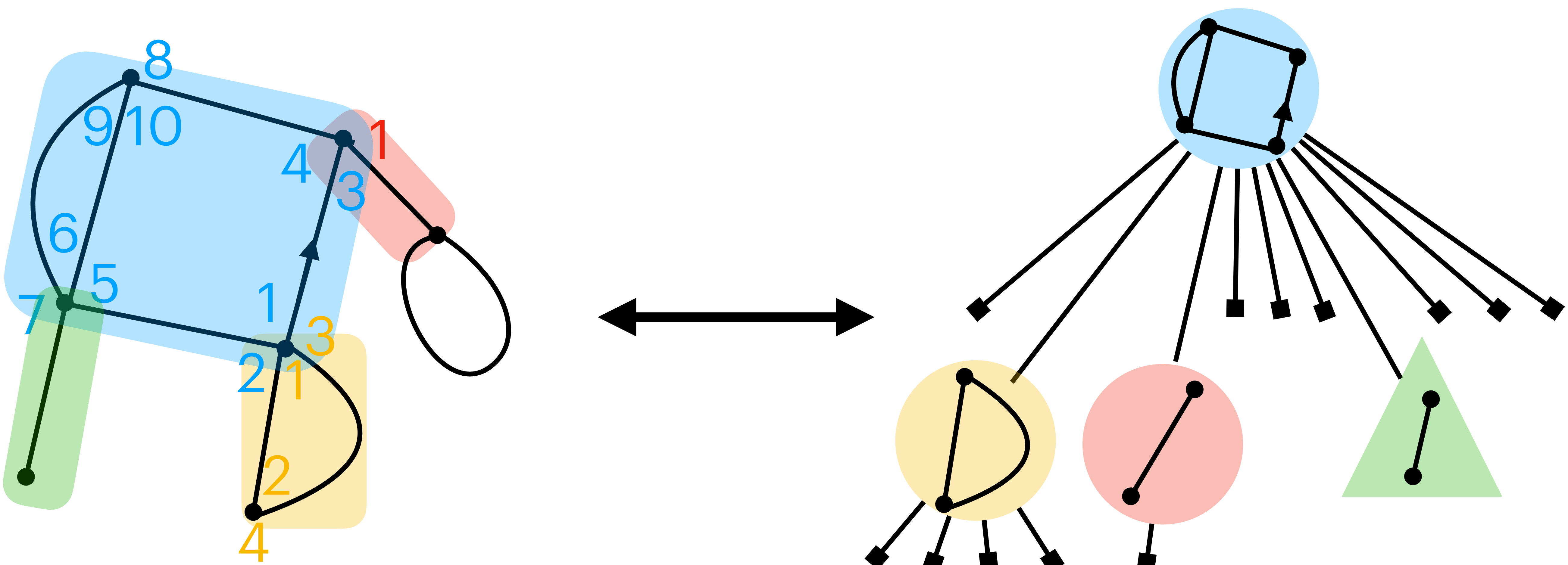
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

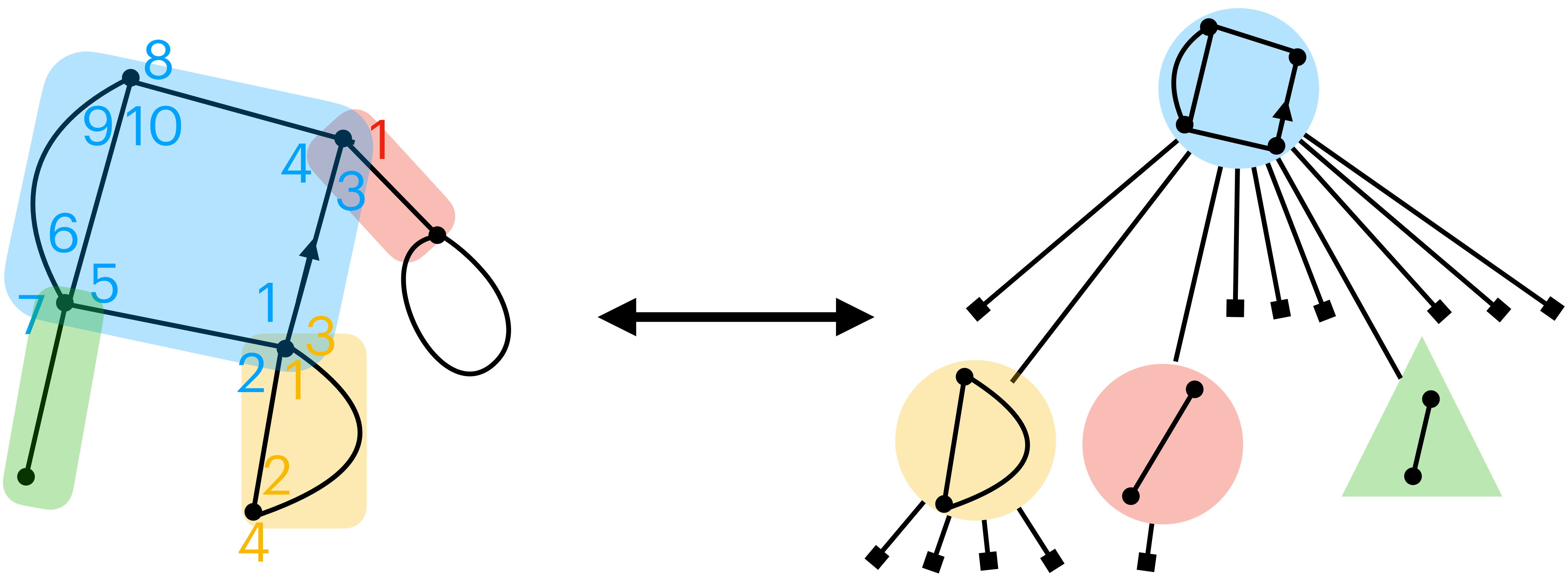
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



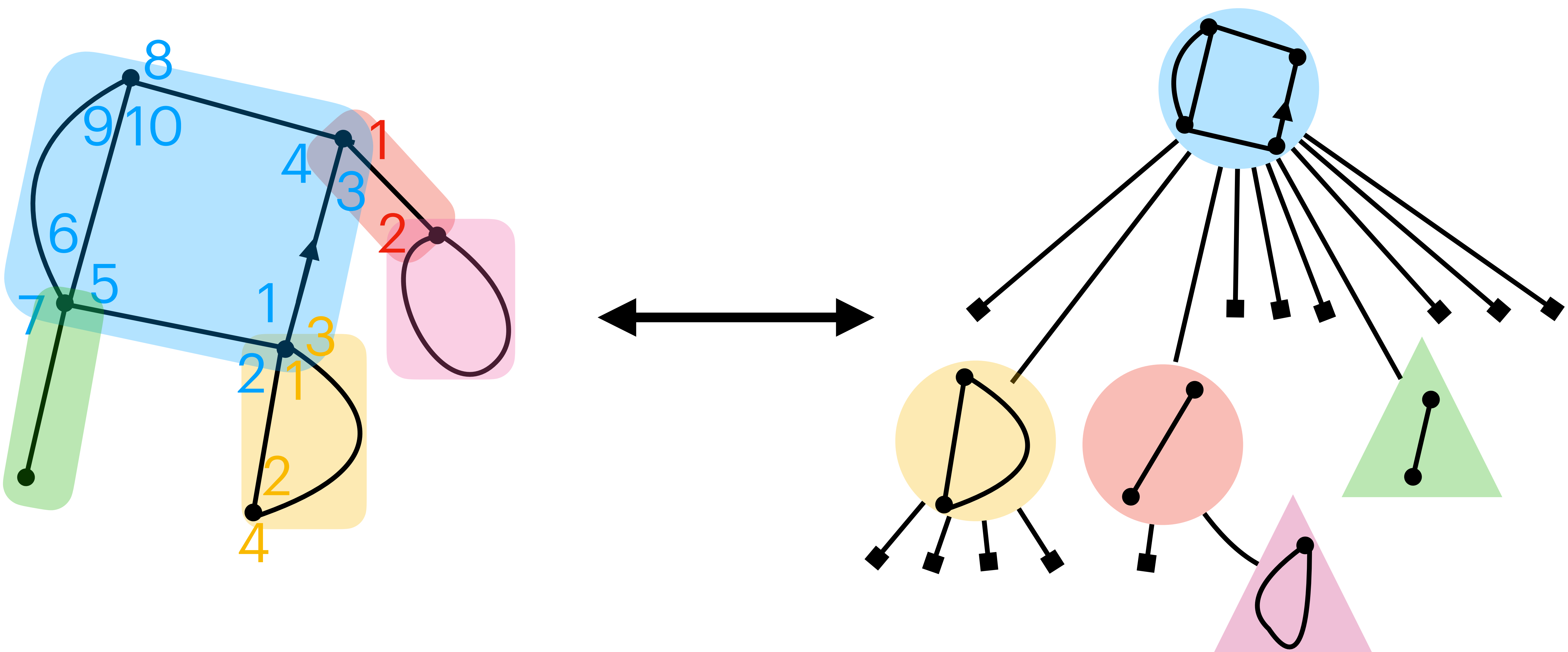
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



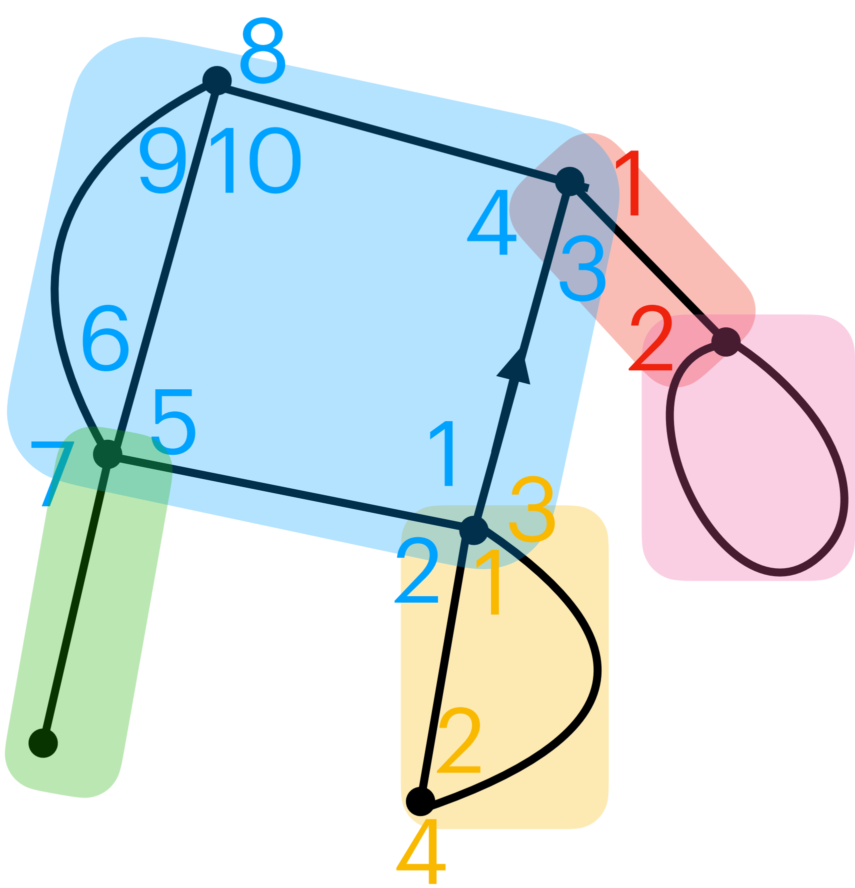
GS of 2-connected maps

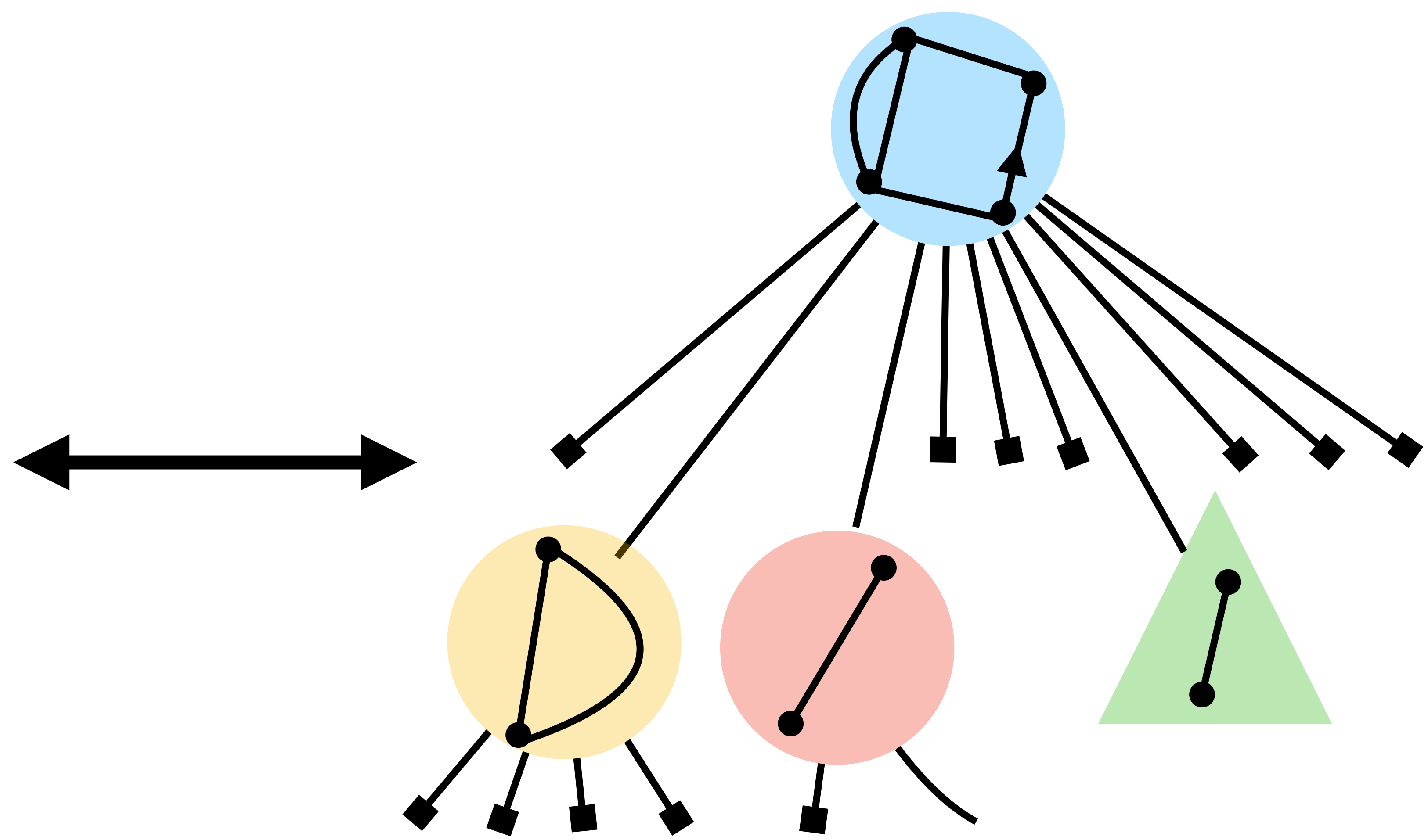
With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$





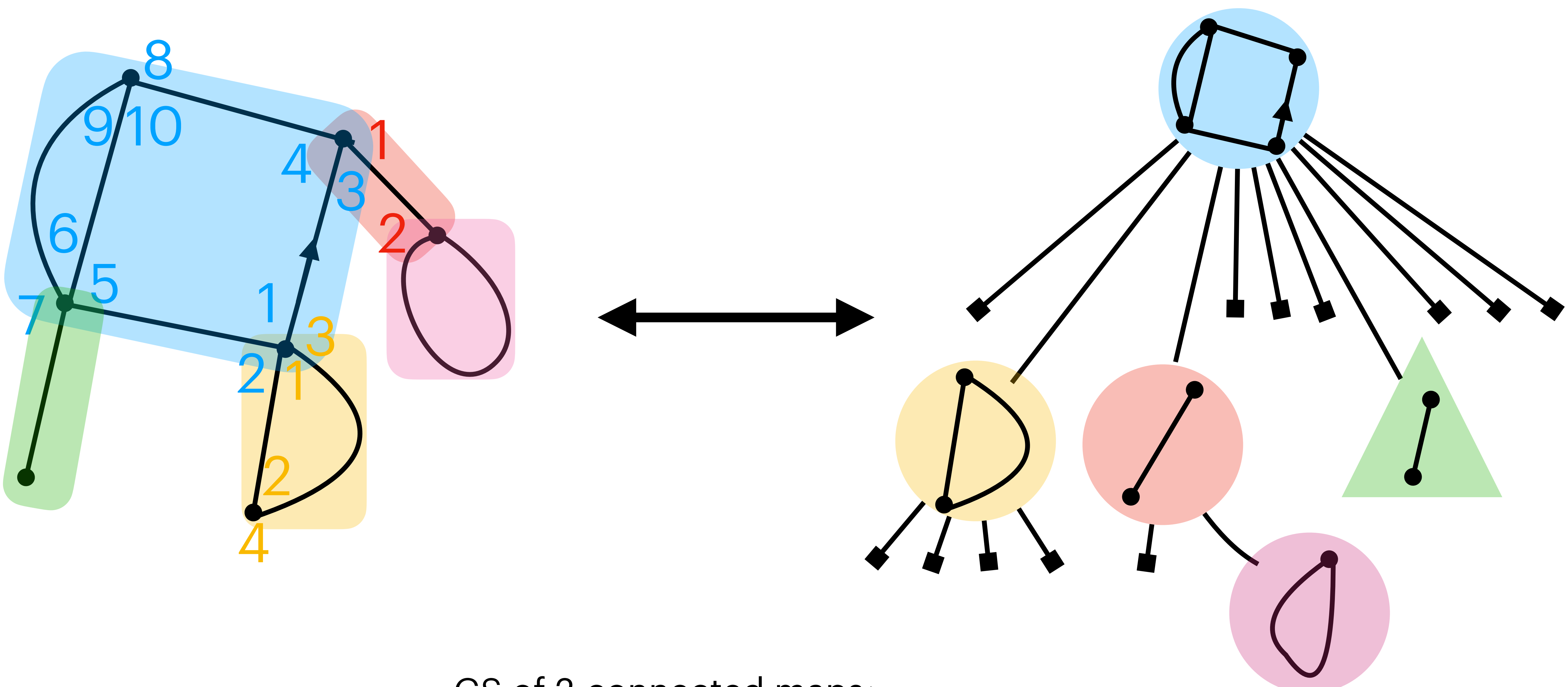
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



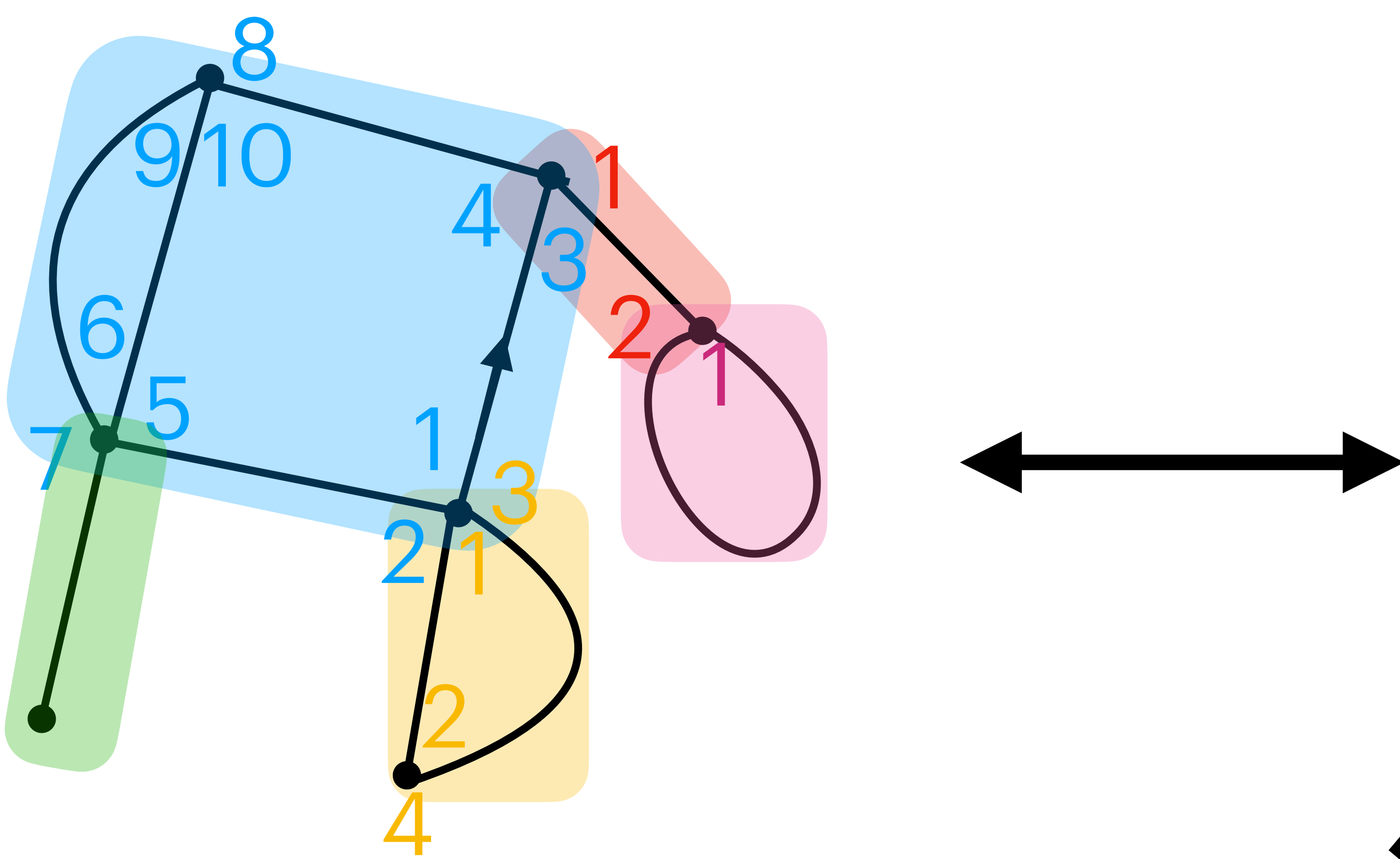
GS of 2-connected maps

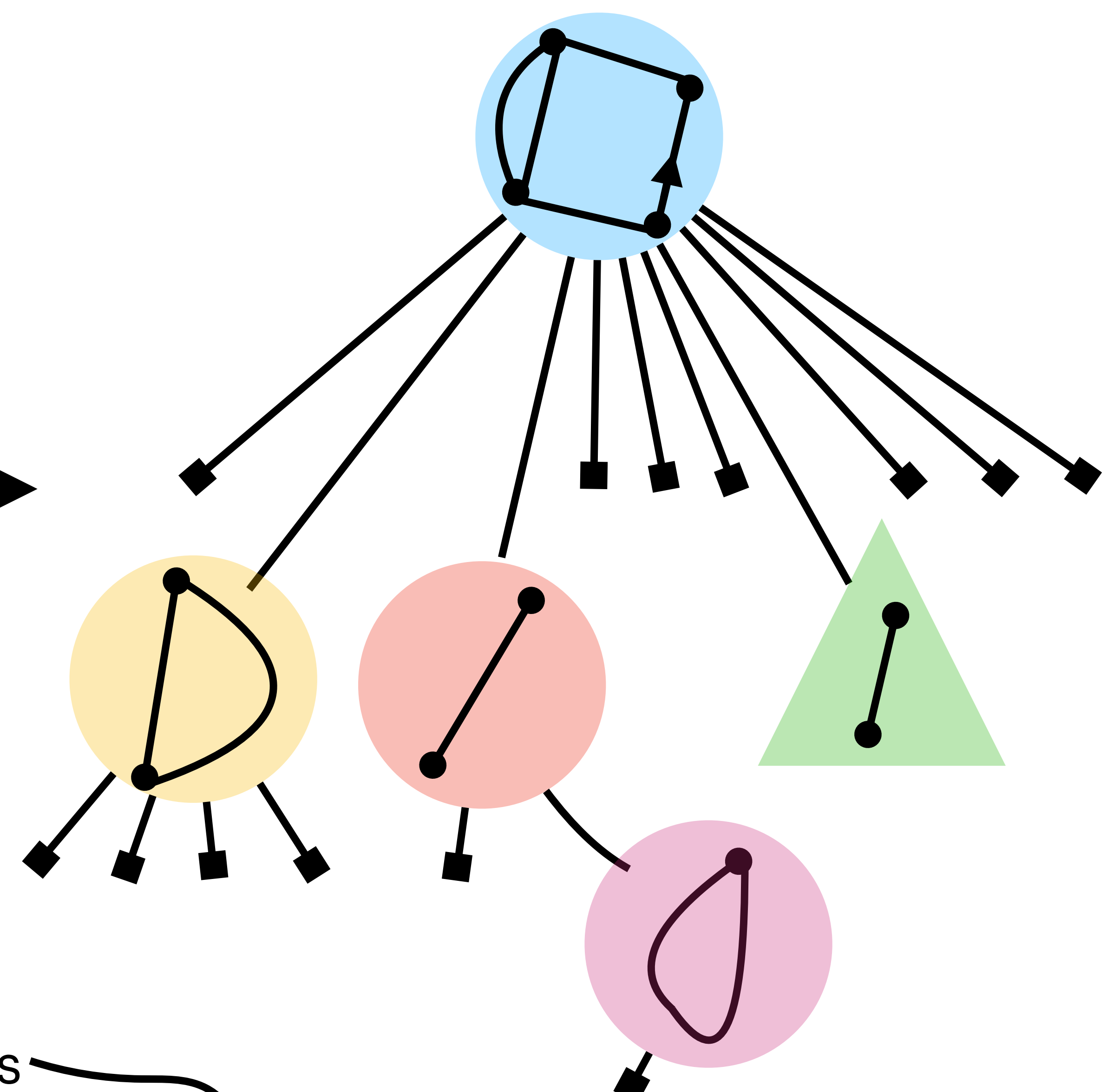
With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$





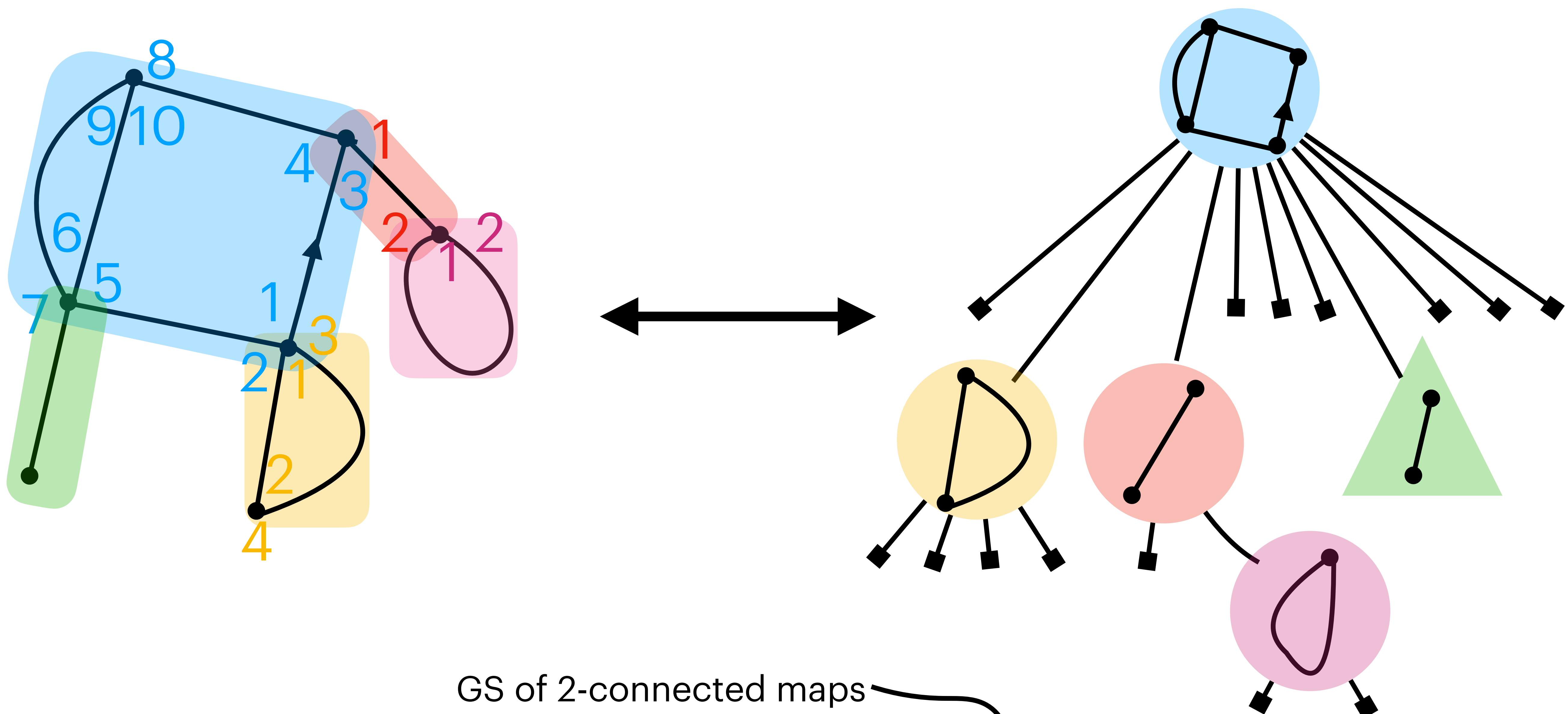
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m} \in \mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



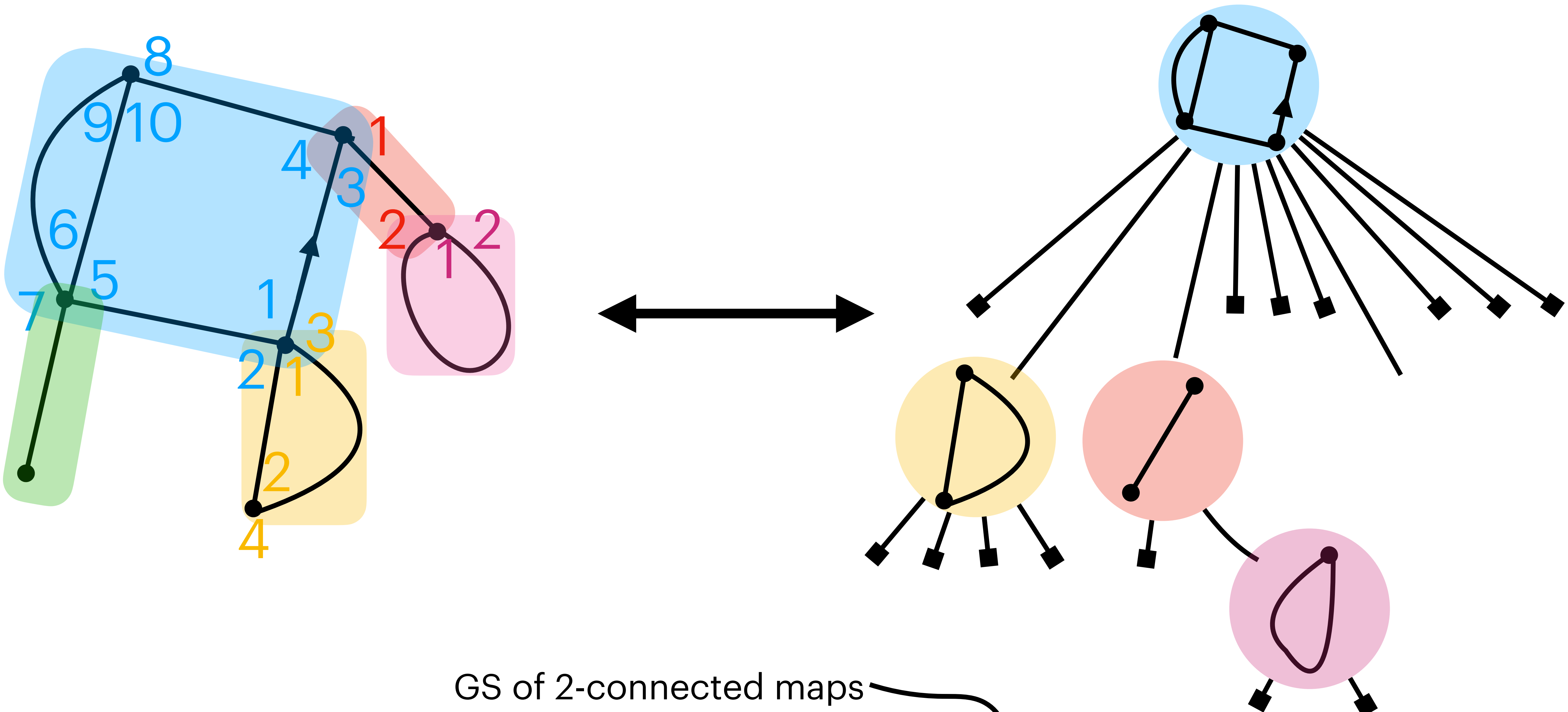
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



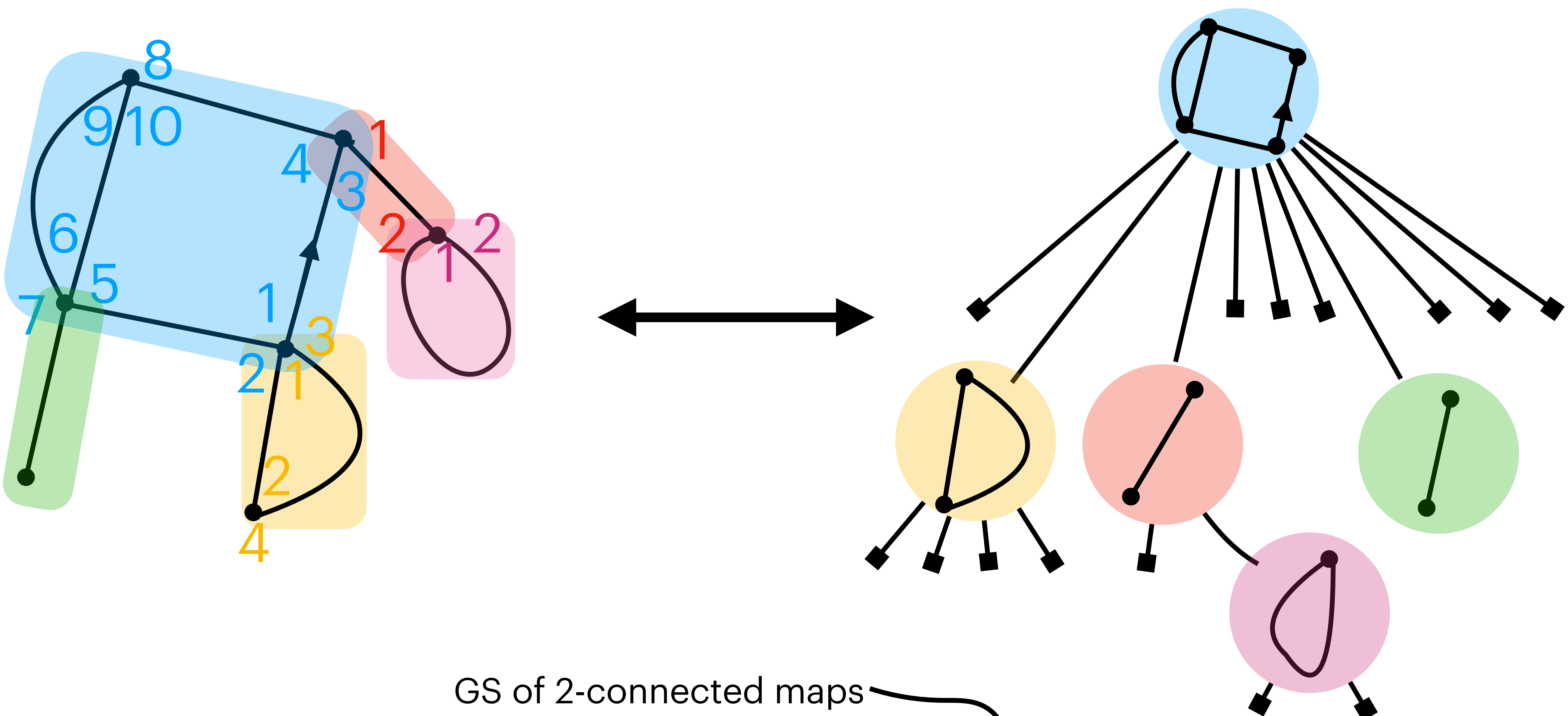
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



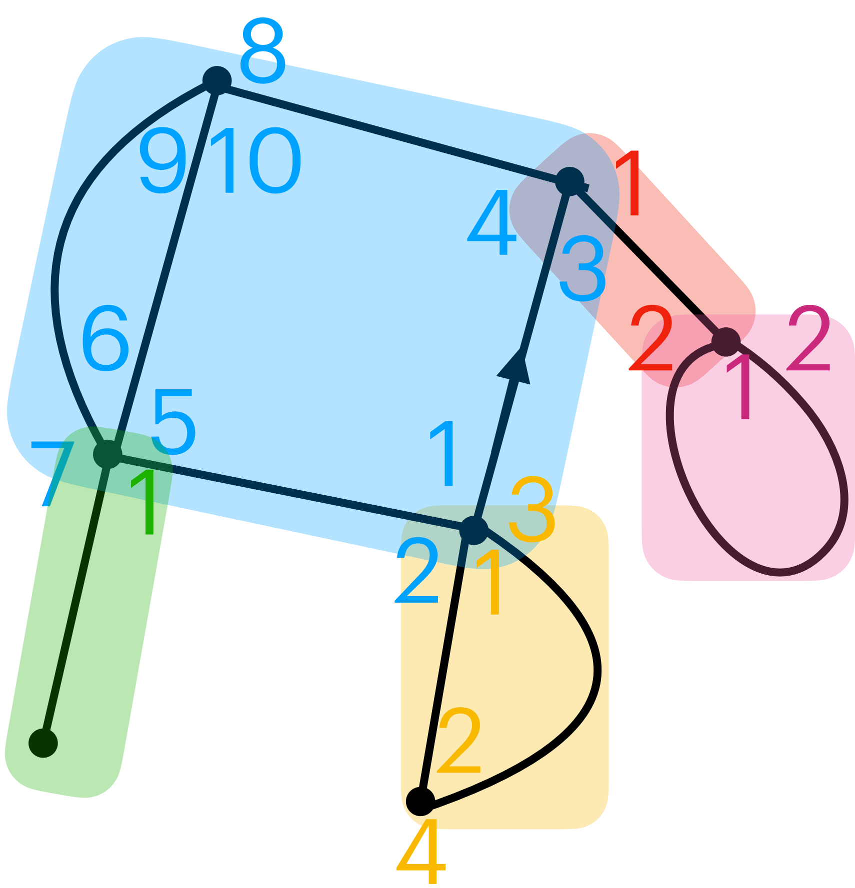
GS of 2-connected maps

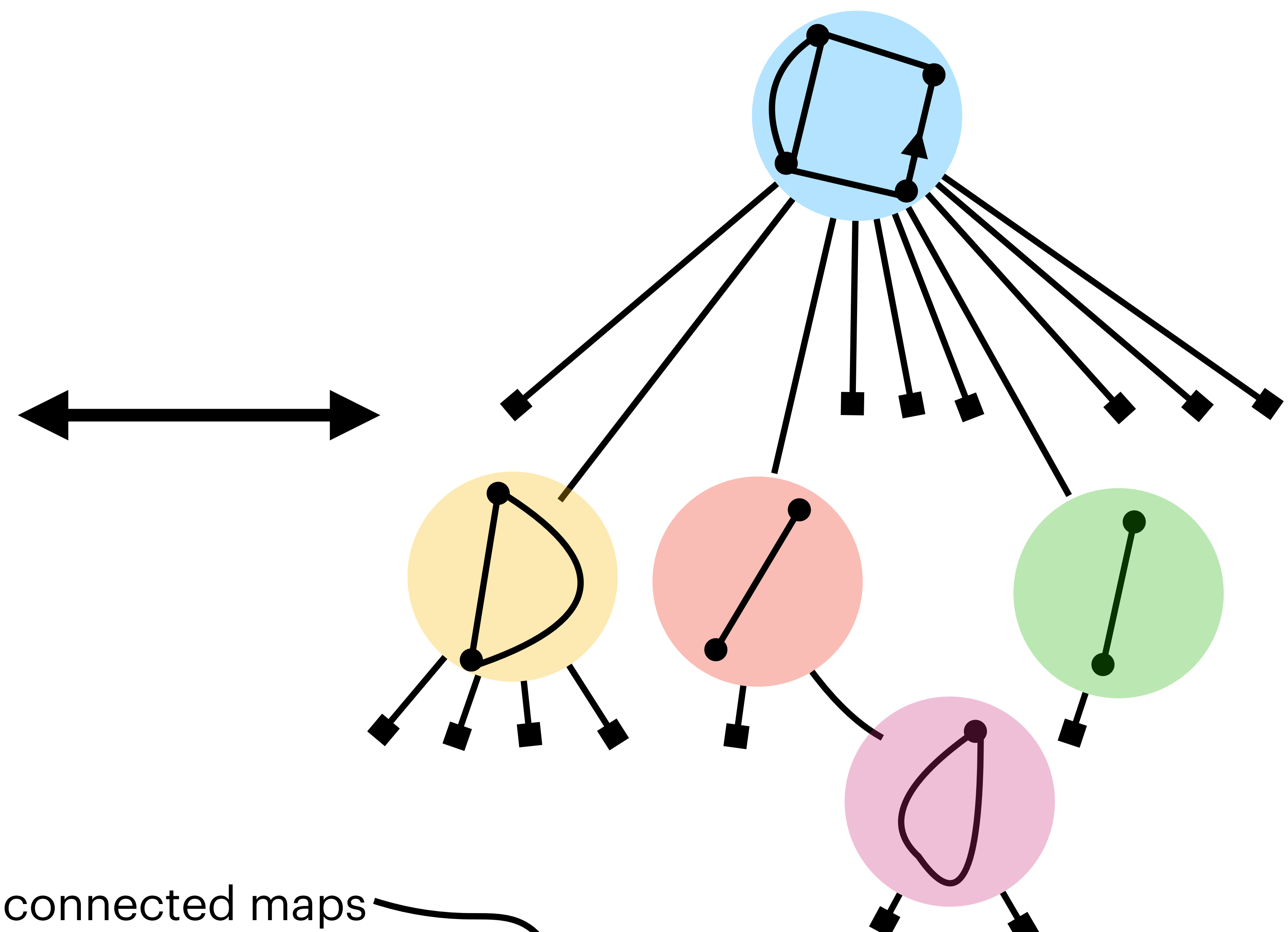
With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$





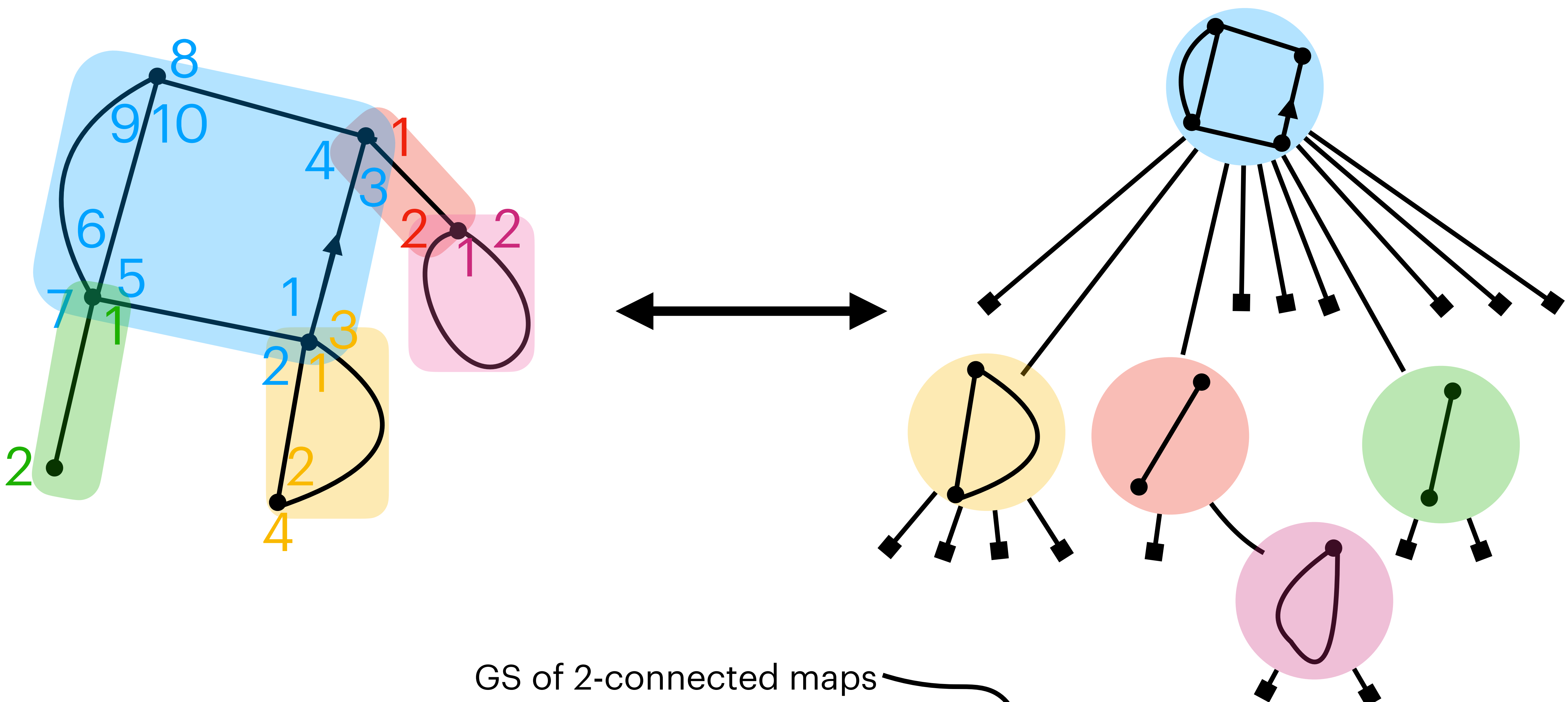
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

Inspiration from [Tutte 1963]

$$M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$



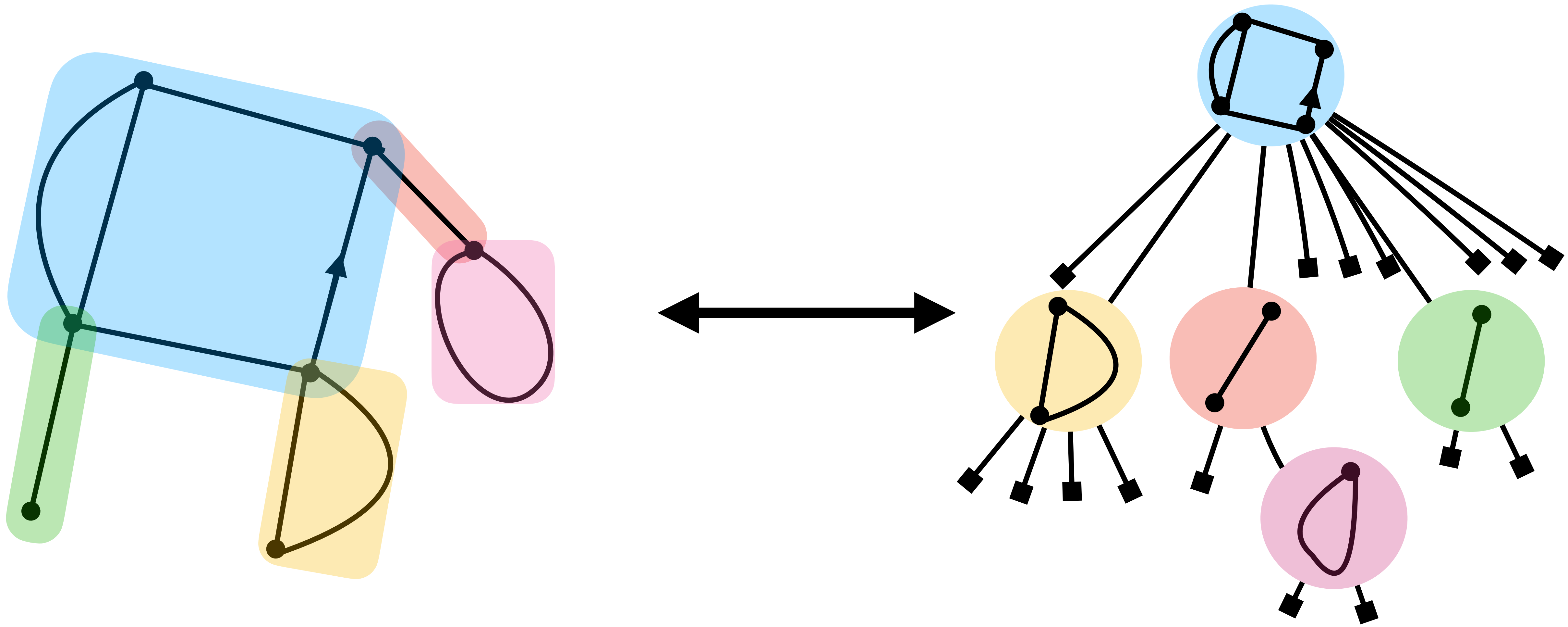
GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Decomposition of a map into blocks: properties



- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$;
- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$.

T_{M_n} **gives the block sizes of a random map** M_n.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

<u>Theorem</u>

If $M_n \hookrightarrow \mathbb{P}_{n,u}$, then T_{M_n} has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k\neq 0}}}{uB(y) + 1 - u}.$$

$u > 0$

$y \in (0, 4/27]$

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

<u>Theorem</u>

If $M_n \hookrightarrow \mathbb{P}_{n,u}$, then T_{M_n} has the law of a Galton-Watson tree of reproduction law $\mu^{y,u}$ conditioned to be of size $2n$, with

$$\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^{\mathbf{1}_{k\neq 0}}}{uB(y) + 1 - u}.$$

$u > 0$

$y \in (0, 4/27]$

=> Choice of y?

Phase transition

When is $\mu^{y,u}$ critical? (= $\mathbb{E}(\mu) = 1$?)

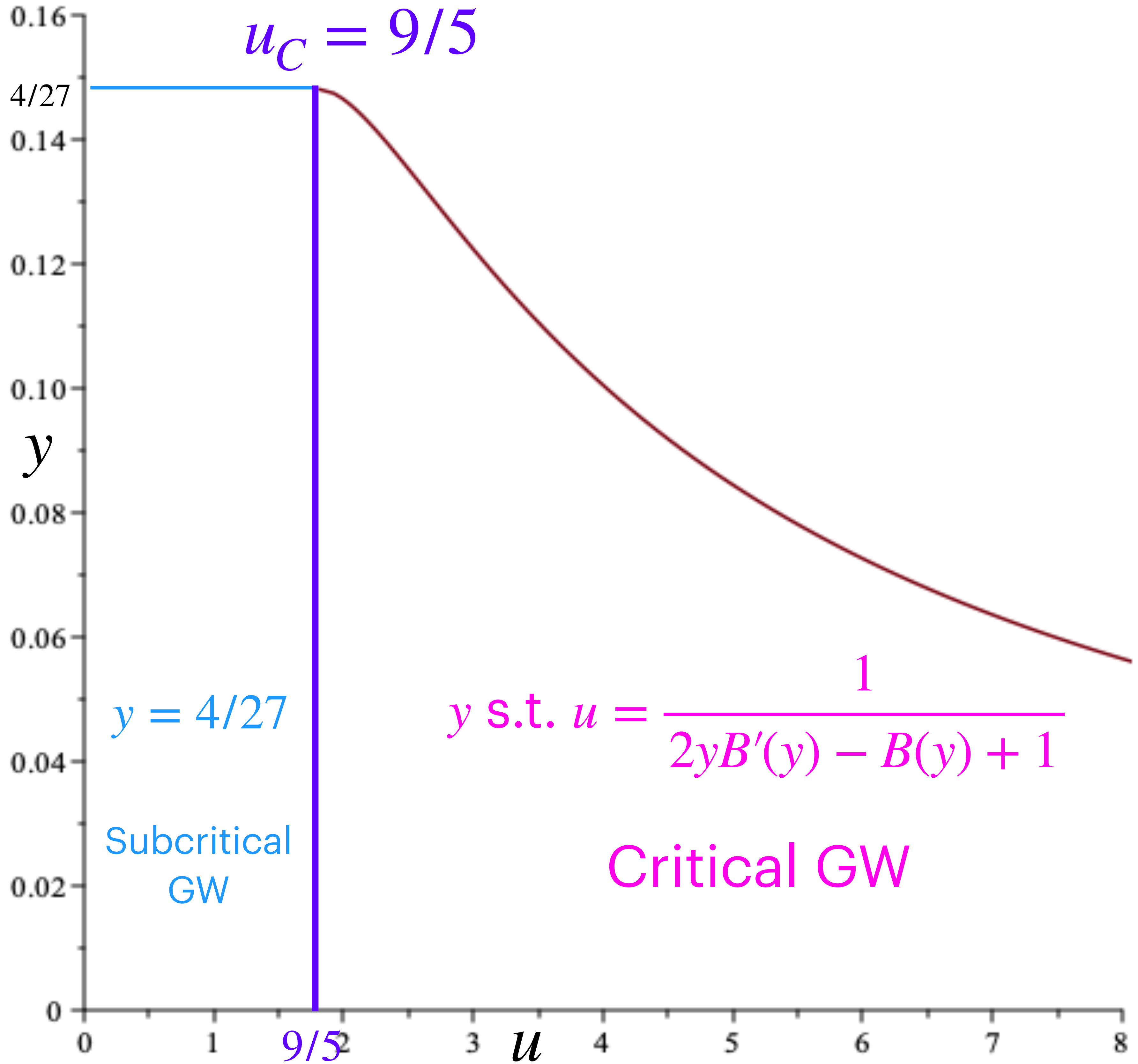
$$\mathbb{E}(\mu^{y,u}) = 1 \Leftrightarrow u = \frac{1}{2yB'(y) - B(y) + 1}$$

covers $[9/5, +\infty)$ when y covers $(0, \rho_B = 4/27]$.

<u>Theorem</u>

- If $u < 9/5$, then $\mathbb{E}(\mu^{y,u}) < 1$. The mean is maximal for $y = 4/27$ and then $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;
- If $u = 9/5$ and $y = 4/27$, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u k^{-5/2}$;
- If $u > 9/5$ and y is well chosen, then $\mathbb{E}(\mu^{y,u}) = 1$ and $\mu^{y,u}(2k) \sim c_u \pi_u^k k^{-5/2}$.

Phase transition



"Map regime"

"Tree regime"

II. Largest blocks

Properties of T_{M_n}

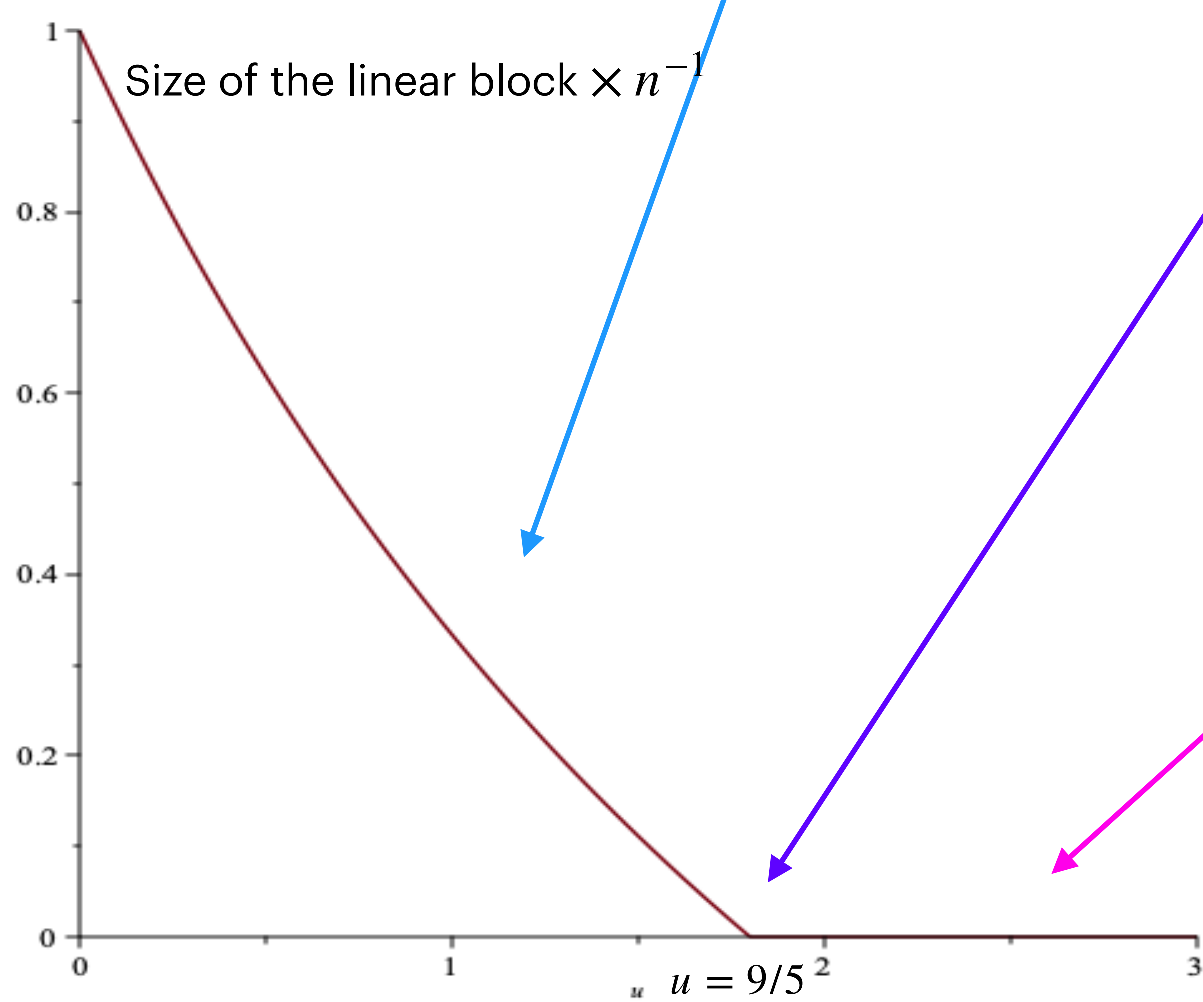
	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Variance	∞		$< \infty$
Galton-Watson tree	subcritical	critical	

Tool: [Janson 2012] = extensive study of the degrees in Galton-Watson trees

Properties on trees give properties of maps.

Size $L_{n,k}$ of the k-th largest block

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$ [Stufler 2020]		



Rough intuition

	$u < 9/5$	$u = 9/5$	$u > 9/5$
$\mu^{y(u),u}(\{2k\})$	$\sim c_u k^{-5/2}$		$\sim c_u \pi_u^k k^{-5/2}$
Galton-Watson tree	subcritical	critical	

Dichotomy between situations:

- Subcritical: condensation, cf [Jonsson Stefánsson 2011];
- Supercritical: behaves as maximum of independent variables.

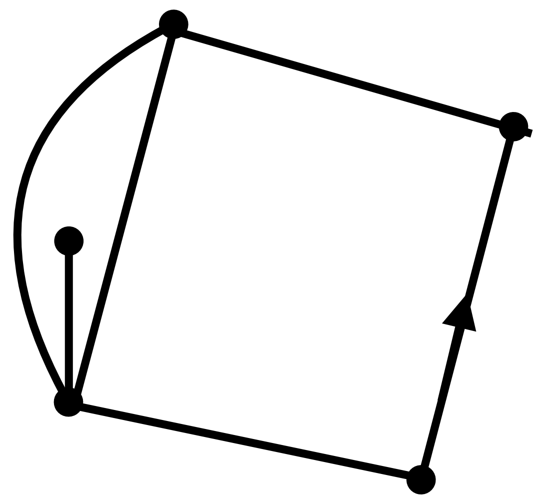
Results

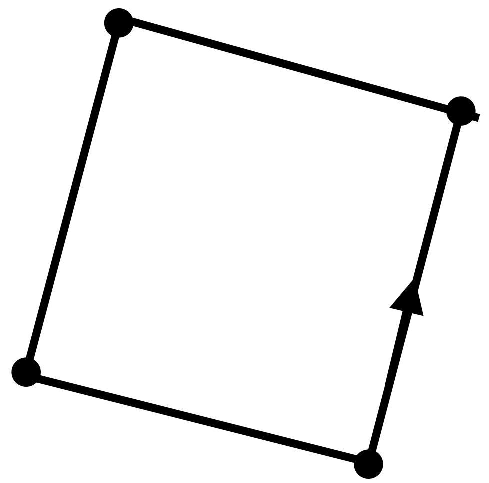
For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016]	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

III. Similar model: quadrangulations

Quadrangulations

Def: map with all faces of degree 4.



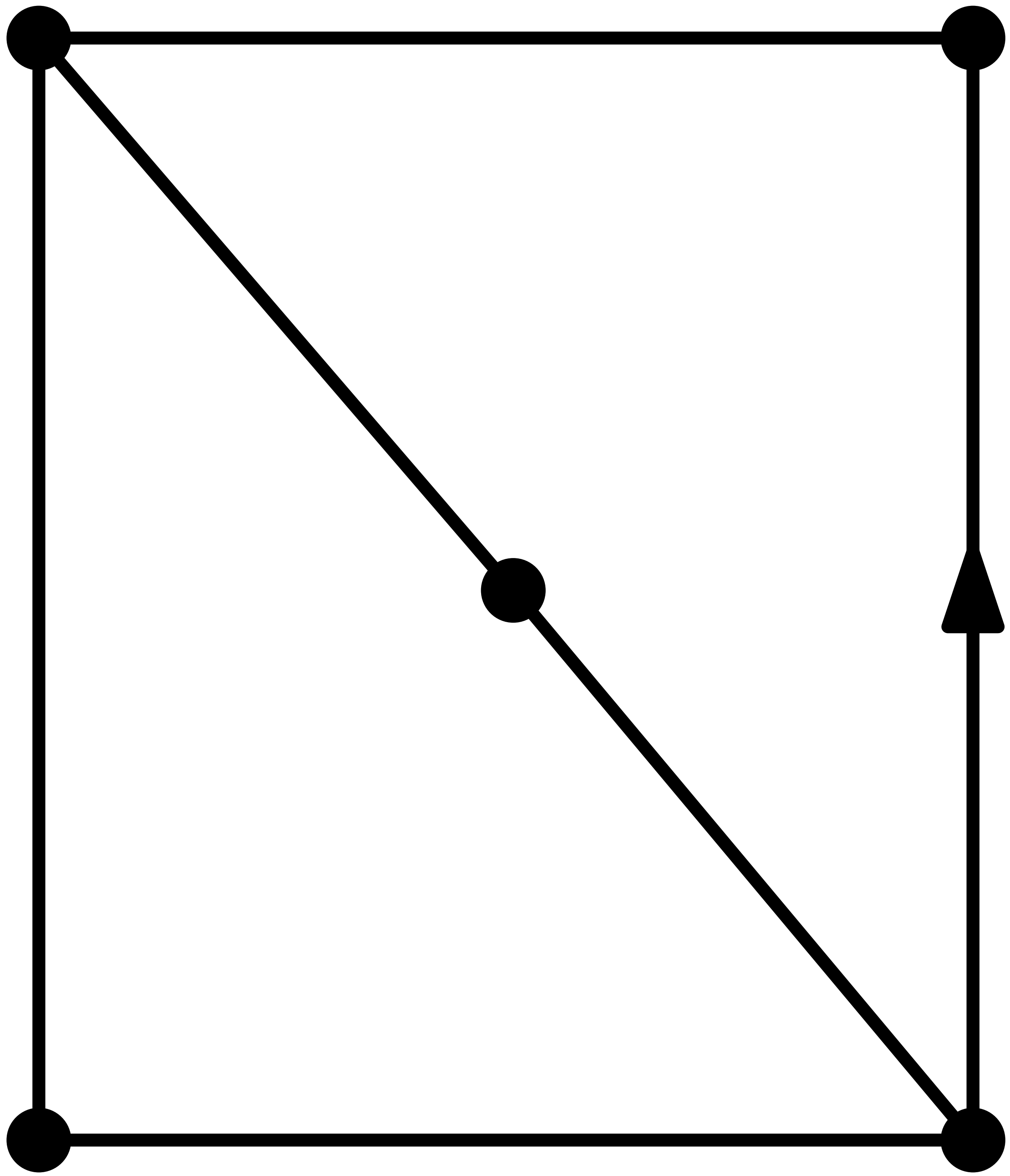


Simple quadrangulation = no multiple edges.

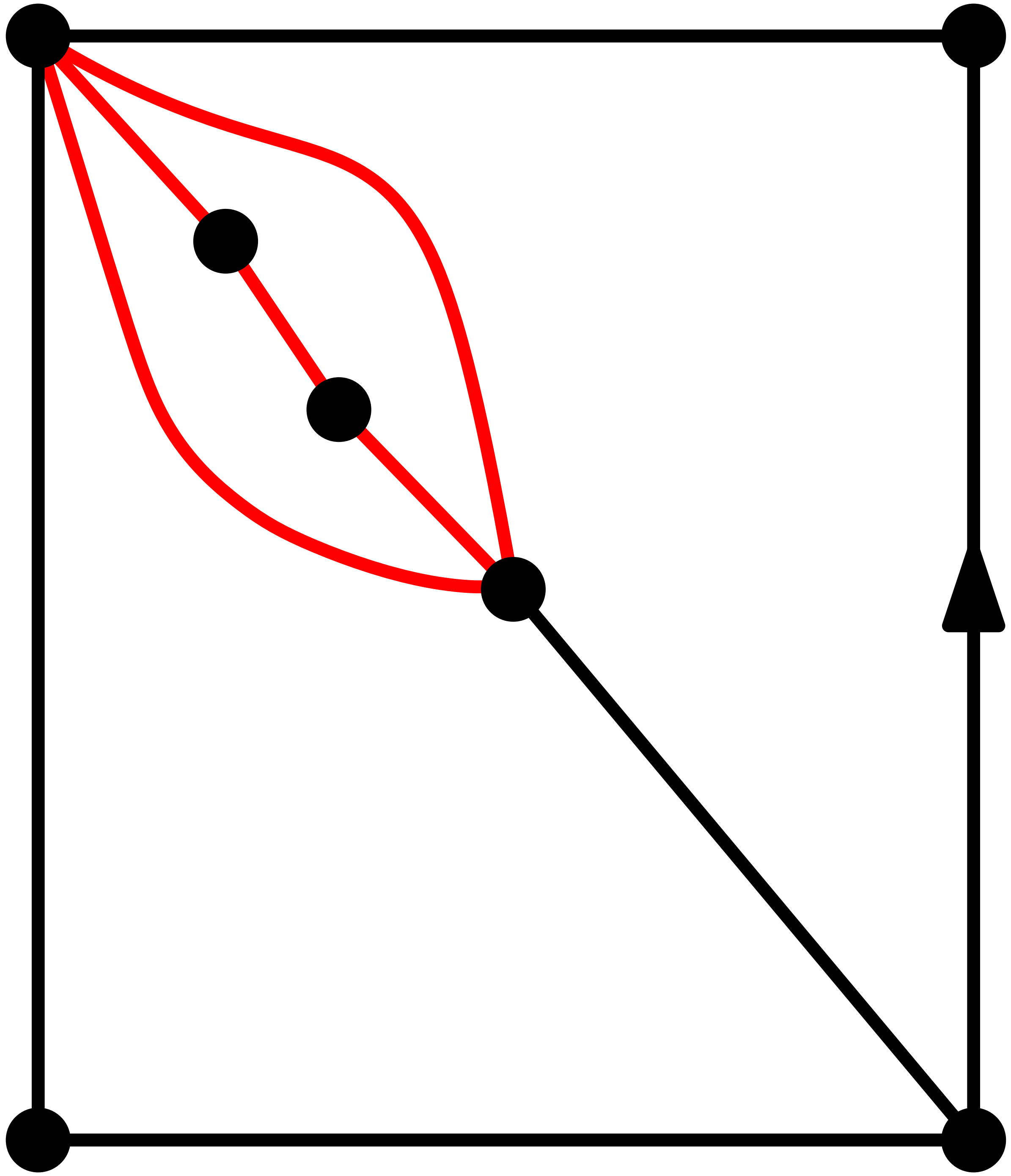
Size $|\mathbf{q}|$ = number of *faces*.

$$|V(\mathbf{q})| = |\mathbf{q}| + 2, \ |E(\mathbf{q})| = 2|\mathbf{q}|.$$

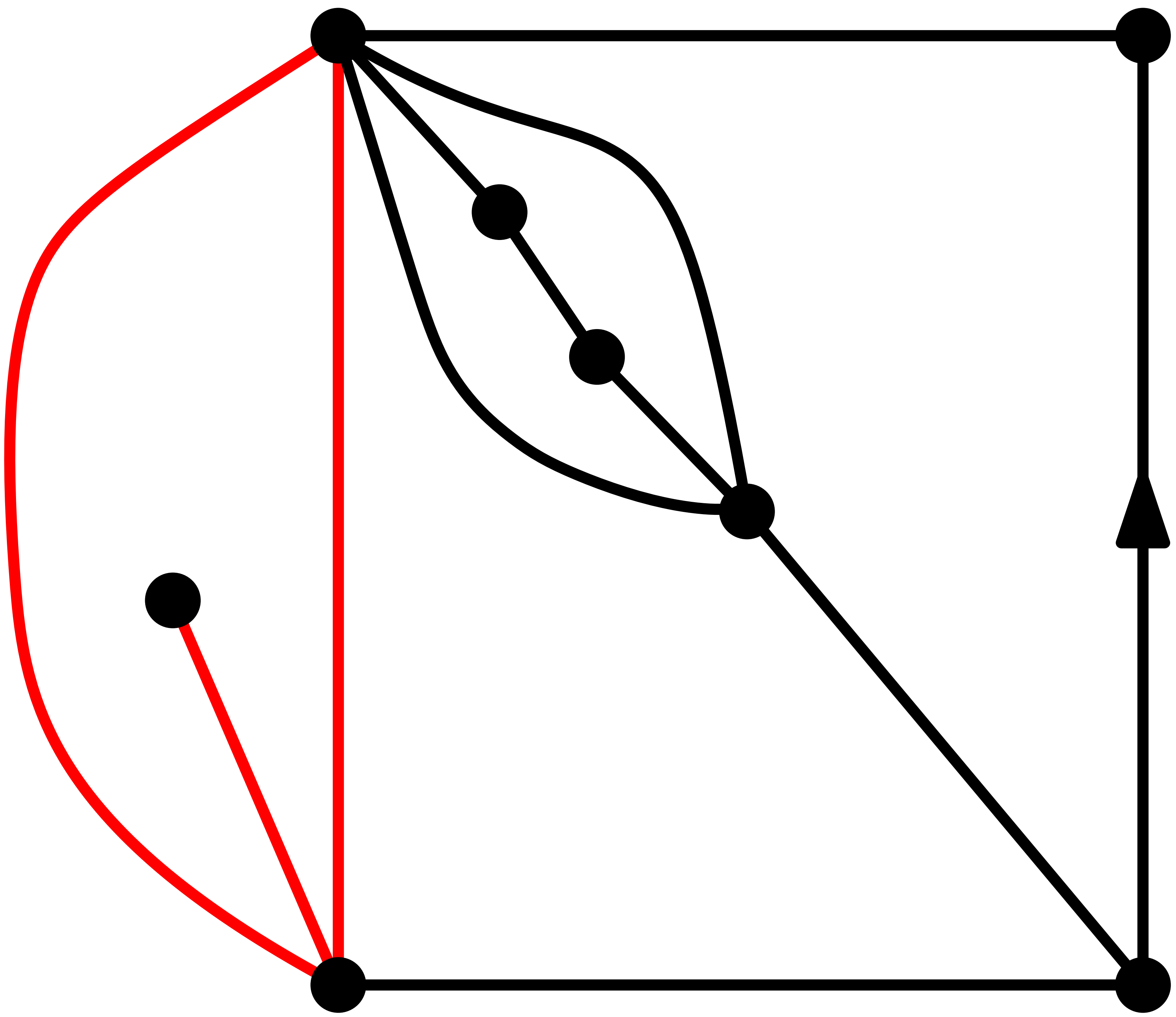
Construction of a quadrangulation from a simple core



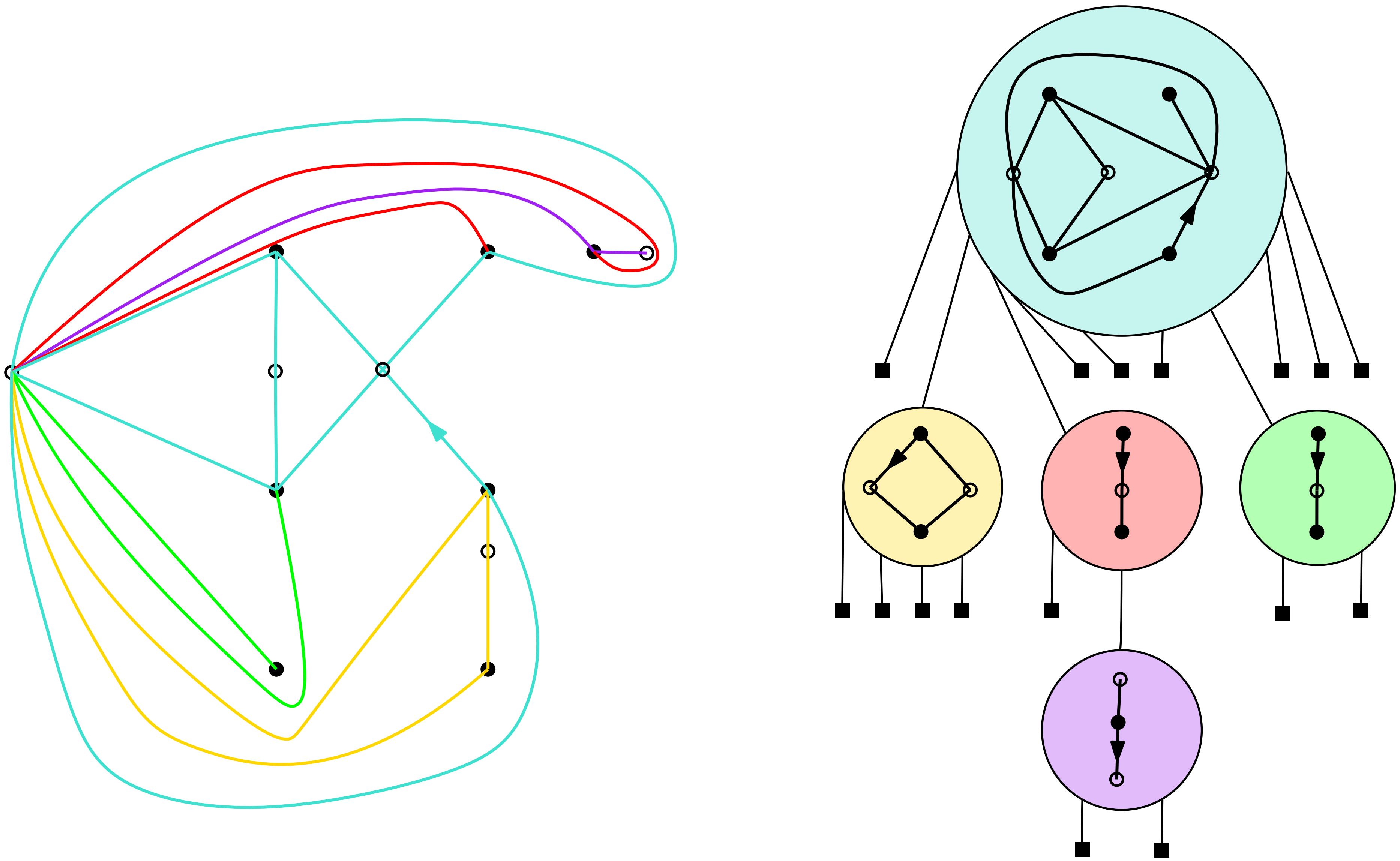
Construction of a quadrangulation from a simple core



Construction of a quadrangulation from a simple core



Block tree for a quadrangulation

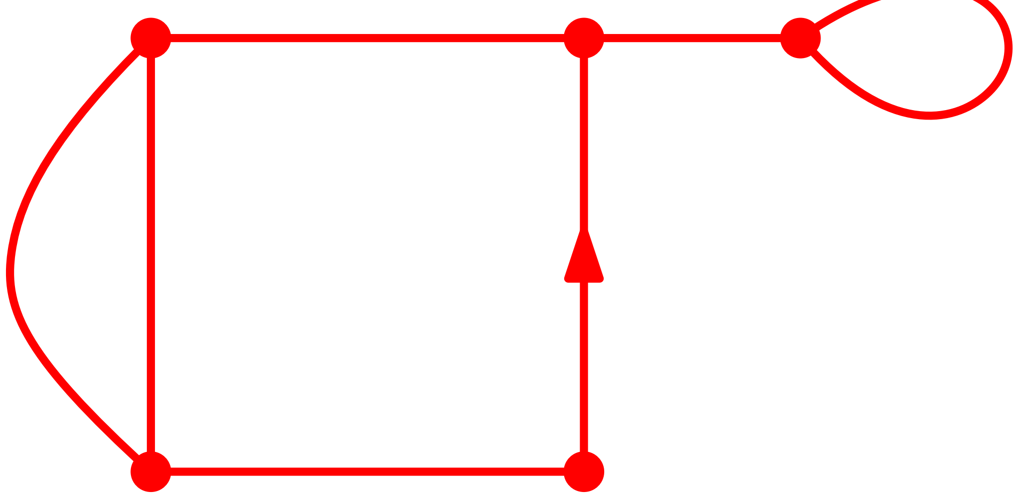


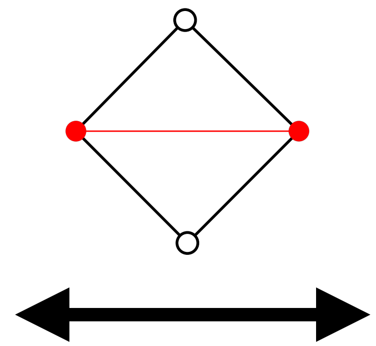
With a weight u on blocks: $Q(z,u) = uS(zQ^2(z,u)) + 1 - u$

Remember: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

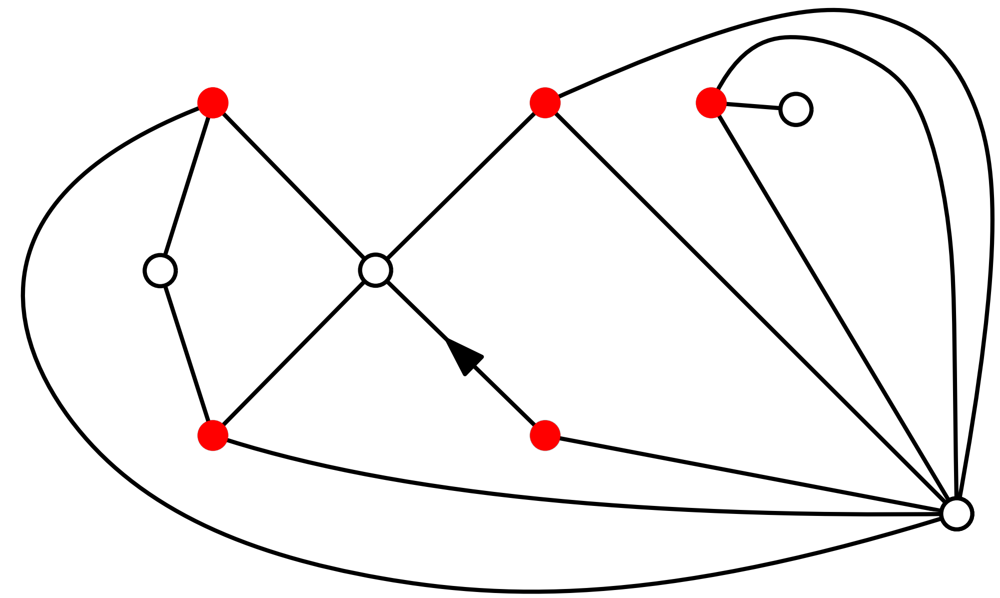
Tutte's bijection

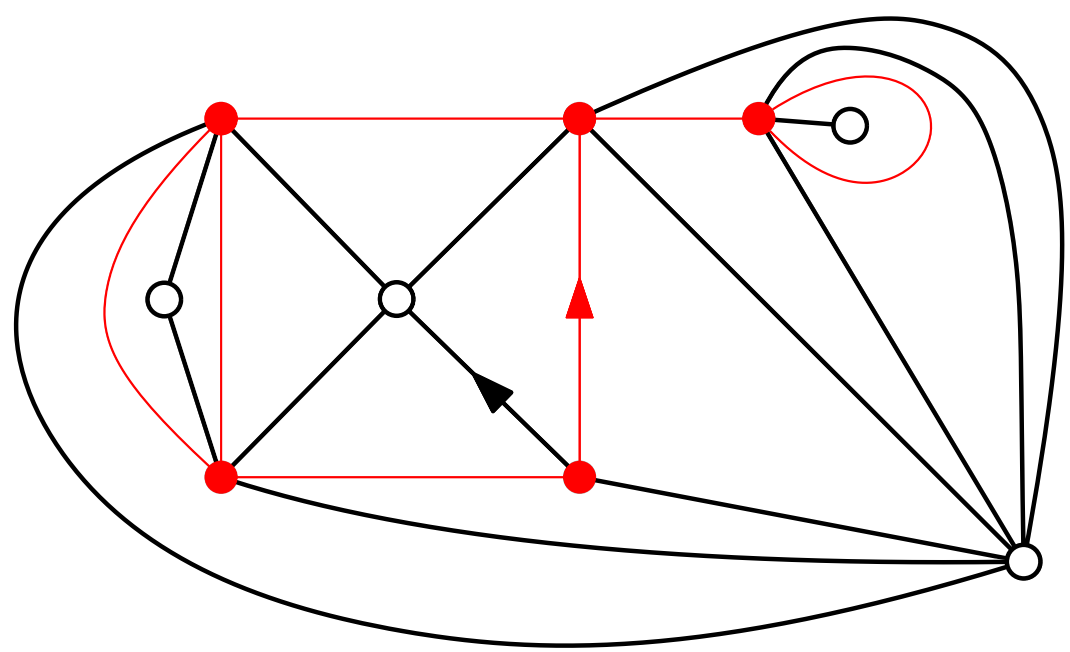
Map





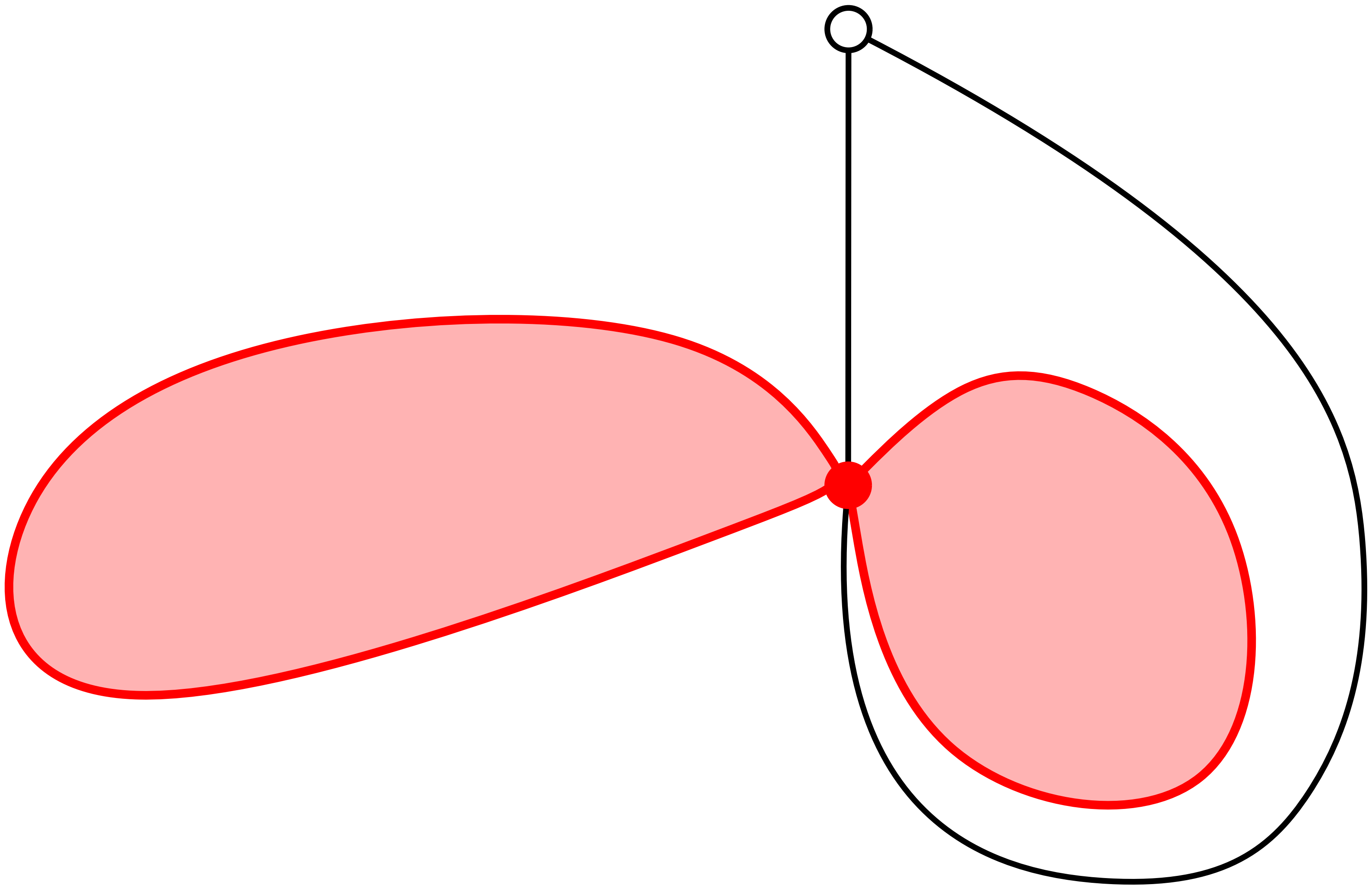
Quadrangulation





[Tutte 1963]

Tutte's bijection for 2-connected maps

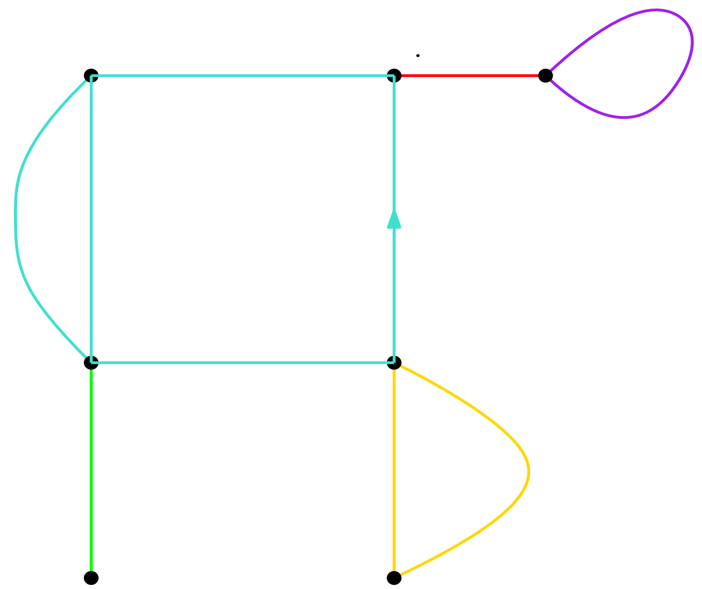


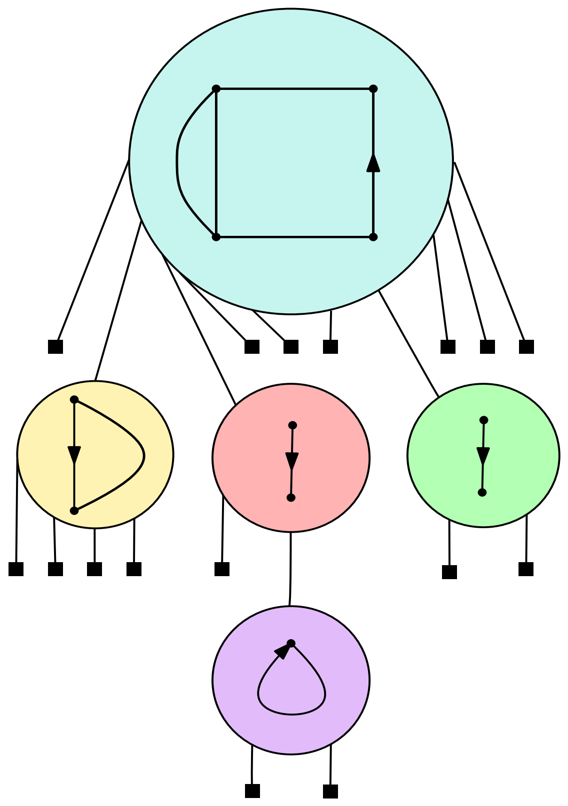
Cut vertex => multiple edge

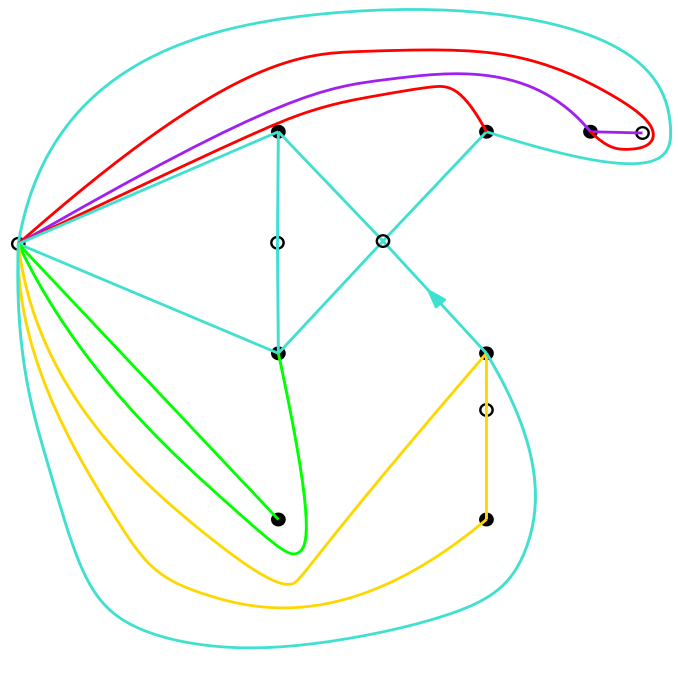
2-connected maps <=> simple quadrangulations

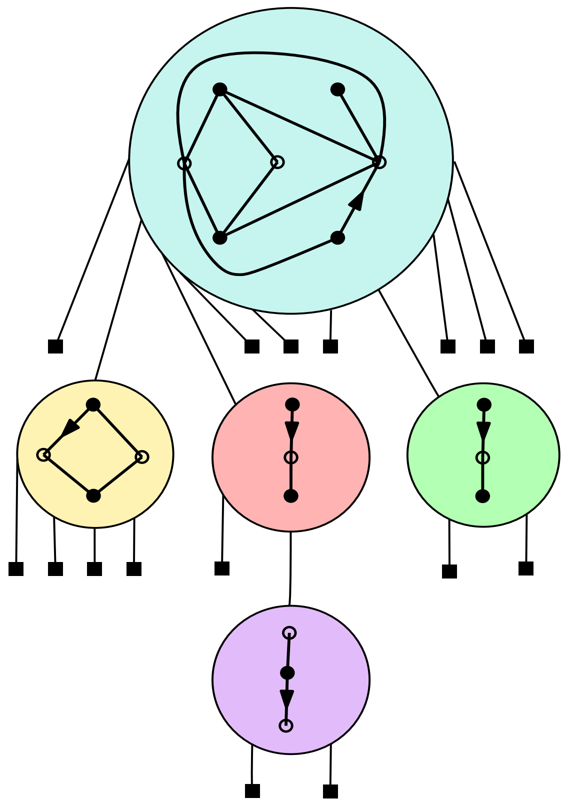
[Brown 1965]

Block trees under Tutte's bijection









Implications on results

We choose: $\mathbb{P}_{n,u}(\mathfrak{q}) = \dfrac{u^{\#blocks(\mathfrak{q})}}{Z_{n,u}}$ where

$u > 0$,

$\mathcal{Q}_n$ = {quadrangulations of size n},

$\mathfrak{q} \in \mathcal{Q}_n$,

$Z_{n,u} =$ normalisation.

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.

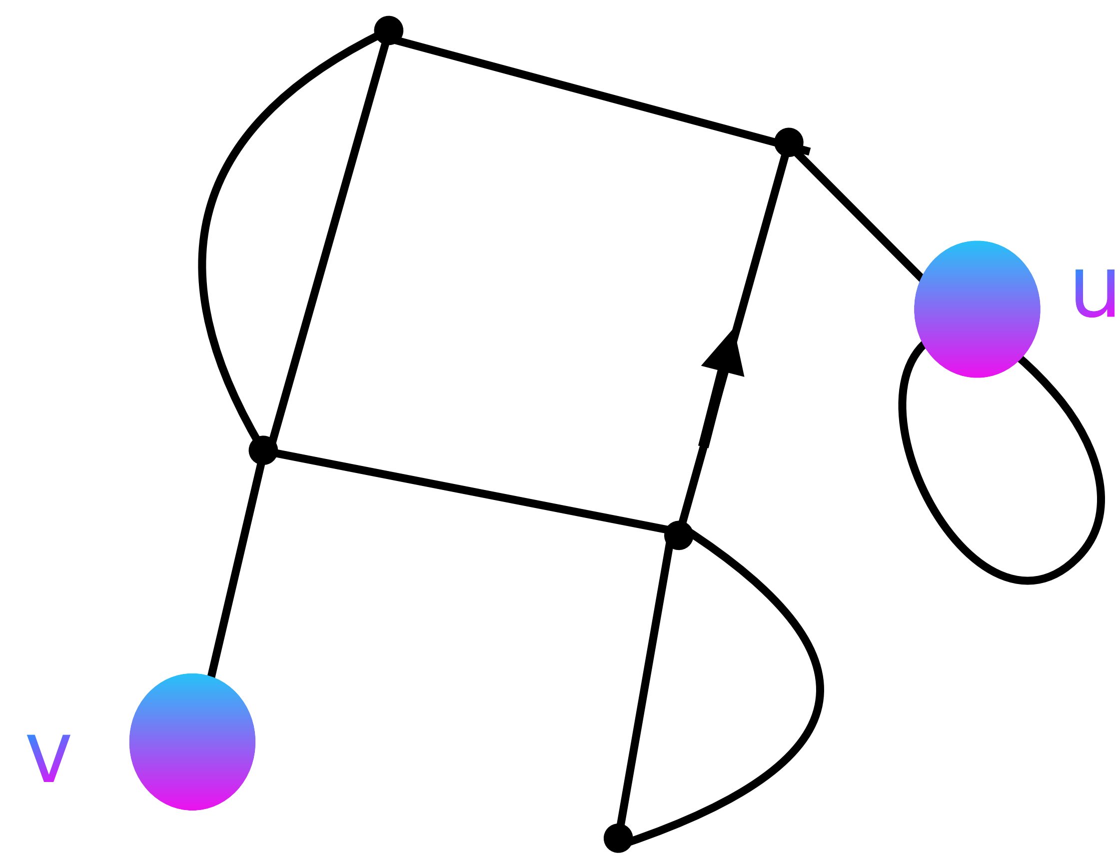
Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration [Bonzom 2016] for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n			

IV. Scaling limits

Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.



$d(u, v) = 4$

$M_n \hookrightarrow \mathbb{P}_{n,u}$

What is the limit of the sequence of metric spaces $((M_n, d/n^?))_{n\in\mathbb{N}}$?

(Convergence for Gromov-Hausdorff metric)

Scaling limits of Galton-Watson trees

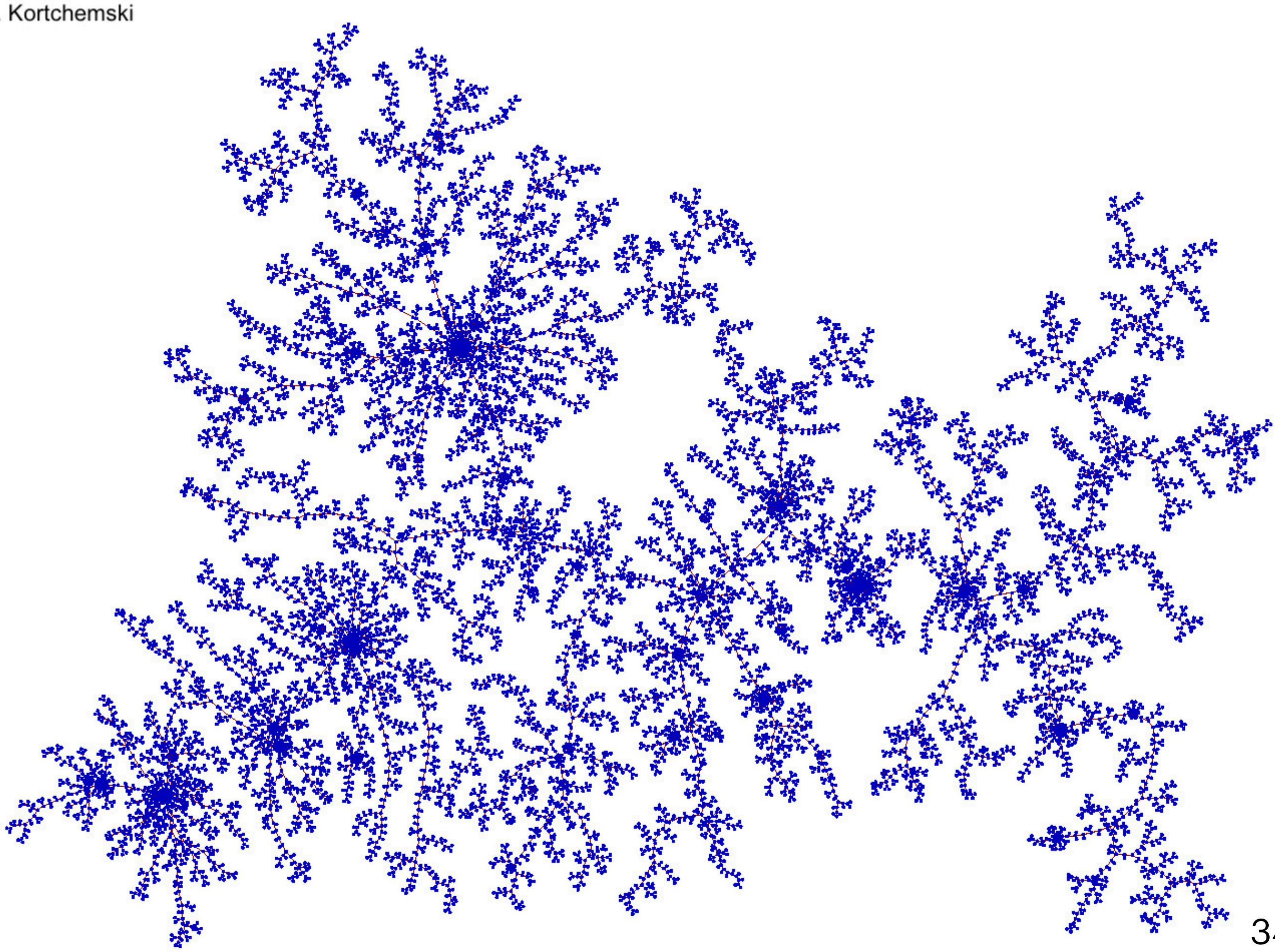
<u>Theorem</u> For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

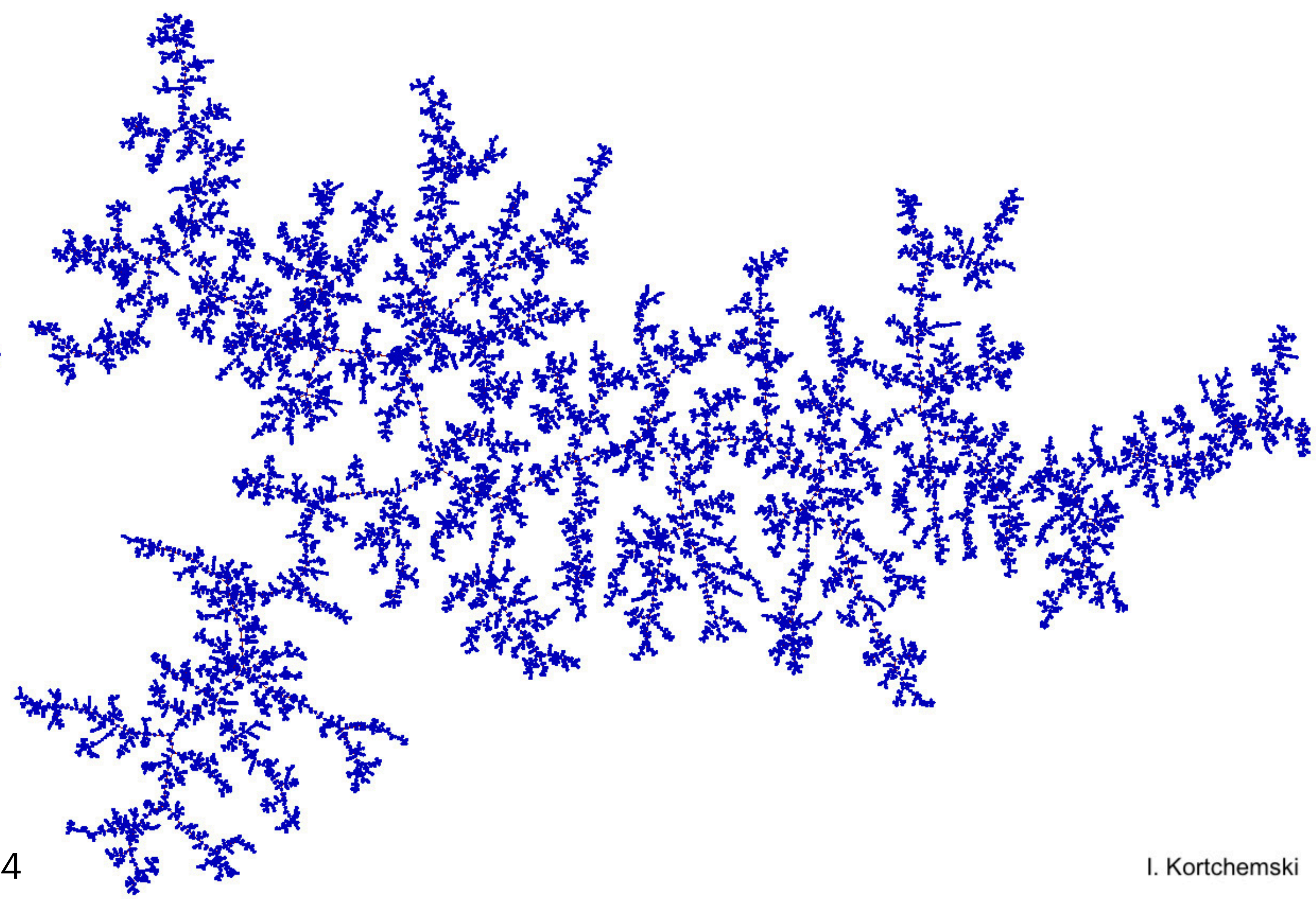
- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}$.

Stable tree of index 3/2 $\mathcal{T}_{3/2}$

Brownian tree $\mathcal{T}_e$ (Aldous's CRT)

I. Kortchemski





I. Kortchemski

Scaling limits of Galton-Watson trees

Theorem For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$, $\frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e$.
- If $u = 9/5$, $\frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}$.

Proof

- Scaling limit of critical Galton-Watson trees with finite variance [Aldous 1993, Le Gall 2006];
- Scaling limit of critical Galton-Watson with infinite variance and nice tails [Duquesne 2003].

Scaling limit of supercritical and critical maps

<u>Theorem</u> For $M_n \hookrightarrow \mathbb{P}_{n,u}$,

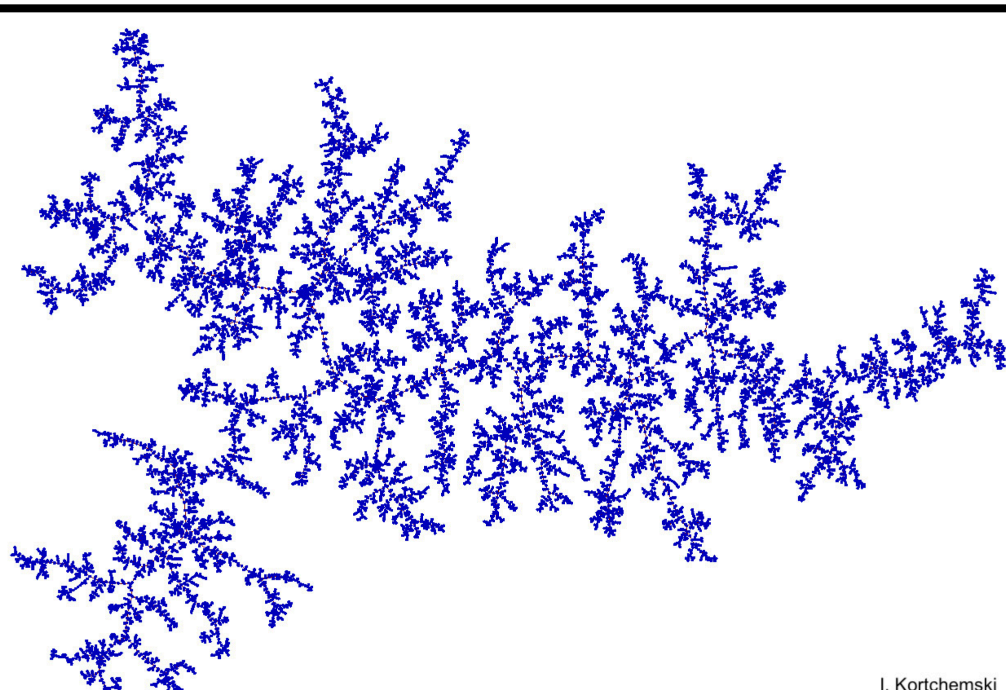
- If $u > 9/5$,

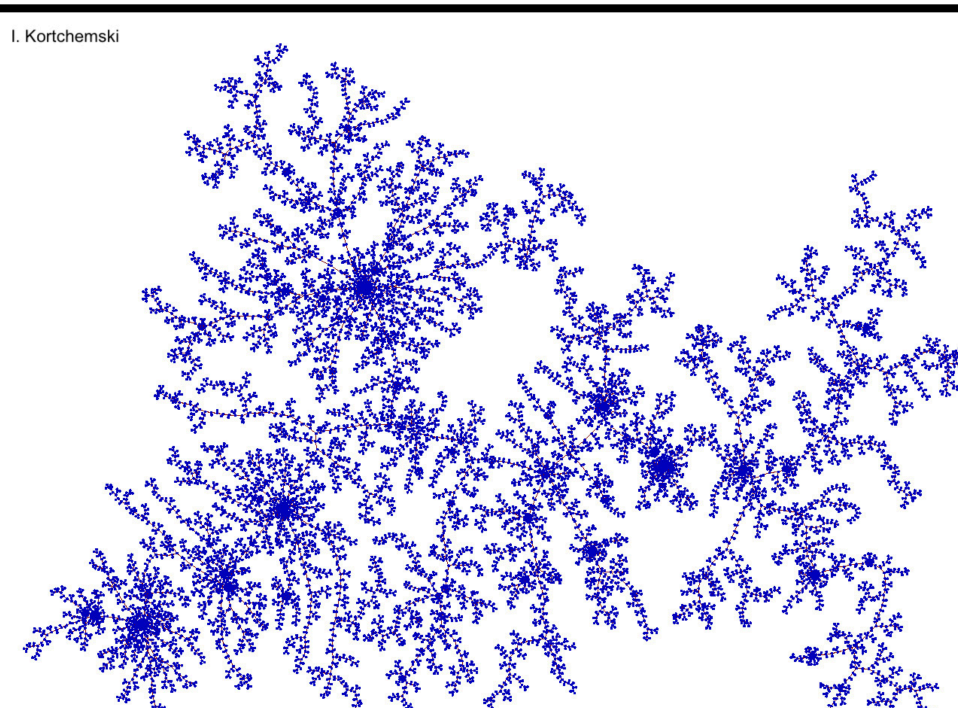
$$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e.$$

- If $u = 9/5$,

$$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}.$$

[Stufler 2020]

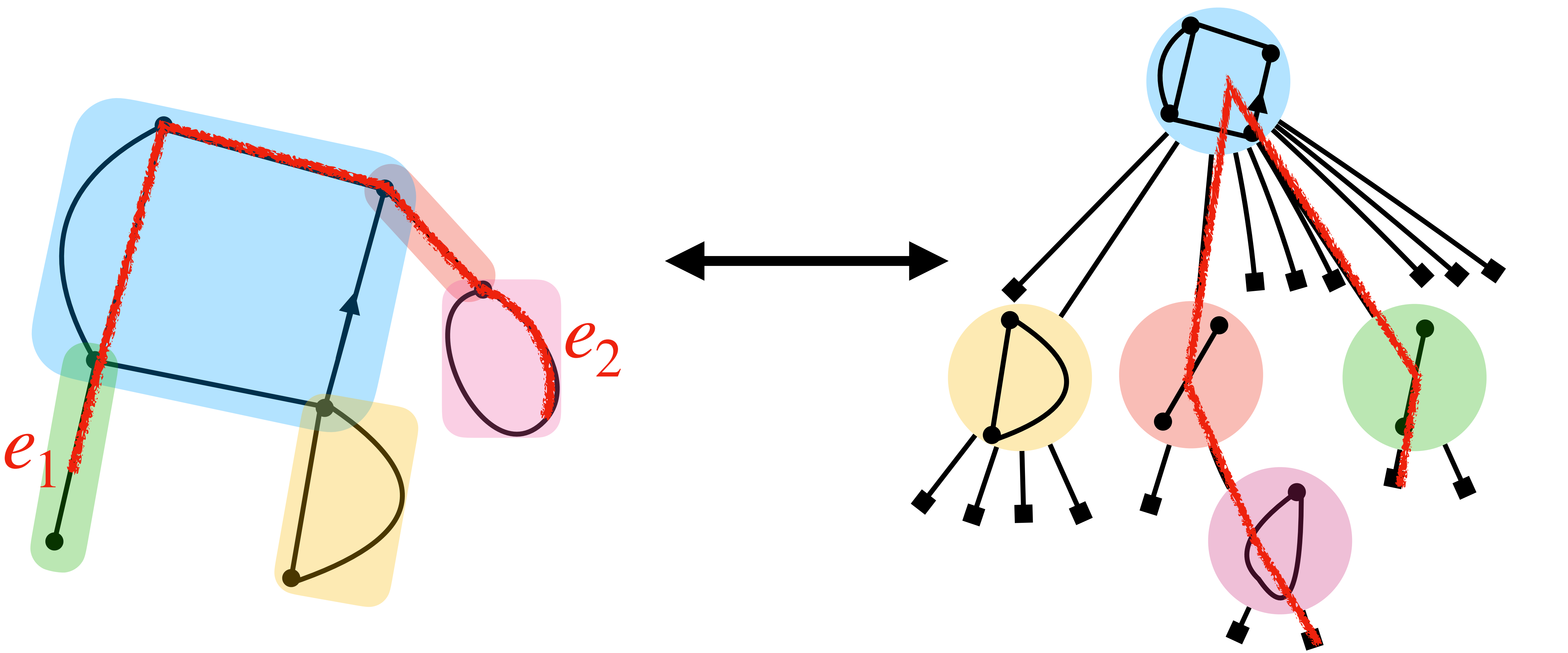




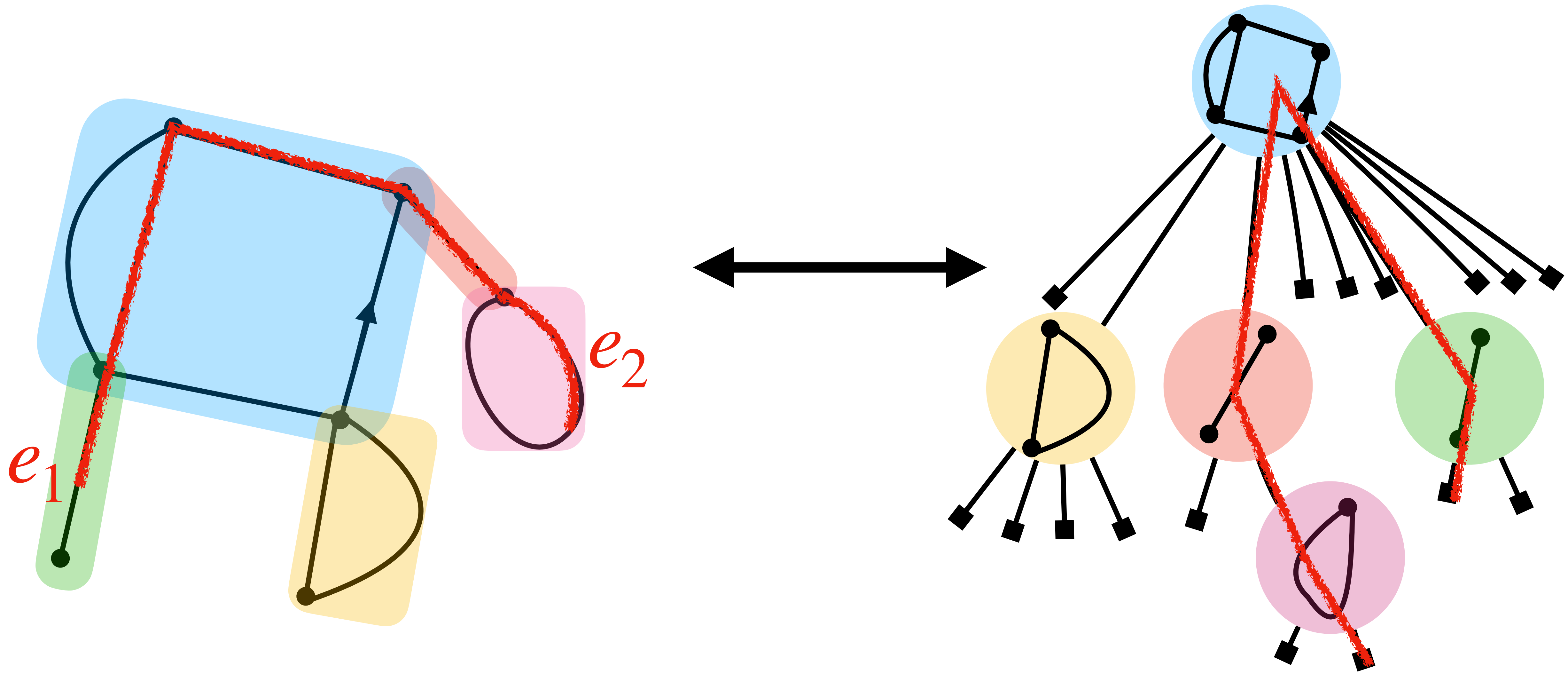
Supercritical and critical cases (1)

Difficult part = show that distances in $\mathfrak{m}$ behave like distances in $T_{\mathfrak{m}}$. We show

$$\forall e_1, e_2 \in \overrightarrow{E}(M_n), d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$



Supercritical and critical cases (2)



Let $\kappa = \mathbb{E}$("diameter" bipointed block). By a “law of large numbers”-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2)\,.$$

Difficult for the critical case => use diameter estimates

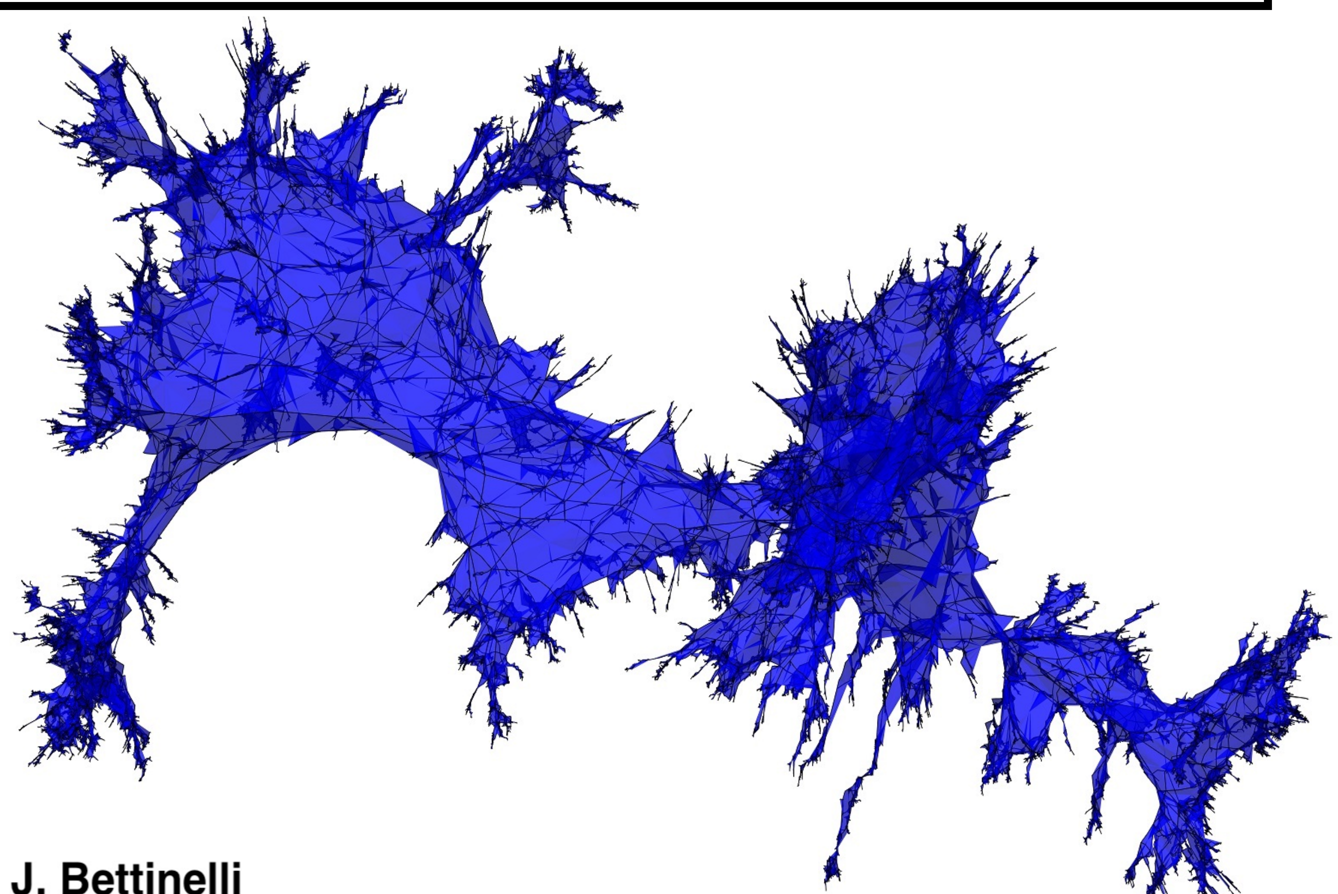
Scaling limits of subcritical maps

<u>Theorem</u> If $u < 9/5$, for $M_n \hookrightarrow \mathbb{P}_{n,u}$ a quadrangulation,

$$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e.$$

Moreover, M_n and its simple core converge jointly to the same Brownian sphere.

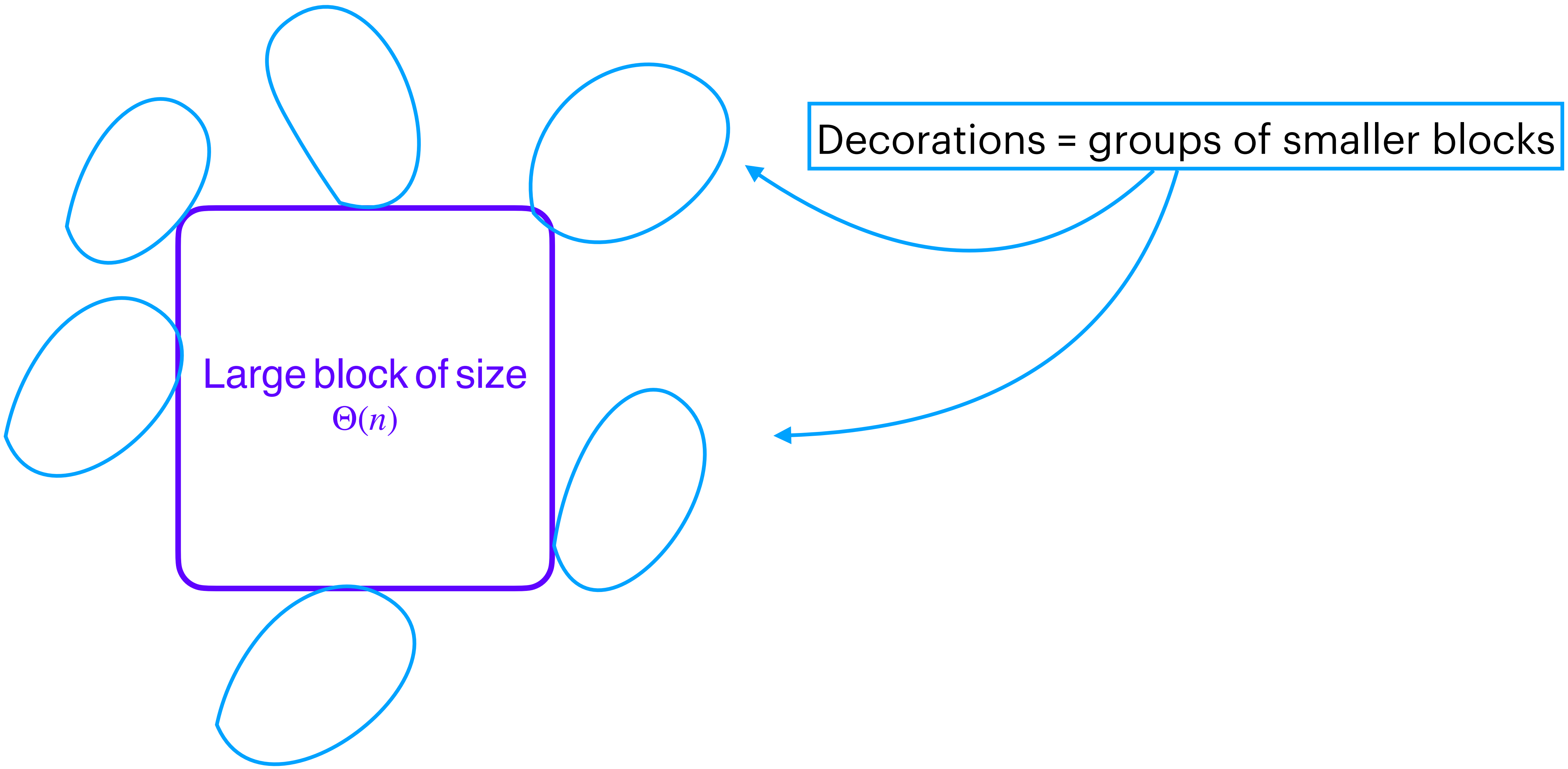
We expect the same scaling limits for maps but the scaling limit of 2-connected maps is not yet proved.



J. Bettinelli

See [Addario-Berry, Wen 2019] for a similar result and method

Subcritical case (1)



Diameter of a decoration ≤ number of blocks × max diameter of blocks

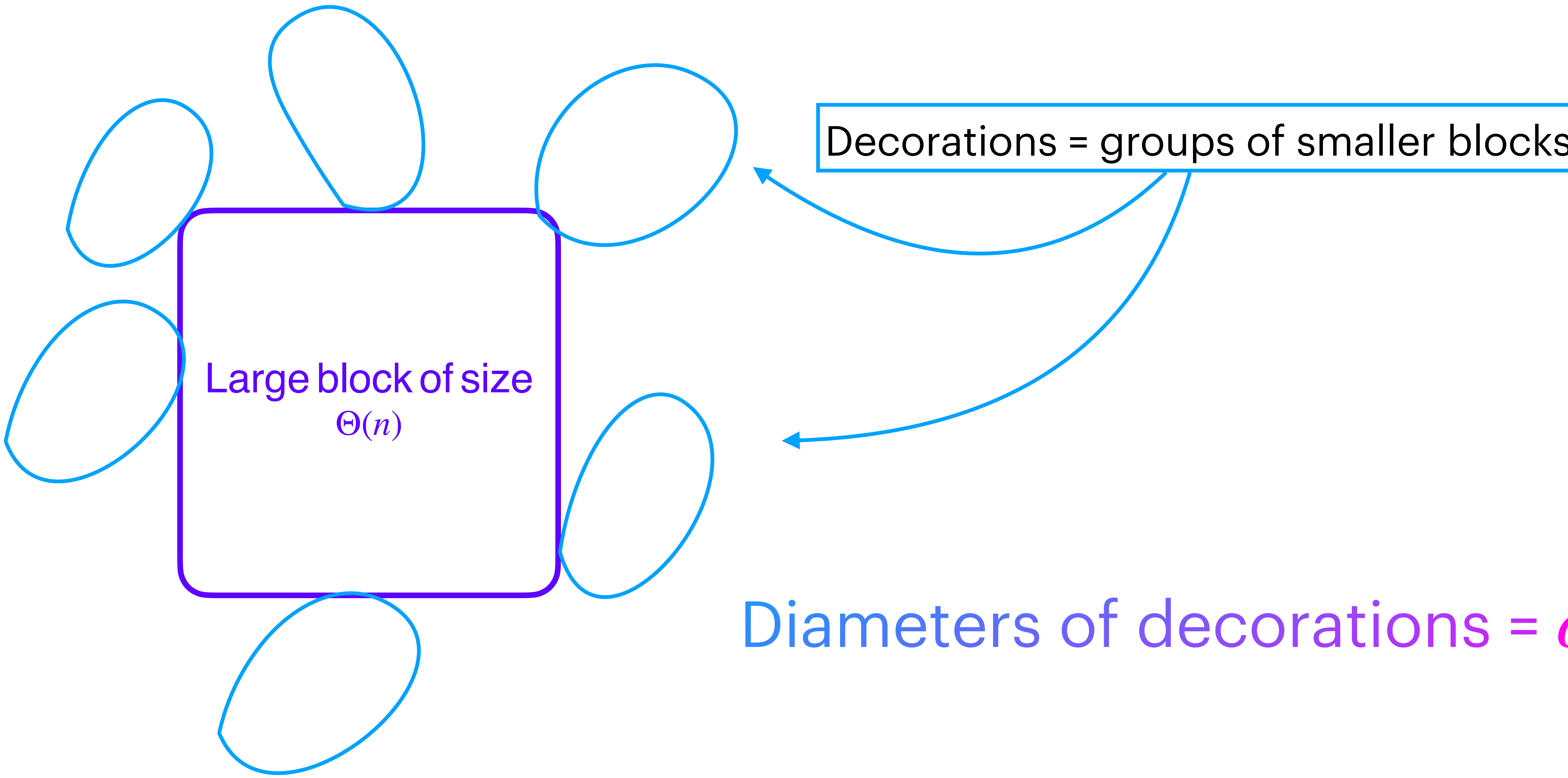
$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) \,.$$

T_{M_n} **is a subcritical Galton-Watson tree**

[Chapuy Fusy Giménez Noy 2015]

Subcritical case (1)



Diameters of decorations = $o(n^{1/4})$.

Diameter of a decoration ≤ number of blocks × max diameter of blocks

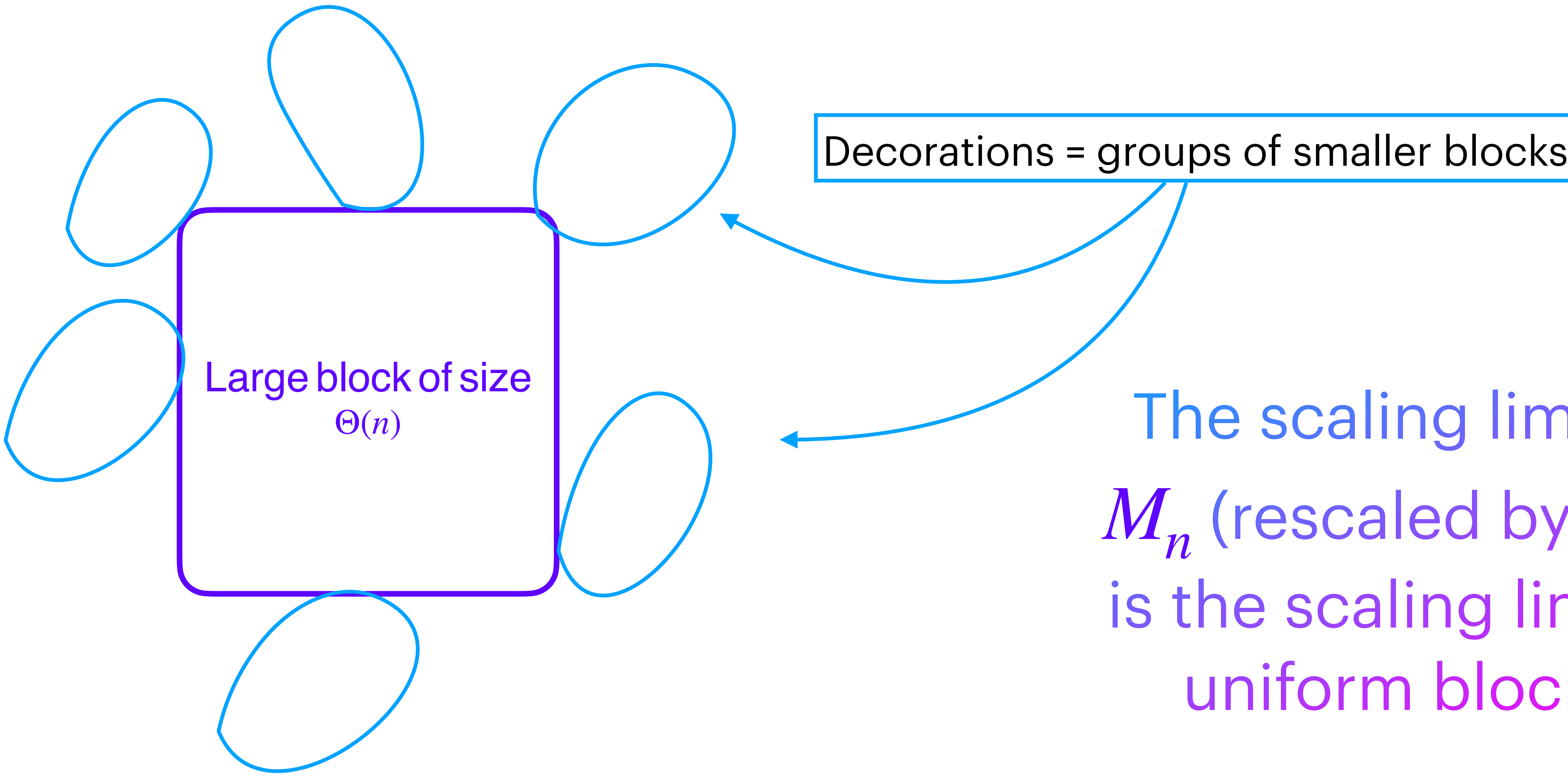
$$\leq diam(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = diam(T_{M_n}) \times O(n^{1/6+\delta})$$

$$= O(n^{1/6+2\delta}) = o(n^{1/4}) .$$

T_{M_n} **is a subcritical Galton-Watson tree**

[Chapuy Fusy Giménez Noy 2015]

Subcritical case (2)

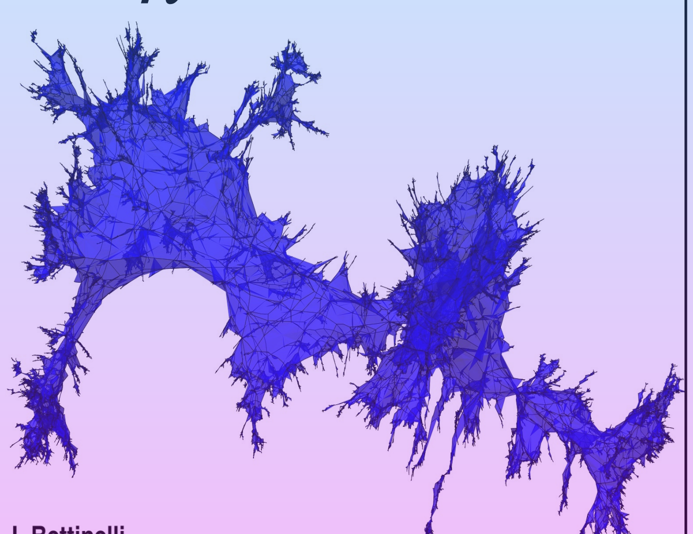
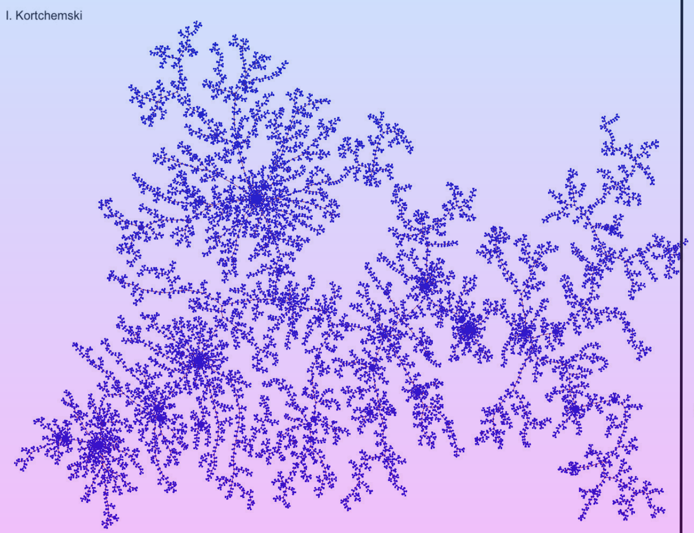
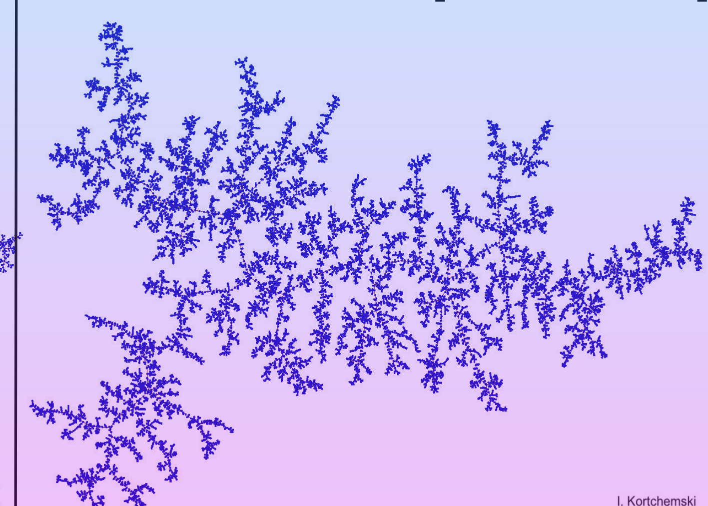


The scaling limit of M_n (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!

Scaling limit of uniform ~ (rescaled by $n^{1/4}$)

- 2-connected maps = brownian sphere (assumed);
- Simple quadrangulations = Brownian sphere [Addario-Berry Albenque 2017].

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u = 9/5$	$u > 9/5$
Enumeration Bonzom 2016 for 2-c case	$\rho(u)^{-n} n^{-5/2}$	$\rho(u)^{-n} n^{-5/3}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$ $\Theta(n^{2/3})$ [Stufler 2020]	$\Theta(n^{2/3})$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e$  J. Bettinelli Assuming the convergence of 2-connected maps towards the brownian sphere	$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$ I. Kortchemski 	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$ [Stufler 2020]  I. Kortchemski

V. Perspectives

Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

TABLE 3. Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$	coreless, $D(z)$
all, $M_1(z)$	bridgeless, or loopless $M_2(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$
loopless $M_2(z)$	simple $M_3(z)$	$z(1+M)$	–
all, $M_1(z)$	nonsep., $M_4(z)$	$z(1+M)^2$	–
nonsep. $M_4(z) - z$	nonsep. simple $M_5(z)$	$z(1+M)$	–
nonsep. $M_4(z)/z - 2$	3-connected $M_6(z)$	M	$z + 2M^2/(1+M)$
bipartite, $B_1(z)$	bip. simple, $B_2(z)$	$z(1+M)$	–
bipartite, $B_1(z)$	bip. bridgeless, $B_3(z)$	$z/(1 - z(1+M))^2$	$z(1+M)^2$
bipartite, $B_1(z)$	bip. nonsep., $B_4(z)$	$z(1+M)^2$	–
bip. nonsep., $B_4(z)$	bip. ns. smpl, $B_5(z)$	$z(1+M)$	–
singular tri., $T_1(z)$	triang., $z + zT_2(z)$	$z(1+M)^3$	–
triangulations, $T_2(z)$	irreducible tri., $T_3(z)$	$z(1+M)^2$	–

16/7

64/37

Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n)$, $\varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, conjecture $L_{n,1} \sim 2.7648\, \varepsilon^{-2} \ln(\varepsilon^3 n)$ when $\varepsilon^3 n \to \infty$ (analogous to [Bollobás 1984]'s result for Erdős-Rényi graphs!);
- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?

Perspectives

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < 9/5$	$u_n = 9/5 - \varepsilon(n)$ $\varepsilon^3 n \to \infty$	$u = 9/5$	$u_n = 9/5 + \varepsilon(n)$ $\varepsilon^3 n \to \infty$	$u > 9/5$
$L_{n,1}$	$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$		$\Theta(n^{2/3})$	$\sim 2.7648\, \varepsilon^{-2} \ln(\varepsilon^3 n)$	$\frac{\ln(n)}{2\ln\left(\frac{4}{27y}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y}\right)} + O(1)$
$L_{n,2}$	$\Theta(n^{2/3})$				
Scaling limit of M_n	$\frac{C_1(u)}{n^{1/4}} M_n \to \mathcal{S}_e$ Admitting the convergence of 2-connected maps towards the brownian map	$\varepsilon(n) = n^{-\alpha}$ $\frac{C_4}{n^{(1-\alpha)/4}} M_n \to \mathcal{S}_e$	$\frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}$	stable tree ?	$\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$

Pink = work in progress

Thank you!