Topology of the arc complex

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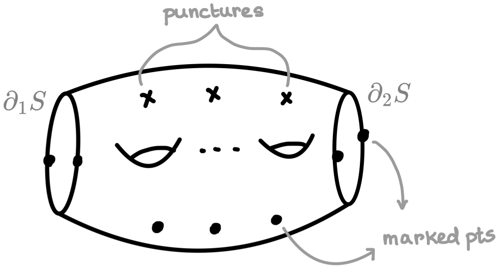
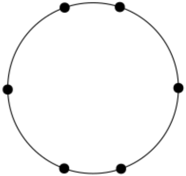
Marked surfaces

Setting: Let S be a finite-type, possibly non-orientable surface with finitely many marked points such that

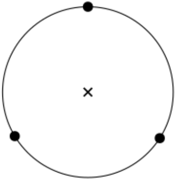
- if $\partial S \neq \emptyset$, then there is at least one marked point on every boundary component;
- interior points can be marked.



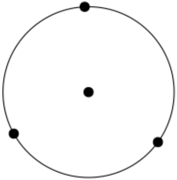
Examples of marked surfaces



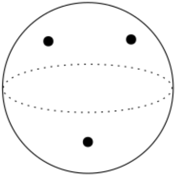
Convex polygon, $\mathcal{P}_n$



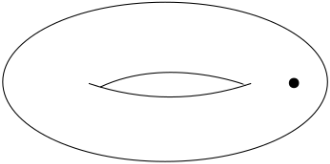
Once-punctured polygon, $\mathcal{P}_n^{\times}$



Orientable crown, $\mathcal{P}_n^{\circledcirc}$



Three-holed sphere



One-holed torus

Arcs

Definition

An arc on S is defined as $\alpha : [0,1] \hookrightarrow S$ such that $\alpha([0,1]) \cap S = \{\alpha(0), \alpha(1)\} \subset \mathcal{P}$.

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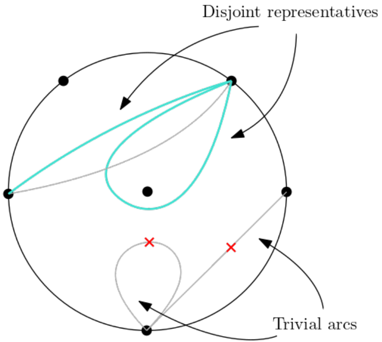
- we consider isotopy classes of non-trivial arcs;
- 2 classes are *disjoint* if they have two disjoint representatives.



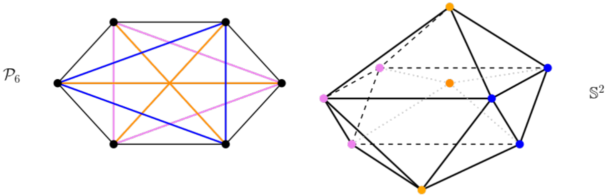
Arc complex

$(S, \mathcal{P})$: a marked surface.

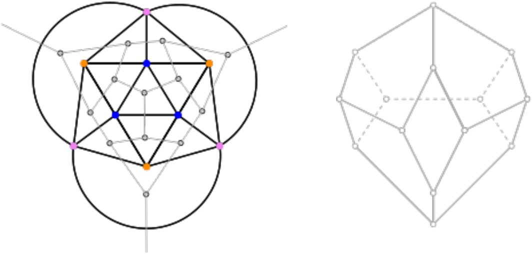
$\mathcal{A}(S)$: a flag, pure simplicial complex constructed in the following way:

- 0-simplices $\longleftrightarrow$ isotopy classes of embedded arcs,
- For $k \geq 1$, k-simplices $\longleftrightarrow$ $(k+1)$ pairwise disjoint and distinct classes.

Example: a convex polygon $\mathcal{P}_n$, for $n \geq 4$



(a) The arc complex of a hexagon



(b) Two-dimensional associahedron

The arc complex

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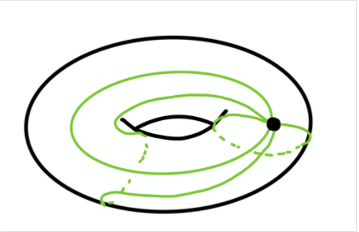
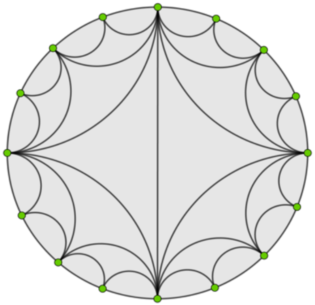
- The arcs of top-dimensional simplices divide the surface into triangles and at most one once-punctured disk.
- The arc complex is connected.
- For a "generic" surface, the arc complex is locally non-compact with infinite diameter.

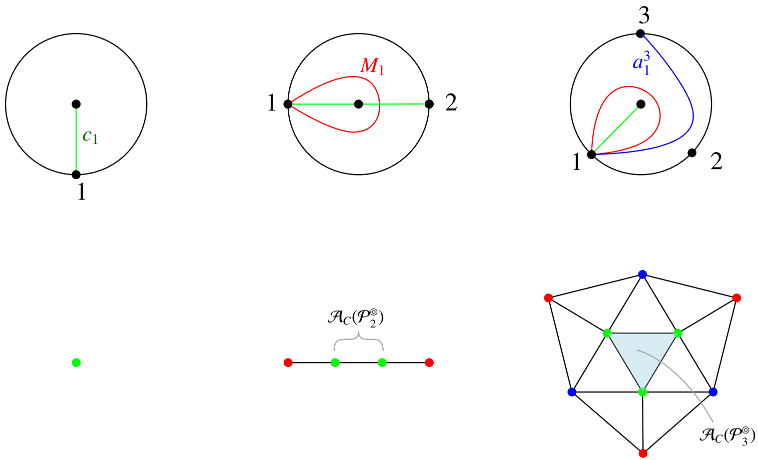
"Generic" example: One-holed torus





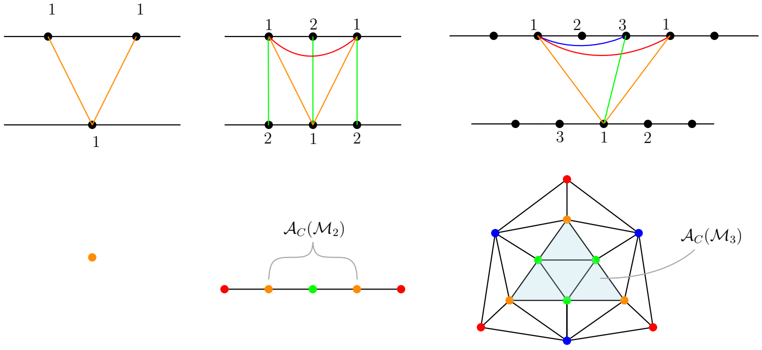
The arc complex $\mathcal{A}(S_{1,1})$

Example: orientable crown $\mathcal{P}_n^{\circledcirc}$, for $n \geq 1$



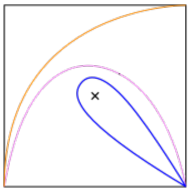
$\mathcal{A}_C(\mathcal{P}_1^{\circledcirc}) = \mathcal{A}(\mathcal{P}_1^{\circledcirc})$ $\quad\quad$ $\mathcal{A}(\mathcal{P}_2^{\circledcirc})$ $\quad\quad$ $\mathcal{A}(\mathcal{P}_3^{\circledcirc})$

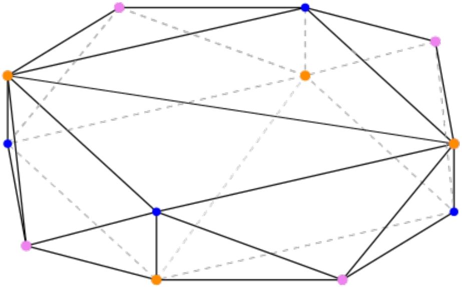
Example: non-orientable crown $\mathcal{M}_n$, for $n \geq 1$



$\mathcal{A}(\mathcal{M}_1) = \mathcal{A}_C(\mathcal{M}_1)$ $\mathcal{A}(\mathcal{M}_2)$ $\mathcal{A}(\mathcal{M}_3)$

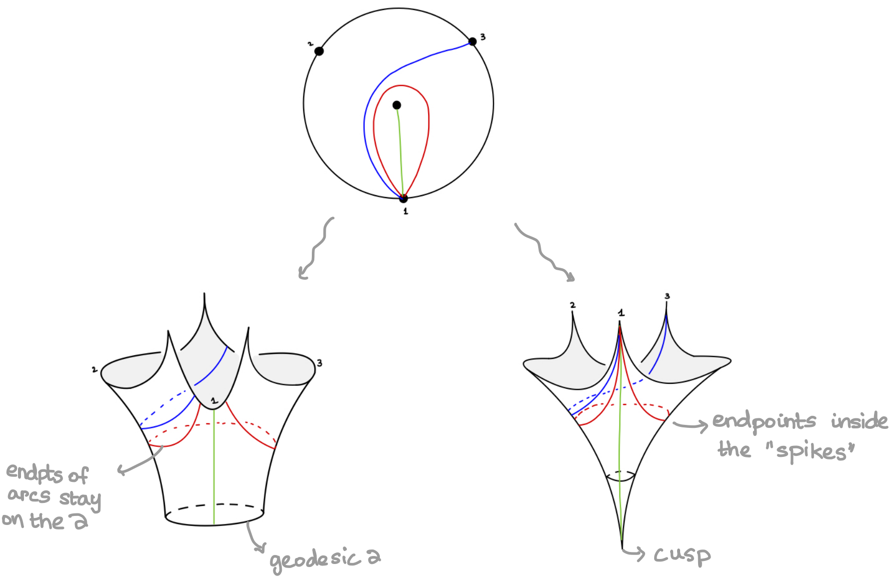
Example: Once-punctured polygon $\mathcal{P}_n^{\times}$, for $n \geq 2$





$$\mathcal{A}\left(\mathcal{P}_n^{\times}\right) \simeq \partial\mathcal{A}\left(\mathcal{P}_n^{\circledcirc}\right) \simeq \partial\mathcal{A}\left(\mathcal{M}_n\right)$$

Crowned hyperbolic surfaces



Classical result: For $n \geq 4$, the arc complex $\mathcal{A}(\mathcal{P}_n)$ of a polygon is a PL-sphere of dimension $n - 4$.

Theorem (Penner)

- *The arc complex $\mathcal{A}(\Pi_n)$ of an ideal polygon Π_n ($n \geq 4$) is a PL-sphere of dimension $n - 4$.*
- *The arc complex $\mathcal{A}(\Pi_n^\times)$ of an once-punctured ideal polygon $\Pi_n^\times$ ($n \geq 2$) is a PL-sphere of dimension $n - 2$.*

Penner gave a list of surfaces for which the *quotient* arc complex is a sphere.

Topology of the arc complex: generic case

- **Hatcher**: for S orientable, $\mathscr{A}(S)$ is contractible. (Hatcher flow, combinatorics)

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open dense subset of $\mathscr{A}(S)$

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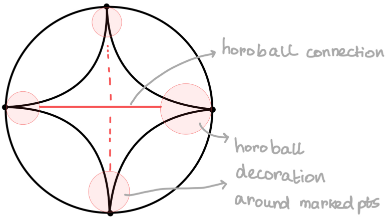
- **Fomin-Schapiro-Thurston**: for S orientable, the arc complex is a subset of the associated cluster complex. (combinatorics, hyperbolic geometry)

Decorated Teichmüller Theory

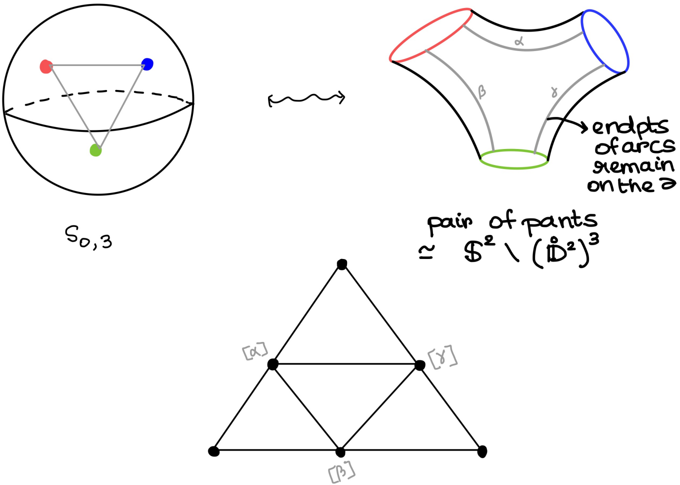
Introduced by Penner to study Teichmüller theory of surfaces decorated with horoballs using combinatorial methods.



- "lambda" lengths of h.c parametrise $\mathfrak{D}(S)$
- the a.c gives a cellular decomposition of $\mathfrak{D}(S)$
- lambda lengths behave like cluster variables

One particular application

Let $S_{0,3}$ be the three-holed sphere.



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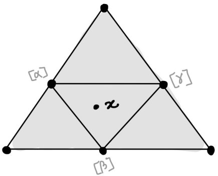
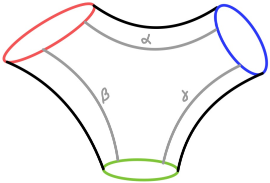
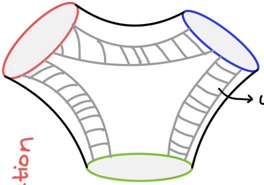


$x = c_1[\alpha] + c_2[\beta] + c_3[\gamma]$



add ⇝ strips



admissible deformation { lengthens every curve uniformly

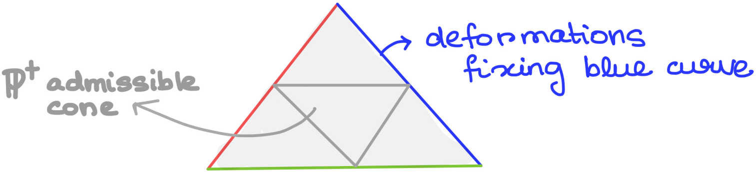
$m \in D(S) \longmapsto m' \in D(S)$

One particular application

Theorem (Danciger-Guéritaud-Kassel)

Let S be a compact hyperbolic surface with totally geodesic boundary. Let $m = ([\rho]) \in \mathfrak{D}(S)$ be a metric. Fix a choice of strip template $\{(\alpha_g, p_\alpha, w_\alpha)\}_{\alpha \in \mathcal{K}}$ with respect to m. Then the restriction of the projectivised infinitesimal strip map $\mathbb{P}f : \mathcal{PA}(S) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.

Here the admissible cone $\Lambda(m)$ consists of all infinitesimal deformations that uniformly lengthen every non-trivial closed geodesic.



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- **D–G–K**: The arc complex parametrises Margulis spacetimes.

Applications: decorated surfaces

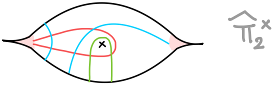
Let S be a decorated hyperbolic surface.

Aim: To parametrise *decorated* Margulis spacetimes using the arc complex of decorated hyperbolic surfaces.

Theorem (P.)

Let S be a finite-type decorated surface with a metric $m \in \mathfrak{D}(\widehat{\Pi_n})$. Then the projectivised strip map $\mathbb{P}f : \mathcal{PA}(S) \longrightarrow \mathbb{P}^+(T_m\mathfrak{D}(S))$ is a homeomorphism on its image $\mathbb{P}^+(\Lambda(m))$.

Here $\Lambda(m)$ is the set of deformations uniformly lengthening all horoball connections.

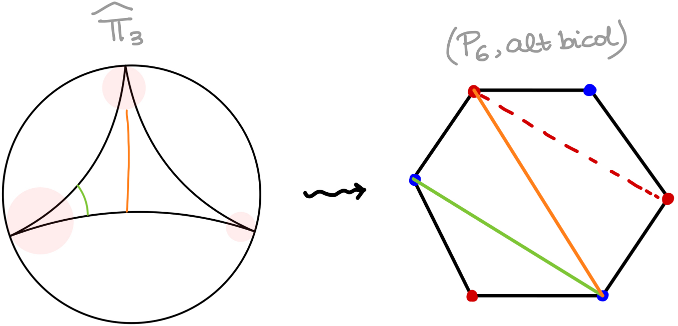


- 4 h.c.s
- 4 edge-to-edge arcs
- 4 m.p-to-edge arcs

} new arcs

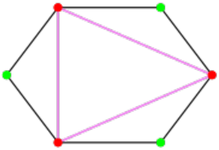
Decorated surfaces to bicolourings

Non-trivial bicolouring of marked points with blue and red: at least one R-R diagonal.

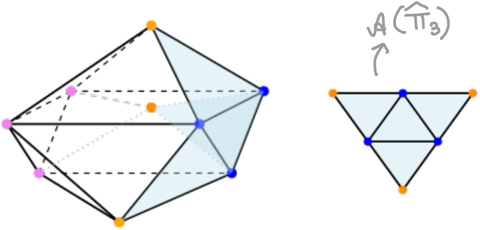


The subcomplex $\mathcal{Y}$ generated by $G - G$, $R - G$ diagonals is isomorphic to the arc complex of the decorated surface

Examples

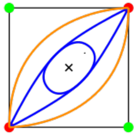


Rejected R-R diagonals



The subcomplex $\mathcal{Y}(\mathcal{P}_6)$

Examples



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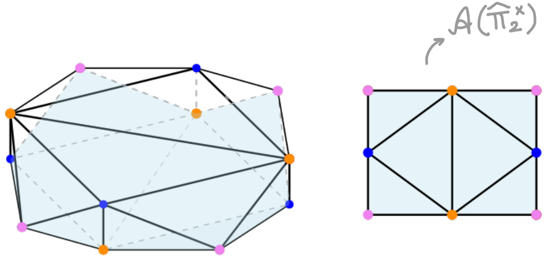


The subcomplex $\mathcal{Y}(\mathcal{P}_4^\times)$

Contributions

Theorem (P.)

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^{\times}$) be a polygon with a non-trivial bicolouring. Then the subcomplex $\mathcal{Y}(\mathcal{P}_n)$ (resp. $\mathcal{Y}(\mathcal{P}_n^{\times})$) is a shellable closed ball of dimension $2n-4$ (resp. $2n-2$).

Theorem (P.)

Let $S = \mathcal{P}_n^{\circledcirc}, \mathcal{M}_n$, where $n \geq 1$ with **any** *bicoloring. Then, the subcomplex $\mathcal{Y}(S)$ is a collapsible combinatorial ball of dimension $n-1$.*

In fact, we show something stronger...

Shellability

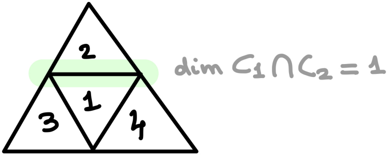
Let X be a pure simplicial complex of dimension n.

Definition

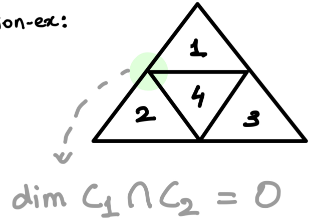
A shelling order is an ordering of the maximal simplices $\{C_1, C_2 \ldots\}$ of X such that $C_k \cap (\bigcup_{i=1}^{k-1} C_i)$ is a pure simplicial complex of dimension $n-1$.

A complex is called *shellable* if there exists a shelling order.

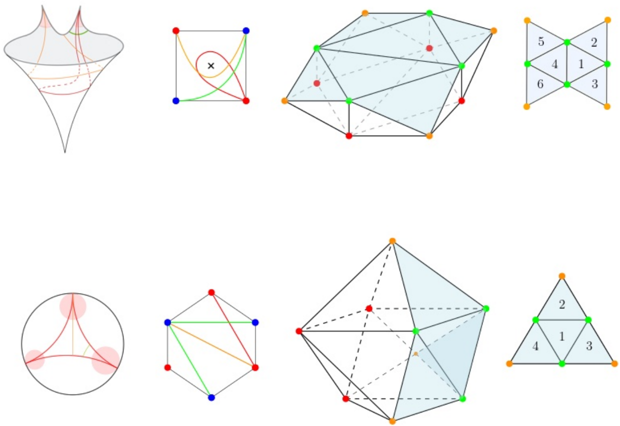
Ex:



Non-ex:



Shellability: Example



Danaraj-Klee: Any shellable pseudomanifold with boundary is PL-homeomorphic to a closed ball.

Shellability of the arc complex

Theorem (P.)

Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^{\times}$) be a polygon with a non-trivial bicolouring. Then the subcomplex $\mathcal{Y}(\mathcal{P}_n)$ (resp. $\mathcal{Y}(\mathcal{P}_n^{\times})$) is a shellable closed ball of dimension $2n-4$ (resp. $2n-2$).

Corollary

- *For $n \geq 3$, the arc complex of a decorated polygon is a closed ball of dimension $2n-4$.*
- *For $n \geq 1$, the arc complex of a decorated once-punctured polygon is a closed ball of dimension $2n-2$.*

Collapsibility

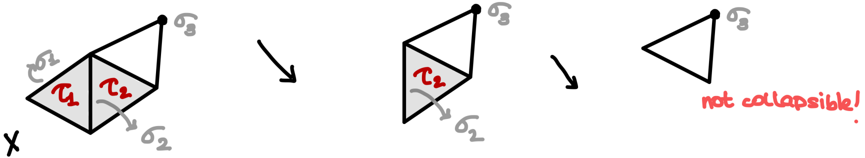
Let X be a finite simplicial complex.

Definition

Let σ, τ be two simplces of X such that

- $\sigma \subsetneq \tau$,
- τ is the unique maximal simplex containing σ.

Then X is said to be *collapsing onto* $X \setminus \{\sigma, \tau\}$. A complex X is said to be collapsible if there is a finite sequence of collapses ending at a 0-simplex.



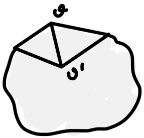
Strong collapsibility

Definition

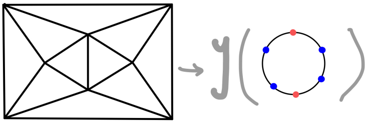
Let X be a finite simplicial complex. A 0-simplex $v \in X$ is *vertex-dominated* by another 0-simplex v' if $\mathrm{Link}(X, v) = v' \bowtie L$. In this case, X is said to *strongly collapse* onto $X \setminus v$.

A finite complex is *strongly collapsible* if there is a finite sequence of strong collapses terminating at a 0-simplex.

In dim 2 :

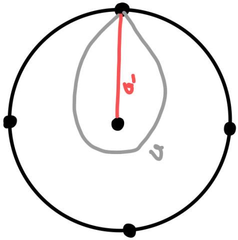


Non ex :

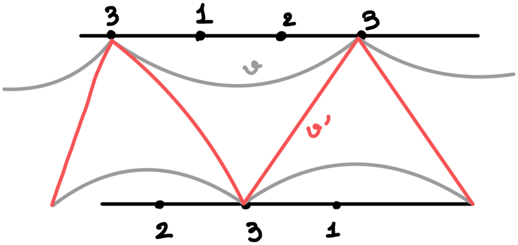


Strong collapsibility and the arcs

An arc v is vertex-dominated by an arc v' if any triangulation containing the arc v also contains v'.



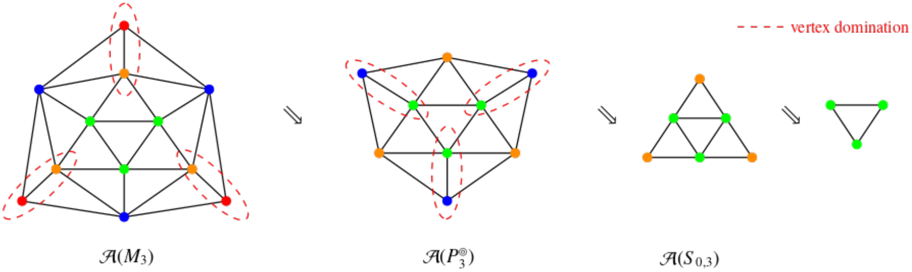
Orientable crown



Non orientable crown

Strong collapsibility: Illustration

A coincidence in dimension two...



Collapsibility of the arc complexes

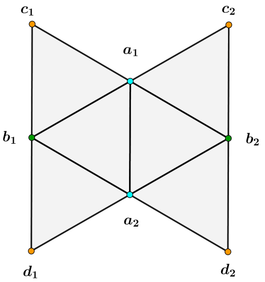
Theorem (P.)

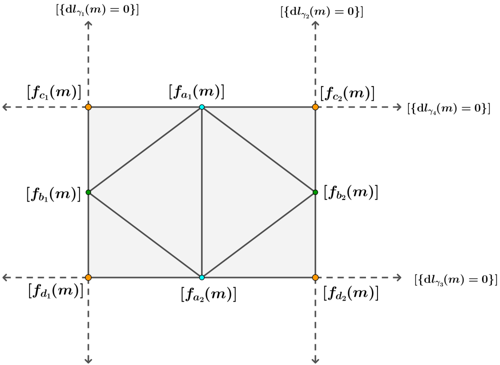
For $n \geq 1$,

- $\mathcal{A}(\mathcal{P}_n^{\circledcirc})$ *is strongly collapsible.*
- $\mathcal{A}(\mathcal{M}_n)$ *collapses onto* $\mathcal{A}_C(\mathcal{M}_n)$.
- $\mathcal{A}_C(\mathcal{M}_n)$ *is strongly collapsible.*
- $\mathcal{A}(\mathcal{M}_n)$ *is collapsible but not strongly collapsible.*

The statements remain true even if we put a bicolouring on the marked points.

Walls of the admissible cone





What next?

- Is $\mathcal{Y}(\mathcal{P}_n)$ or $\mathcal{Y}(\Pi_n^{\times})$ collapsible for any bicolouring?
- Collapsibility of infinite arc complexes: arborescence (Adiprasito–Funar).
- How to interpret collapsibility in terms of hyperbolic geometry?