On the High Complexity of Petri Nets ω-Languages

Olivier Finkel

Equipe de Logique Mathématique
Institut de Mathématiques de Jussieu - Paris Rive Gauche
CNRS and Université Paris 7

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Acceptance of infinite words

- The ω-regular languages accepted by Büchi automata and their extensions have been much studied and used for **specification and verification of non terminating systems**.

Complexity of ω-languages

The question naturally arises of the complexity of ω-languages accepted by various kinds of automata.

A way to study the complexity of ω-languages is to consider their topological complexity.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel Hierarchy

Below an arrow $\rightarrow$ represents a strict inclusion between Borel classes.

$$
\begin{array}{ccccccccccc}
 & & \Pi^0_1 & & & & & \Pi^0_\alpha & & \Pi^0_{\alpha+1} & \\
 & \nearrow & & \searrow & \nearrow & & \nearrow & & \searrow \quad \nearrow & & \\
\Delta^0_1 & & & & \Delta^0_2 & \cdots\ \cdots & \Delta^0_\alpha & & \Delta^0_{\alpha+1} & & \cdots \\
 & \searrow & & \nearrow & \searrow & & \searrow & & \nearrow \quad \searrow & & \\
 & & \Sigma^0_1 & & & & & \Sigma^0_\alpha & & \Sigma^0_{\alpha+1} & \\
\end{array}
$$

A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Topological complexity of 1-counter or context free ω-languages

Let $1 - CL_\omega$ be the class of real-time 1-counter ω-languages.

Let $\mathcal{C}$ be a class of ω-languages such that:

$$1 - CL_\omega \subseteq \mathcal{C} \subseteq \text{ Effective-}\Sigma^1_1.$$

(a) **(F. and Ressayre 2003)** There are some Σ^1_1-complete sets in **the class** $\mathcal{C}$**.**

(b) **(F. 2005)** The Borel hierarchy of **the class** $\mathcal{C}$ is equal to the Borel hierarchy of **the class Effective-**Σ^1_1**.**

(c) γ^1_2 is the supremum of the set of Borel ranks of ω-languages in **the class** $\mathcal{C}$**.**

(d) For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in **the class** $\mathcal{C}$**.**

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Wadge Reducibility

Definition (Wadge 1972)

For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, $L \leq_W L'$ iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.

L and L' are Wadge equivalent ($L \equiv_W L'$) iff $L \leq_W L'$ and $L' \leq_W L$. .

The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called Wadge degrees.

Intuitively $L \leq_W L'$ means that L is less complicated than L' because to check whether $x \in L$ it suffices to check whether $f(x) \in L'$ where f is a continuous function.

Wadge Degrees

Hence the Wadge degree of an ω-language is a measure of its topological complexity.

Wadge degrees were firstly studied by Wadge for Borel sets using Wadge games.

The Wadge hierarchy (on Borel sets) is a great refinement of the Borel hierarchy

Petri Nets are used for the description of distributed systems

In Automata Theory, Petri nets may be defined as (partially) blind multicounter automata.

First, one can distinguish between the places of a given Petri net by dividing them into the bounded ones (the number of tokens in such a place at any time is uniformly bounded) and the unbounded ones. Then each unbounded place may be seen as a partially blind counter, and the tokens in the bounded places determine the state of the partially blind multicounter automaton that is equivalent to the initial Petri net.

The infinite behavior of Petri nets was first studied by Valk 1983 and by Carstensen in the case of deterministic Petri nets 1988.

partially blind multicounter Büchi automata

A *k*-counter machine has *k* *counters*, each of which containing a non-negative integer.

The machine cannot test whether the content of a given partially blind counter is zero or not.

This means that if a transition of the machine is enabled when the content of a partially blind counter is zero then the same transition is also enabled when the content of the same counter is a non-zero integer.

We consider partially blind *k*-counter automata over infinite words with Büchi acceptance condition.

Using a simulation of:

– a given real time 1-counter (with zero-test) Büchi automaton $\mathcal{A}$ accepting ω-words x over the alphabet Σ

by

– a real time 4-blind-counter Büchi automaton $\mathcal{B}$ reading some special codes $h(x)$ of the words x, we prove here that ω-languages of *non-deterministic* Petri nets and effective analytic sets have the same topological complexity.

Topological complexity of Petri net ω-languages

Theorem (F. ArXiv 2017)

The Wadge hierarchy of Petri nets ω-languages (accepted by 4-blind-counter automata) is equal to the Wadge hierarchy of ω-languages of 1-counter automata, or of ω-languages of Turing machines.

We also get some non-Borel ω-languages of Petri nets, accepted by 4-blind-counter automata. However one blind-counter is actually sufficient:

Theorem (Skrzypczak 2018)

There exist some Σ_1^1-complete sets accepted by 1-blind-counter automata.

The Axiomatic System ZFC of Set Theory

The axioms of ZFC (Zermelo 1908, Fraenkel 1922) express some natural facts that we consider to hold in the universe of sets.

These axioms are first-order sentences in the logical language of set theory whose only non logical symbol is the membership binary relation symbol $\in$.

The *Axiom of Extensionality* states that two sets x and y are equal iff they have the same elements:

The *Powerset Axiom* states the existence of the set of subsets of a set x.

...

The Topological complexity of a Petri net ω-language depends on the models of ZFC

Theorem (F. 2009-2019)

There exists a 4-blind-counter Büchi automaton $\mathcal{A}$ such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system **ZFC**.

1. *There is a model V_1 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is an analytic but non Borel set.*
2. *There is a model V_2 of* **ZFC** *in which the ω-language $L(\mathcal{A})$ is a G_δ-set (i.e. Π^0_2-set).*

High undecidability of the topological complexity of a Petri net ω-language

Theorem (F. 2017)

Let $\alpha \geq 2$ be a countable ordinal. Then

1. $\{z \in N \mid L(P_z)$ *is in the Borel class* $\Sigma^0_\alpha\}$ *is* Π^1_2*-hard.*
2. $\{z \in N \mid L(P_z)$ *is in the Borel class* $\Pi^0_\alpha\}$ *is* Π^1_2*-hard.*
3. $\{z \in N \mid L(P_z)$ *is a Borel set*$\}$ *is* Π^1_2*-hard.*

High undecidability of the equivalence and the inclusion problems for ω-languages of Petri nets

Theorem (F. 2017)

The equivalence and the inclusion problems for ω-languages of Petri nets, or even for ω-languages of 4-blind-counter automata, are Π_2^1-complete.

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Concluding remarks

- In some sense our results show that the infinite behavior of Petri nets is closer to the infinite behavior of Turing machines than to the infinite behavior of finite automata.
- Except that the emptiness problem is decidable for ω-languages of Petri nets and Σ_1^1-complete for ω-languages of Turing machines.
- It remains open to determine the Borel and Wadge hierarchies of ω-languages accepted by automata with less than four blind counters.
- Do the highly undecidable problems about four blind counter automata remain highly undecidable for less than four counters ?

THANK YOU !

Wadge Games

Definition (Wadge 1972)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'.

The two players alternatively write letters a_n of X for Player 1 and b_n of Y for player 2.
Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps.

After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written $b \in Y^\omega$.
Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff :

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L')].$$

Wadge Games

Theorem (Wadge)

Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. Then $L \leq_W L'$ iff Player 2 has a winning strategy in the game $W(L, L')$.

By Martin's Theorem, the Wadge game $W(L, L')$, for Borel sets L and L', is determined: One of the two players has a winning strategy.

$\longrightarrow$ Study of the Wadge hierarchy on Borel sets.

Determinacy of Wadge games

Theorem (F. 2017)

The determinacy of Wadge games between two players in charge of ω-languages of Petri nets is equivalent to the (effective) analytic determinacy, which is known to be a large cardinal assumption, and thus is not provable in the axiomatic system ZFC.

The ordinal γ_2^1 may depend on set theoretic axioms

The ordinal γ_2^1 is the least basis for subsets of ω_1 which are Π_2^1 in the codes.

It is the least ordinal such that whenever $X \subseteq \omega_1$, $X \neq \emptyset$, and $\hat{X} \subseteq WO$ is Π_2^1, there is $\beta \in X$ such that $\beta < \gamma_2^1$.

The least ordinal which is not a Δ_n^1-ordinal is denoted δ_n^1.

Theorem (Kechris, Marker and Sami 1989)

- (ZFC) $\delta_2^1 < \gamma_2^1$
- (V = L) $\gamma_2^1 = \delta_3^1$
- (Π_1^1-Determinacy) $\gamma_2^1 < \delta_3^1$

Are there effective analytic sets of every Borel rank $\alpha < \gamma_2^1$?

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- **ω-regular languages** accepted by **deterministic Büchi automata** are Π_2^0-sets.
- **ω-regular languages** are boolean combinations of Π_2^0-sets hence Δ_3^0-sets.

deterministic Turing machines

- **ω-languages** accepted by **deterministic Büchi Turing machines** are Π_2^0-sets.
- **ω-languages** accepted by **deterministic Muller Turing machines** are boolean combinations of Π_2^0-sets hence Δ_3^0-sets.

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