# Schützenberger's factorization on the (completed) Hopf algebra of $q-$stuffle product

C. Bui[♭], G. H. E. Duchamp[♯], V. Hoang Ngoc Minh[♢,♯]

[♭]Hué University - College of sciences, 77 - Nguyen Hue street - Hué city, Viêt Nam
[♯]Institut Galilée, LIPN - UMR 7030, CNRS - Université Paris 13, F-93430 Villetaneuse, France,
[♢]Université Lille II, 1, Place Déliot, 59024 Lille, France

**Abstract.** In order to extend the Schützenberger's factorization, the combinatorial Hopf algebra of the $q$-stuffles product is developed systematically in a parallel way with that of the shuffle product and and in emphasizing the Lie elements as studied by Ree. In particular, we will give here an effective construction of pair of bases in duality. [ 21-01-2015 16:34]

**Keywords** : Shuffle, Lyndon words, Lie elements, transcendence bases.

## 1 Introduction

Schützenberger's factorization [23, 22] has been introduced and plays a central role in the renormalization [18] of associators[1] which are formal power series in non commutative variables [1]. The coefficients of these power series are polynomial at positive integral multi-indices of Riemann's zêta function[2] [13, 26] and they satisfy quadratic relations [5] which can be explained through the Lyndon words [2, 14, 6, 20]. These quadratic relations can be obtained by identification of the local coordinates, in infinite dimension, on a bridge equation connecting the Cauchy and Hadamard algebras of the polylogarithmic functions and using the factorizations, by Lyndon words, of the non commutative generating series of polylogarithms [16] and of harmonic sums [18]. This bridge equation is mainly a consequence of the double isomorphy between these algebraic structures to respectively the shuffle [16] and quasi-shuffle (or stuffle) [17] algebras both admitting the Lyndon words as a transcendence basis[3] [20, 15].
In order to better understand the mechanisms of the shuffle product and to obtain algorithms on quasi-shuffle products, we will examine, in the section below, the commutative $q$-stuffle product interpolating between the shuffle [21], quasi-shuffle (or stuffle [15]) and minus-stuffle products [7, 8], obtained for[4] $q = 0, 1$ and $-1$ respectively. We will extend the Schützenberger's factorization by developping the combinatorial Hopf algebra of this product in a parallel way with that of the shuffle and in emphasizing the Lie elements studied by Ree [21]. In particular, we will give an effective construction (implemented in Maple [4]) of pair of bases in duality (see Propositions 4 and 6).
This construction uses essentially an adapted version of the Eulerian projector and its adjoint [22] in order to obtain the primitive elements of the $q$-stuffle Hopf algebra (see Definition 1). They are obtained thanks to the computation of the logarithm of the diagonal series (see Proposition 1). This study completes the treatement for the stuffle [18] and boils down to the shuffle case for $q = 0$ [22].
Let us remark that it is quite different from other studies [9, 19] concerning non commutative $q$-shuffle products interpolating between the concatenation and shuffle products, for $q = 0$ and 1 respectively and using the $q$-deformation theory of non commutative symmetric functions[5] [9].

1 The associators were introduced in quantum field theory by Drinfel'd [10, 11] and the universal Drinfel'd associator, *i.e.* $\Phi_{KZ}$, was obtained, in [13], with explicit coefficients which are polyzêtas and regularized polyzêtas (see [18] for the computation of the other associators involving only convergent polyzêtas as local coordinates, and for three algorithmical process to regularize the divergent polyzêtas).
2 These values are usually abbreviated MZV's by Zagier [26] and are also called polyzêtas by Cartier [5].
3 Our method applies also to any other transcendence basis built by duality from PBW, see below.
4 In [7], the letter $\lambda$ is used instead of $q$.
5 Recall also that the algebra of non commutative symmetric functions, denoted by **Sym** is the Solomon descent algebra [24] and it is dual to the algebra of quasi-symmetric functions, denoted by **QSym** which is isomorphic to the quasi-shuffle algebra [15].
Thus our construction of pair of bases in duality are also suitable for **Sym** and **QSym** (and their deformations, provided they remain graded connected cocommutative Hopf algebras).

## 2 $q$-deformed stuffle

### 2.1 Results for the $q$-deformed stuffle

Let $\mathbf{k}$ be a unitary $\mathbb{Q}$-algebra containing $q$. Let also $Y = \{y_s\}_{s\geq 1}$ be an alphabet with the total order

$$y_1 > y_2 > \cdots . \tag{1}$$

One defines the $q$-stuffle, by a recursion or by its dual co-product $\Delta_{\uplus_q}$, as follows. For any $y_s, y_t \in Y$ and for any $u, v \in Y^*$,

$$u \uplus_q 1_{Y^*} = 1_{Y^*} \uplus_q u = u \text{ and } y_s u \uplus_q y_t v = y_s(u \uplus_q y_t v) + y_t(y_s u \uplus_q v) + q y_{s+t}(u \uplus_q v), \tag{2}$$

$$\Delta_{\uplus_q}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*} \text{ and } \Delta_{\uplus_q}(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q \sum_{s_1+s_2=s} y_{s_1} \otimes y_{s_2}. \tag{3}$$

This product is commutative, associative and unital (the neutral being the empty word $1_{Y^*}$). With the co-unit defined by, for any $P \in \mathbf{k}\langle Y \rangle$,

$$\epsilon(P) = \langle P \mid 1_{Y^*} \rangle \tag{4}$$

one gets $\mathcal{H}_{\uplus_q} = (\mathbf{k}\langle Y \rangle, \texttt{conc}, 1_{Y^*}, \Delta_{\uplus_q}, \epsilon)$ and $\mathcal{H}^{\vee}_{\uplus_q} = (\mathbf{k}\langle Y \rangle, \uplus_q, 1_{Y^*}, \Delta_{\texttt{conc}}, \epsilon)$ which are mutually dual bialgebras and, in fact, Hopf algebras because they are $\mathbb{N}$-graded by the weight, defined by

$$\forall w = y_{i_1} \ldots y_{i_r} \in Y^+,\ (w) = i_1 + \ldots + i_r. \tag{5}$$

**Lemma 1 (Friedrichs criterium).** *Let $S \in \mathbf{k}\langle\langle Y \rangle\rangle$ (for (2), we suppose in addition that $\langle S \mid 1_{Y^*} \rangle = 1$). Then,*

1. *$S$ is* primitive, *i.e. $\Delta_{\uplus_q} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S$, if and only if, for any $u, v \in Y^+$, $\langle S \mid u \uplus_q v \rangle = 0$.*
2. *$S$ is* group-like, *i.e. $\Delta_{\uplus_q} S = S \otimes S$, if and only if, for any $u, v \in Y^+$, $\langle S \mid u \uplus_q v \rangle = \langle S \mid u \rangle \langle S \mid v \rangle$.*

*Proof.* The expected equivalence is due respectively to the following facts

$$\Delta_{\uplus_q} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S - \langle S \mid 1_{Y^*} \otimes 1_{Y^*} \rangle 1_{Y^*} \otimes 1_{Y^*} + \sum_{u,v \in Y^+} \langle S \mid u \uplus_q v \rangle u \otimes v,$$

$$\Delta_{\uplus_q} S = \sum_{u,v \in Y^*} \langle S \mid u \uplus_q v \rangle u \otimes v \quad \text{and} \quad S \otimes S = \sum_{u,v \in Y^*} \langle S \mid u \rangle \langle S \mid v \rangle u \otimes v.$$

**Lemma 2.** *Let $S \in \mathbf{k}\langle\langle Y \rangle\rangle$ such that $\langle S \mid 1_{Y^*} \rangle = 1$. Then, for the co-product $\Delta_{\uplus_q}$, $S$ is group-like if and only if $\log S$ is primitive.*

*Proof.* Since $\Delta_{\uplus_q}$ and the maps $T \mapsto T \otimes 1_{Y^*}, T \mapsto 1_{Y^*} \otimes T$ are continous homomorphisms then if $\log S$ is primitve then, by Lemma 1, $\Delta_{\uplus_q}(\log S) = \log S \otimes 1_{Y^*} + 1_{Y^*} \otimes \log S$. Since $\log S \otimes 1_{Y^*}, 1_{Y^*} \otimes \log S$ commute then

$$\begin{aligned}
\Delta_{\uplus_q} S &= \Delta_{\uplus_q}(\exp(\log S)) \\
&= \exp(\Delta_{\uplus_q}(\log S)) \\
&= \exp(\log S \otimes 1_{Y^*}) \exp(1_{Y^*} \otimes \log S) \\
&= (\exp(\log S) \otimes 1_{Y^*})(1_{Y^*} \otimes \exp(\log S)) \\
&= S \otimes S.
\end{aligned}$$

This means $S$ is group-like. The converse can be obtained in the same way.

**Lemma 3.** *Let $S_1, \ldots, S_n$ be* proper *formal power series in $\mathbf{k}\langle\langle Y \rangle\rangle$. Let $P_1, \ldots, P_m$ be primitive elements in $\mathbf{k}\langle Y \rangle$, for the co-product $\Delta_{\uplus}$.*

1. *If $n > m$ then $\langle S_1 \uplus_q \ldots \uplus_q S_n \mid P_1 \ldots P_m \rangle = 0$.*
2. *If $n = m$ then*

$$\langle S_1 \uplus_q \ldots \uplus_q S_n \mid P_1 \ldots P_n \rangle = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \langle S_i \mid P_{\sigma(i)} \rangle.$$

*3. If $n < m$ then, by considering the language $\mathcal{M}$ over the new alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$*

$$\mathcal{M} = \{w \in \mathcal{A}^* | w = a_{j_1} \ldots a_{j_{|w|}}, j_1 < \ldots < j_{|w|}, |w| \geq 1\}$$

*and the morphism $\mu : \mathbb{Q}\langle\mathcal{A}\rangle \longrightarrow \mathbf{k}\langle Y\rangle$ given by, for any $i = 1, \ldots, m, \mu(a_i) = P_i$, one has :*

$$\langle S_1 \uplus_q \ldots \uplus_q S_n \mid P_1 \ldots P_m\rangle = \sum_{\substack{w_1, \ldots, w_m \in \mathcal{M} \\ \mathrm{supp}(w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_m) \ni a_1 \ldots a_m}} \prod_{i=1}^{n} \langle S_i \mid \mu(w_i)\rangle.$$

*Proof.* On the one hand, since the $P_i$'s are primitive then

$$\Delta_{\uplus_q}^{(n-1)}(P_i) = \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q}.$$

On the other hand,

$$\Delta_{\uplus_q}^{(n-1)}(P_1 \ldots P_m) = \Delta_{\uplus_q}^{(n-1)}(P_1) \ldots \Delta_{\uplus_q}^{(n-1)}(P_m)$$

and

$$\langle S_1 \uplus_q \ldots \uplus_q S_n \mid P_1 \ldots P_m\rangle = \langle S_1 \otimes \ldots \otimes S_n \mid \Delta_{\uplus_q}^{(n-1)}(P_1 \ldots P_m)\rangle.$$

Hence,

$$\langle S_1 \uplus_q \ldots \uplus_q S_n \mid P_1 \ldots P_m\rangle = \langle \bigotimes_{i=1}^{n} S_i \mid \prod_{i=1}^{m} \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q}\rangle.$$

1. For $n > m$, by expanding $\Delta_{\uplus_q}^{(n-1)}(P_1) \ldots \Delta_{\uplus_q}^{(n-1)}(P_m)$, one obtains a sum of tensors containing at least one factor equal to $1_{Y^*}$. For $i = 1, .., n$, $S_i$ is proper and the result follows immediately.
2. For $n = m$, since

$$\prod_{i=1}^{n} \Delta_{\uplus_q}^{(n-1)}(P_i) = \sum_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^{n} P_{\sigma(i)} + Q,$$

where $Q$ is sum of tensors containing at least one factor equal to 1 and the $S_i$'s are proper then $\langle S_1 \otimes \ldots \otimes S_n \mid Q\rangle = 0$. Thus, the result follows.
3. For $n < m$, since, for $i = 1, .., n$, the power series $S_i$ is proper then the expected result follows by expanding the product

$$\prod_{i=1}^{m} \Delta_{\uplus_q}^{(n-1)}(P_i) = \prod_{i=1}^{m} \sum_{p+q=n-1} 1_{Y^*}^{\otimes p} \otimes P_i \otimes 1_{Y^*}^{\otimes q}.$$

**Definition 1.** *Let $\pi_1$ and $\check{\pi}_1$ be the mutually adjoint projectors degree-preserving linear endomorphisms of $\mathbf{k}\langle Y\rangle$ given by, for any $w \in Y^+$,*

$$\pi_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \uplus_q \ldots \uplus_q u_k\rangle u_1 \ldots u_k,$$

$$\check{\pi}_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \mid u_1 \ldots u_k\rangle u_1 \uplus_q \ldots \uplus_q u_k.$$

*In particular, for any $y_k \in Y$, the polynomials $\pi_1(y_k)$ and $\check{\pi}_1(y_k)$ are given by*

$$\pi_1(y_k) = y_k + \sum_{l \geq 2} \frac{(-q)^{l-1}}{l} \sum_{\substack{j_1, \ldots, j_l \geq 1 \\ j_1 + \ldots + j_l = k}} y_{j_1} \ldots y_{j_l} \text{ and } \check{\pi}_1(y_k) = y_k.$$

**Proposition 1.** *Let $\mathcal{D}_Y$ be the diagonal series over $Y$ :*

$$\mathcal{D}_Y = \sum_{w \in Y^*} w \otimes w.$$

*Then*

*1.* $\log\mathcal{D}_Y = \sum_{w\in Y^+} w\otimes\pi_1(w) = \sum_{w\in Y^+}\check{\pi}_1(w)\otimes w.$

*2. For any $w\in Y^*$, we have*

$$w=\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1 ⧢_q\ldots ⧢_q u_k\rangle\pi_1(u_1)\ldots\pi_1(u_k)$$
$$=\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1\ldots u_k\rangle\check{\pi}_1(u_1) ⧢_q\ldots ⧢_q\check{\pi}_1(u_k).$$

*In particular, for any $y_s\in Y$, we have*

$$y_s=\sum_{k\geq 1}\frac{q^{k-1}}{k!}\sum_{s'_1+\cdots+s'_k=s}\pi_1(y_{s'_1})\ldots\pi_1(y_{s'_k})\text{ and } y_s=\check{\pi}_1(y_s).$$

*Proof.* 1. Expanding by different ways the logarithm, it follows the results :

$$\log\mathcal{D}_Y=\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\Big(\sum_{w\in Y^+}w\otimes w\Big)^k$$
$$=\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}(u_1 ⧢_q\ldots ⧢_q u_k)\otimes u_1\ldots u_k$$
$$=\sum_{w\in Y^+}w\otimes\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1 ⧢_q\ldots ⧢_q u_k\rangle u_1\ldots u_k.$$
$$\log\mathcal{D}_Y=\sum_{w\in Y^+}\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1\ldots u_k\rangle u_1 ⧢_q\ldots ⧢_q u_k\otimes w.$$

2. Since $\mathcal{D}_Y=\exp(\log(\mathcal{D}_Y))$ then, by the previous results, one has separately,

$$\mathcal{D}_Y=\sum_{k\geq 0}\frac{1}{k!}\Big(\sum_{w\in Y^+}w\otimes\pi_1(w)\Big)^k$$
$$=\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}(u_1 ⧢_q\ldots ⧢_q u_k)\otimes(\pi_1(u_1)\ldots\pi_1(u_k))$$
$$=\sum_{w\in Y^+}w\otimes\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1 ⧢_q\ldots ⧢_q u_k\rangle\pi_1(u_1)\ldots\pi_1(u_k).$$
$$\mathcal{D}_Y=\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}(\check{\pi}_1(u_1) ⧢_q\ldots ⧢_q\check{\pi}_1(u_k))\otimes(u_1\ldots u_k)$$
$$=\sum_{w\in Y^+}\sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots,u_k\in Y^+}\langle w\mid u_1\ldots u_k\rangle\check{\pi}_1(u_1) ⧢_q\ldots ⧢_q\check{\pi}_1(u_k)\otimes w.$$

It follows then the expected result.

**Lemma 4.** *For any $w\in Y^+$, one has $\Delta_{⧢_q}\pi_1(w)=\pi_1(w)\otimes 1_{Y^*}+1_{Y^*}\otimes\pi_1(w)$.*

*Proof.* Let $\alpha$ be the alphabet duplication isomorphism defined by, for any $\bar{y}\in\bar{Y}$, $\bar{y}=\alpha(y)$
Applying the tensor product of algebra isomorphisms $\alpha\otimes\mathrm{Id}$ to the diagonal series $\mathcal{D}_Y$, we obtain, by Lemma 1, a group-like element and then applying the logarithm of this element (or equivalently, applying $\alpha\otimes\pi_1$ to $\mathcal{D}_Y$) we obtain $\mathcal{S}$ which is, by Lemma 2, a primitive element :

$$(\alpha\otimes\mathrm{Id})\mathcal{D}_Y=\sum_{w\in Y^*}\alpha(w)\,w\text{ and }\mathcal{S}=(\alpha\otimes\pi_1)\mathcal{D}_Y=\sum_{w\in Y^*}\alpha(w)\,\pi_1(w).$$

The two members of the identity $\Delta_{⧢_q}\mathcal{S}=\mathcal{S}\otimes 1_{Y^*}+1_{Y^*}\otimes\mathcal{S}$ give respectively

$$\sum_{w\in Y^*}\alpha(w)\,\Delta_{⧢_q}\pi_1(w)\text{ and }\sum_{w\in Y^*}\alpha(w)\,\pi_1(w)\otimes 1_{Y^*}+\sum_{w\in Y^*}\alpha(w)\,1_{Y^*}\otimes\pi_1(w).$$

Since $\{w\}_{w\in\bar{Y}^*}$ is a basis for $\mathbb{Q}\langle\bar{Y}\rangle$ then identifying the coefficients in the previous expressions, we get $\Delta_{⧢_q}\pi_1(w)=\pi_1(w)\otimes 1_{Y^*}+1_{Y^*}\otimes\pi_1(w)$ meaning that $\pi_1(w)$ is primitive.

### 2.2 Pair of bases in duality on $q$-deformed stuffle algebra

Let $\mathcal{P}=\{P\in\mathbb{Q}\langle Y\rangle \mid \Delta_{\text{⧢}_q}P=P\otimes 1_{Y^*}+1_{Y^*}\otimes P\}$ be the set of primitive polynomials [3]. Since, in virtue of Lemma 4., $\mathrm{Im}(\pi_1)\subseteq\mathcal{P}$, we can state the following

**Definition 2.** *Let $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ be the family of $\mathcal{P}$ and*[6] $\mathbf{k}\langle Y\rangle$ *obtained as follows*

$$\begin{array}{rcl} \Pi_{y_k} &=& \pi_1(y_k) \quad \text{for } k\geq 1,\\ \Pi_l &=& [\Pi_s,\Pi_r] \quad \text{for } l\in\mathcal{L}ynX,\ \text{standard factorization of } l=(s,r),\\ \Pi_w &=& \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k} \text{ for } w=l_1^{i_1}\ldots l_k^{i_k}, l_1>\ldots>l_k, l_1\ldots,l_k\in\mathcal{L}ynY. \end{array}$$

**Proposition 2.** *1. For $l\in\mathcal{L}ynY$, the polynomial $\Pi_l$ is upper triangular and homogeneous in weight :*

$$\Pi_l = l+\sum_{v>l,(v)=(l)} c_v v,$$

*where for any $w\in Y^+$, $(w)$ denotes the weight of $w$ with $(y_k)=\deg(y_k)=k$.*

*2. The family $\{\Pi_w\}_{w\in Y^*}$ is upper triangular and homogeneous in weight :*

$$\Pi_w = w+\sum_{v>w,(v)=(w)} c_v v.$$

*Proof.* 1. Let us prove it by induction on the length of $l$ : the result is immediate for $l\in Y$. The result is suppose verified for any $l\in\mathcal{L}ynY\cap Y^k$ and $0\leq k\leq N$. At $N+1$, by the standard factorization $(l_1,l_2)$ of $l$, one has $\Pi_l=[\Pi_{l_1},\Pi_{l_2}]$ and $l_2l_1>l_1l_2=l$. By induction hypothesis,

$$\Pi_{l_1}=l_1+\sum_{v>l_1,(v)=(l_1)} c_v v \text{ and } \Pi_{l_2}=l_2+\sum_{u>l_2,(u)=(l_2)} d_u u,$$
$$\Rightarrow\ \Pi_l=l+\sum_{w>l,(w)=(l)} e_w w,$$

getting $e_w$'s from $c_v$'s and $d_u$'s.

2. Let $w=l_1\ldots l_k$, with $l_1\geq\ldots\geq l_k$ and $l_1,\ldots,l_k\in\mathcal{L}ynY$. One has

$$\Pi_{l_i}=l_i+\sum_{v>l_i,(v)=(l_i)} c_{i,v}v \text{ and } \Pi_w=l_1\ldots l_k+\sum_{u>w,(u)=(w)} d_u u,$$

where the $d_u$'s are obtained from the $c_{i,v}$'s. Hence, the family $\{\Pi_w\}_{w\in Y^*}$ is upper triangular and homogeneous in weight. As the grading by weight is in finite dimensions, this family is a basis of $\mathbf{k}\langle Y\rangle$.

**Definition 3.** *Let $\{\Sigma_w\}_{w\in Y^*}$ be the family of the quasi-shuffle algebra (viewed as a $\mathbb{Q}$-module) obtained by duality with $\{\Pi_w\}_{w\in Y^*}$ :*

$$\forall u,v\in Y^*,\quad \langle\Sigma_v\mid\Pi_u\rangle=\delta_{u,v}.$$

**Proposition 3.** *The family $\{\Sigma_w\}_{w\in Y^*}$ is lower triangular and homogeneous in weight. In other words,*

$$\Sigma_w = w+\sum_{v<w,(v)=(w)} d_v v.$$

*Proof.* By duality with $\{\Pi_w\}_{w\in Y^*}$ (see Proposition 2), we get the expected result.

**Theorem 1.** *1. The family $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ forms a basis of $\mathcal{P}$.*

*2. The family $\{\Pi_w\}_{w\in Y^*}$ forms a basis of $\mathbf{k}\langle Y\rangle$.*

*3. The family $\{\Sigma_w\}_{w\in Y^*}$ generate freely the quasi-shuffle algebra.*

*4. The family $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ forms a transcendence basis of $(\mathbf{k}\langle Y\rangle, \text{⧢}_q)$.*

[6] Due to the fact this Hopf algebra is cocommutative and graded, then by the theorem of CQMM, $\mathbf{k}\langle Y\rangle\simeq\mathcal{U}(\mathcal{P})$.

*Proof.* The family $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ of primitive upper triangular homogeneous in weight polynomials is free and the first result follows. The second is a direct consequence of the Poincaré-Birkhoff-Witt theorem. By the Cartier-Quillen-Milnor-Moore theorem, we get the third one and the last one is obtained as consequence of the constructions of $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ and $\{\Sigma_w\}_{w\in Y^*}$.

To decompose any letter $y_s \in Y$ in the basis $\{\Pi_w\}_{w\in Y^*}$, one can use its expression in Proposition 1. Now, using the mutually adjoint projectors $\pi_1$ and $\check{\pi}_1$ given in Definition 1 and are determinded by Proposition 1, let us clarify the basis $\{\Sigma_w\}_{w\in Y^*}$ and then the transcendence basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ of the quasi-shuffle algebra $(\mathbf{k}\langle Y\rangle, ⧢_q, 1_{Y^*})$ as follows

**Proposition 4.** *We have*

1. *For* $w = 1_{Y^*}$, $\Sigma_w = 1$.
2. *For any* $w = l_1^{i_1}\ldots l_k^{i_k}$, *with* $l_1,\ldots,l_k \in \mathcal{L}ynY$ *and* $l_1 > \ldots > l_k$,

$$\Sigma_w = \frac{\Sigma_{l_1}^{⧢_q i_1} ⧢_q \ldots ⧢_q \Sigma_{l_k}^{⧢_q i_k}}{i_1!\ldots i_k!}.$$

3. *For any* $y \in Y$,

$$\Sigma_y = y = \check{\pi}_1(y).$$

*Proof.* 1. Since $\Pi_{1_{Y^*}} = 1$ then $\Sigma_{1_{Y^*}} = 1$.

2. Let $u = u_1\ldots u_n = l_1^{i_1}\ldots l_k^{i_k}, v = v_1\ldots v_m = h_1^{j_1}\ldots h_p^{j_p}$ with $l_1\ldots,l_k,h_1,\ldots,h_p,u_1,\ldots,u_n$ and $v_1,\ldots,v_m \in \mathcal{L}ynY, l_1>\ldots>l_k, h_1>\ldots>h_p, u_1\geq\ldots\geq u_n$ and $v_1\geq\ldots\geq v_m$ and $i_1+\ldots+i_k = n$, $j_1+\ldots+j_p = m$. Hence, if $m\geq 2$ (resp. $n\geq 2$) then $v\notin\mathcal{L}ynY$ (resp. $u\notin\mathcal{L}ynY$).
Since

$$\langle \Sigma_{u_1} ⧢_q \ldots ⧢_q \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{v_i}\rangle = \langle \Sigma_{u_1}\otimes\ldots\otimes\Sigma_{u_n} \mid \Delta_{⧢_q}^{(n-1)}(\Pi_{v_1}\ldots\Pi_{v_m})\rangle$$

then many cases occur :
(a) Case $n > m$. By Lemma 3(1), one has

$$\langle \Sigma_{u_1} ⧢_q \ldots ⧢_q \Sigma_{u_n} \mid \Pi_{v_1}\ldots\Pi_{v_m}\rangle = 0.$$

(b) Case $n = m$. By Lemma 3(2), one has

$$\begin{aligned}\langle \Sigma_{u_1} ⧢_q \ldots ⧢_q \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{v_i}\rangle &= \sum_{\sigma\in\Sigma_n}\prod_{i=1}^n\langle\Sigma_{u_i}\mid\Pi_{v_{\sigma(i)}}\rangle\\ &= \sum_{\sigma\in\Sigma_n}\prod_{i=1}^n\delta_{u_i,v_{\sigma(i)}}.\end{aligned}$$

Thus, if $u \neq v$ then $(u_1,\ldots,u_n)\neq(v_1,\ldots,v_n)$ then the second member is vanishing else, *i.e.* $u = v$, the second member equals 1 because the factorization by Lyndon words is unique.
(c) Case $n < m$. By Lemma 3(3), let us consider the following language over the new alphabet $\mathcal{A} := \{a_1,\ldots,a_m\}$ :

$$\mathcal{M} = \{w\in\mathcal{A}^* | w = a_{j_1}\ldots a_{j_{|w|}}, j_1<\ldots<j_{|w|}, |w|\geq 1\},$$

and the morphism $\mu : \mathbb{Q}\langle\mathcal{A}\rangle \longrightarrow \mathbf{k}\langle Y\rangle$ given by, for any $i = 1,\ldots,m, \mu(a_i) = \Pi_{v_i}$. We get :

$$\begin{aligned}\langle \Sigma_{u_1} ⧢_q \ldots ⧢_q \Sigma_{u_n} \mid \prod_{i=1}^n \Pi_{v_i}\rangle &= \sum_{\substack{w_1,\ldots,w_n\in\mathcal{M}\\ \mathrm{supp}(w_1⧢\cdots⧢ w_n)\ni a_1\ldots a_m}}\prod_{i=1}^n\langle\Sigma_{u_i}\mid\mu(w_i)\rangle\\ &= 0.\end{aligned}$$

Because in the right side of the first equality, on the one hand, there is at least one $w_i, |w_i|\geq 2$, corresponding to $\mu(w_i) = \Pi_{v_{j_1}}\ldots\Pi_{v_{j_{|w_i|}}}$ such that $v_{j_1}\geq\ldots\geq v_{j_{|w_i|}}$ and on the other hand, $\nu_i := v_{j_1}\ldots v_{j_{|w_i|}} \notin \mathcal{L}ynY$ and $u_i\in\mathcal{L}ynY$.

By consequent,

$$\begin{aligned}\langle \Sigma_u \mid \Pi_v\rangle &= \frac{1}{i_1!\ldots i_k!}\langle \Sigma_{l_1}^{\uplus_q i_1} \uplus_q \ldots \uplus_q \Sigma_{l_k}^{\uplus_q i_k} \mid \Pi_{h_1}^{j_1}\ldots \Pi_{h_p}^{j_p}\rangle\\ &= \delta_{u,v}.\end{aligned}$$

3. For any $y \in Y$, by Proposition 3, $\Sigma_y = y = \check{\pi}_1(y)$. The directe computation prove that, for any $w \in Y^*$ and for any $y \in Y$, one has $\langle \Pi_w \mid \Sigma_y\rangle = \delta_{w,y}$.

**Proposition 5.** *1. For $w \in Y^+$, the polynomial $\Sigma_w$ is proper and homogeneous of degree $(w)$, for $\deg(y_i) = i$, and with rational positive coefficients.*

*2.* $\mathcal{D}_Y = \sum_{w\in Y^*} \Sigma_w \otimes \Pi_w = \prod^{\searrow}_{l\in\mathcal{L}ynY} \exp(\Sigma_l \otimes \Pi_l).$

*3. The family $\mathcal{L}ynY$ forms a transcendence basis of the quasi-shuffle algebra and the family of proper polynomials of rational positive coefficients defined by, for any $w = l_1^{i_1}\ldots l_k^{i_k}$ with $l_1 > \ldots > l_k$ and $l_1,\ldots,l_k \in \mathcal{L}ynY$,*

$$\chi_w = \frac{1}{i_1!\ldots i_k!} l_1^{\uplus_q i_1} \uplus_q \ldots \uplus_q l_k^{\uplus_q i_k}$$

*forms a basis of the quasi-shuffle algebra.*

*4. Let $\{\xi_w\}_{w\in Y^*}$ be the basis of the envelopping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ obtained by duality with $\{\chi_w\}_{w\in Y^*}$ :*

$$\forall u, v \in Y^*, \quad \langle \chi_v \mid \xi_u\rangle = \delta_{u,v}.$$

*Then the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ forms a basis of the free Lie algebra $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$.*

*Proof.* 1. The proof can be done by induction on the length of $w$ using the fact that the product $\uplus_q$ conserve the property, l'homogenity and rational positivity of the coefficients.

2. Expressing $w$ in the basis $\{\Sigma_w\}_{w\in Y^*}$ of the quasi-shuffle algebra and then in the basis $\{\Pi_w\}_{w\in Y^*}$ of the envelopping algebra, we obtain successively

$$\begin{aligned}\mathcal{D}_Y &= \sum_{w\in Y^*}\left(\sum_{u\in Y^*}\langle \Pi_u \mid w\rangle \Sigma_u\right)\otimes w\\ &= \sum_{u\in Y^*}\Sigma_u \otimes \left(\sum_{w\in Y^*}\langle \Pi_u \mid w\rangle w\right)\\ &= \sum_{u\in Y^*}\Sigma_u\otimes \Pi_u\\ &= \sum_{\substack{l_1>\ldots>l_k\\ i_1,\ldots,i_k\geq 1}} \frac{1}{i_1!\ldots i_k!}\Sigma_{l_1}^{\uplus_q i_1}\uplus_q \ldots \uplus_q \Sigma_{l_k}^{\uplus_q i_k}\otimes \Pi_{l_1}^{i_1}\ldots \Pi_{l_k}^{i_k}\\ &= \prod^{\searrow}_{l\in\mathcal{L}ynY}\sum_{i\geq 0}\frac{1}{i!}\Sigma_l^{\uplus_q i}\otimes \Pi_l^i\\ &= \prod^{\searrow}_{l\in\mathcal{L}ynY}\exp(\Sigma_l\otimes \Pi_l).\end{aligned}$$

3. For $w = l_1^{i_1}\ldots l_k^{i_k}$ with $l_1,\ldots,l_k \in \mathcal{L}ynY$ and $l_1 > \ldots > l_k$, by Proposition 2, the proper polynomial of positive coefficients $\Sigma_w$ is lower triangular :

$$\begin{aligned}\Sigma_w &= \frac{1}{i_1!\ldots i_k!}\Sigma_{l_1}^{\uplus_q i_1}\uplus_q\ldots\uplus_q \Sigma_{l_k}^{\uplus_q i_k}\\ &= w + \sum_{v<w,(v)=(w)} c_v v.\end{aligned}$$

In particular, for any $l_j \in \mathcal{L}ynY$, $\Sigma_{l_j}$ is lower triangular :

$$\Sigma_{l_j} = l_j + \sum_{v<l_j,(v)=(l_j)} c_v v.$$

Hence, $\Sigma_w = \chi_w + \chi'_w$, where $\chi'_w$ is a proper polynomial of $\mathbf{k}\langle Y\rangle$ of rational positive coefficients. We deduce then the support of $\chi_w$ contains words which are less than $w$ and $\langle \chi_w \mid w\rangle = 1$. Thus, the proper polynomial $\chi_w$ of rational positive coefficients is lower triangular :

$$\chi_w = w + \sum_{v<w,(v)=(w)} c_v v,$$
$$\Rightarrow \quad \forall l \in \mathcal{L}ynY, \quad \chi_l = l + \sum_{v<l,(v)=(l)} c_v v.$$

It follows then expected results.

4. By duality, for $w \in Y^*$, the proper polynomial $\xi_w$ is upper triangular. In particular, for any $l \in \mathcal{L}ynY$, the proper polynomial $\xi_l$ is upper triangular :

$$\xi_l = l + \sum_{v>l,(v)=(l)} d_v v.$$

Hence, the family $\{\xi_l\}_{l\in\mathcal{L}ynY}$ is free and its elements verify an analogous of the generalized criterion of Friedrichs :
 – for $w \in \mathcal{L}ynY$, one has $\langle \chi_w \mid \xi_l\rangle = \delta_{w,l}$,
 – for $w = l_1 \ldots l_n \notin \mathcal{L}ynY$ with $l_1, \ldots, l_n \in \mathcal{L}ynY$ and $l_1 \geq \ldots \geq l_n$, one has (since $l \in \mathcal{L}ynY$)

$$\langle \chi_{l_1} ⧢_q \ldots ⧢_q \chi_{l_n} \mid \xi_l\rangle = \langle \chi_w \mid \xi_l\rangle = 0.$$

The polynomials $\xi_l$'s are primitive. Actually, we have

$$\begin{aligned}\Delta_{⧢_q}\xi_l = &\sum_{u\in Y^+} \langle u ⧢_q 1_{Y^*} \mid \xi_l\rangle u\otimes 1 + \sum_{v\in Y^+} \langle 1_{Y^*} ⧢_q v \mid \xi_l\rangle 1\otimes v + \sum_{u,v\in Y^+} \langle u ⧢_q v \mid \xi_l\rangle u\otimes v\\ &+ \langle 1_{Y^*} ⧢_q 1_{Y^*} \mid \xi_l\rangle 1\otimes 1\\ &= \xi_l\otimes 1 + 1\otimes \xi_l.\end{aligned}$$

Because, after decomposing the words $u$ and $v$ on the transcendence basis $\{\chi_l\}_{l\in\mathcal{L}ynY}$ and by the previous fact, the third sum is vanishing. The last one is also vanishing since the $\xi_l$'s are proper. Hence, it follows the expected result.

### 2.3 Determination of $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$

Following [22], we call a *standard sequence* of Lyndon words to be a sequence

$$S = (l_1, \cdots, l_k), k \geq 1 \tag{6}$$

if for all $i$, either $l_i$ to be a letter or the standard factorization $\sigma(l_i) = (l'_i, l''_i)$ and $l''_i \geq l_{i+1}, \cdots, l_n$. Note that a decreasing sequence of Lyndon words is also a standard sequence. A *rise* of a sequence $S$ is an index $i$ such that $l_i < l_{i+1}$. A *legal rise* of sequence $S$ is a rise of $i$ such that $l_{i+1} \geq l_{i+2}, \cdots, l_k$; with the legal rise $i$, we define

$$\lambda_i(S) = (l_1, \cdots, l_{i-1}, l_i l_{i+1}, l_{i+2}, \cdots, l_n) \text{ and } \rho_i(S) = (l_1, \cdots, l_{i-1}, l_{i+1}, l_i, l_{i+2}, \cdots, l_n) \tag{7}$$

We denote $S \Rightarrow T$ if $T = \lambda_i(S)$ or $T = \rho_i(S)$ for some legal rise $i$; and $S \overset{*}{\Rightarrow} T$, transitive closure of $\Rightarrow$.
A *derivation tree* $\mathcal{T}(S)$ of $S$ to be a labelled rooted tree with the following properties : if $S$ is decreasing, then $\mathcal{T}(S)$ is reduced to its root, labelled $S$; if not, $\mathcal{T}(S)$ is the tree with root labelled $S$, with left and right immediate subtree $\mathcal{T}(S')$ and $\mathcal{T}(S'')$, where $S' = \lambda_i(S)$, $S'' = \rho_i(S)$ for some legal rise $i$ of $S$; we define $\Pi(S) = \Pi_{l_1} \ldots \Pi_{l_n}$ ($\Pi(S) \neq \Pi_{l_1\ldots l_k}$ because $l_1, \cdots, l_k$ can be not a decreasing sequence).
Conversely, we call a *fall* of sequence $S$ is an index $i$ such that $l_1, \cdots, l_i \in Y, l_i > l_{i+1}$. We define

$$\rho_i^{-1}(S) = (l_1, \cdots, l_{i+1}, l_i, \cdots, l_n). \tag{8}$$

We call a *landmark* of sequence $S$ is an index $i$ such that $l_1, \cdots, l_{i-1} \in Y, l_i \in Y^* \setminus Y$, and we define

$$\lambda_i^{-1}(S) = (l_1, \cdots, l_{i-1}, l'_i, l''_i, l_{i+1}, \cdots, l_n), \tag{9}$$

where $\sigma(l_i) = (l'_i, l''_i)$. We will denote by $S \Leftarrow T$ if $T = \rho_i^{-1}(S)$ or $T = \lambda_i^{-1}(S)$ for some fall or landmark $i$; and $S \overset{*}{\Leftarrow} T$, transitive closure of $\Leftarrow$.
Similarly, we call the conversely derivation tree $\mathcal{T}^{-1}(S)$ with root labelled $S$, with left and right immediate subtree $\mathcal{T}^{-1}(S')$ and $\mathcal{T}^{-1}(S'')$, where $S' = \rho_i^{-1}(S)$ for some fall $i$, $S'' = \lambda_i^{-1}(S)$ for some landmark $i$.

**Lemma 5.** *For each standard sequence $S$, $\Pi(S)$ is the sum of all $\Pi(T)$ for $T$ a leaf in a fixed derivation tree of $S$.*

*Proof.* This is a consequence of the definitions of $\lambda_i(S)$ and $\rho_i(S)$ on (7), of $\mathcal{T}(S)$ and $\Pi(S)$, and of the identity $\Pi_{l_i}\Pi_{l_{i+1}} = [\Pi_{l_i}, \Pi_{l_{i+1}}] + \Pi_{l_{i+1}}\Pi_{l_i} = \Pi_{l_i l_{i+1}} + \Pi_{l_{i+1}}\Pi_{l_i}$.

*Example 1.* $\Pi(y_4, y_2, y_1) = \Pi_{y_4y_2y_1} + \Pi_{y_2y_1}\Pi_{y_4} + \Pi_{y_4y_1y_2} + \Pi_{y_2}\Pi_{y_4y_1} + \Pi_{y_1}\Pi_{y_4y_2} + \Pi_{y_1}\Pi_{y_2}\Pi_{y_4}$, we can see the following diagram (note that $y_4 < y_2 < y_1$)

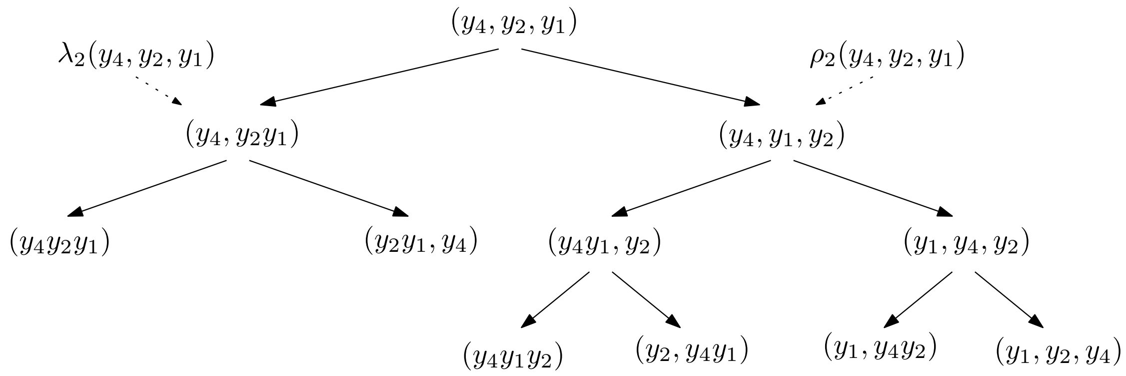


**Fig. 1.** Derivation tree $\mathcal{T}(y_4, y_2, y_1)$

**Proposition 6.** *1. For any Lyndon word $y_{s_1} \dots y_{s_k}$, we have*

$$\Sigma_{y_{s_1}\dots y_{s_k}} = \sum_{\substack{\{s'_1,\cdots,s'_i\}\subset\{s_1,\cdots,s_k\}, l_1\geq\cdots\geq l_n\in\mathcal{L}ynY \\ (y_{s_1}\cdots y_{s_k})\overset{*}{\Leftarrow}(y_{s'_1},\cdots,y_{s'_i},l_1,\cdots,l_n)}} \frac{q^{i-1}}{i!} y_{s'_1+\cdots+s'_i}\Sigma_{l_1\cdots l_n}.$$

*2. In special case, if $y_{s_1} \leq \cdots \leq y_{s_k}$ then*

$$\Sigma_{y_{s_1}\dots y_{s_k}} = \sum_{i=1}^{k} \frac{q^{i-1}}{i!} y_{s_1+\cdots+s_i}\Sigma_{y_{s_{i+1}}\dots y_{s_k}}.$$

*Proof.* At first, we remark this Proposition is equivalent to saying that for any word $u$ and any letter $y_s$,

$$\langle \Sigma_{y_{s_1}\dots y_{s_k}} \mid y_s u\rangle = \sum_{\substack{\{s'_1,\cdots,s'_i\}\subset\{s_1,\cdots,s_k\}, l_1\geq\cdots\geq l_n\in\mathcal{L}ynY \\ (y_{s_1}\cdots y_{s_k})\overset{*}{\Leftarrow}(y_{s'_1},\cdots,y_{s'_n},l_1,\cdots,l_n)}} \frac{q^{i-1}}{i!} \delta_{s'_1+\cdots+s'_i,s}\langle \Sigma_{l_1\dots l_n} \mid u\rangle.$$

One has

$$u = \sum_{w\in Y^*} \langle \Sigma_w \mid u\rangle \Pi_w,$$

multiplying the two members by $y_s$ and by Proposition 1, one obtains

$$\begin{aligned} y_s u &= \sum_{w\in Y^*} \langle \Sigma_w \mid u\rangle \Bigg( \sum_{i\geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1+\cdots+s'_i=s} \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Bigg) \Pi_w \\ &= \sum_{w\in Y^*} \langle \Sigma_w \mid u\rangle \sum_{i\geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1+\cdots+s'_i=s} \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w, \\ \Rightarrow \langle \Sigma_{y_{s_1}\dots y_{s_k}} \mid y_s u\rangle &= \sum_{w\in Y^*} \langle \Sigma_w \mid u\rangle \sum_{i\geq 1} \frac{q^{i-1}}{i!} \sum_{s'_1+\cdots+s'_i=s} \langle \Sigma_{y_{s_1}\dots y_{s_k}} \mid \Pi_{y_{s'_1}} \dots \Pi_{y_{s'_i}} \Pi_w\rangle. \end{aligned}$$

For each $w$ fixed, we write $w$ form factorization of Lyndon words $w = l_1 \ldots l_n, l_1 \geq \cdots \geq l_n$, then we have $S := (y_{s'_1}, \cdots, y_{s'_i}, l_1, \cdots, l_n)$ is a standard sequence, so we obtain from Lemma 5

$$\Pi(S) = \Pi(y_{s'_1}, \cdots, y_{s'_i}, l_1, \cdots, l_n) = \sum_{S \overset{*}{\Rightarrow} T} \alpha_T \Pi(T).$$

Consequently,

$$\langle \Sigma_{y_{s_1} \ldots y_{s_k}} \mid y_s u \rangle = \sum_{l_1 \geq \cdots \geq l_n \in \mathcal{L}ynY} \langle \Sigma_{l_1 \ldots l_n} \mid u \rangle \sum_{i \geq 1} \frac{q^{i-1}}{i!} \sum_{\substack{s'_1 + \cdots + s'_i = s \\ (y_{s'_1}, \cdots, y_{s'_i}, l_n, \cdots, l_n) \overset{*}{\Rightarrow} T}} \alpha_T \langle \Sigma_{y_1 \ldots y_k} \mid \Pi(T) \rangle.$$

Note that, the leaves $T$'s of derivation tree $\mathcal{T}(S)$ are decreasing sequences of Lyndon words with length $\geq 2$ except leaves form $T = (l)$, where $l \in \mathcal{L}ynY$. Therefore $\langle \Sigma_{y_{s_1} \ldots y_{s_k}} | \Pi(T) \rangle \neq 0$ if $T = (y_{s_1} \ldots y_{s_k})$. By maps $\rho^{-1}$ and $\lambda^{-1}$, we construct a conversely derivation tree from the standard sequence of one Lyndon word $S = (y_{s_1} \ldots y_{s_k})$, we take standard sequences form $(y_{s'_1}, \cdots, y_{s'_i}, l_n, \cdots, l_n), i \geq 1$; at that time, for each of these sequences, we get unique leaf $T = (y_{s_1} \ldots y_{s_k})$ in its derivation tree, it means $\alpha_T = 1$. Thus, we get the expected result.
In other words, if $y_{s_1} \leq \cdots \leq y_{s_k}$ then the standard sequence $(y_{s_1} \ldots y_{s_k})$ may only be a leaf of a derivation tree $\mathcal{T}(S)$ after applying map $\lambda_i$ more times, we imply that $\langle \Sigma_{y_{s_1} \ldots y_{s_k}} \mid \Pi_{y_{s'_1}} \ldots \Pi_{y_{s'_i}} \Pi_w \rangle \neq 0$ if and only if $y_{s_1} \ldots y_{s_k} = y_{s'_1} \ldots y_{s'_i} l_1 \ldots l_n$, then $y_{s_1} = y_{s'_1}, \cdots, y_{s'_i} = y_{s_i}$ and $y_{s_{i+1}} \ldots y_{s_k} = l_1 \ldots l_n$. Hence

$$\langle \Sigma_{y_{s_1} \ldots y_{s_k}} | \Pi_{y_{s'_1}} \ldots \Pi_{y_{s'_i}} \Pi_w \rangle = \delta_{s_1 + \cdots + s_i, s} \delta_{y_{s_{i+1}} \ldots y_{s_k}, w},$$

we thus get

$$\langle \Sigma_{y_{s_1} \ldots y_{s_k}} \mid y_s u \rangle = \frac{q^{i-1}}{i!} \delta_{s_1 + \cdots + s_i, s} \langle \Sigma_{y_{s_{i+1}} \ldots y_{s_k}} \mid u \rangle.$$

### 2.4 Examples with Maple

$$\Pi_{y_1} = y_1, \tag{10}$$
$$\Pi_{y_2} = y_2 - \frac{q}{2} y_1^2, \tag{11}$$
$$\Pi_{y_2 y_1} = y_2 y_1 - y_1 y_2, \tag{12}$$
$$\begin{aligned} \Pi_{y_3 y_1 y_2} = {} & y_3 y_1 y_2 - \frac{q}{2} y_3 y_1^3 - q y_2 y_1^2 y_2 + \frac{q^2}{4} y_2 y_1^4 - y_1 y_3 y_2 + \frac{q}{2} y_1 y_3 y_1^2 \\ & + \frac{q}{2} y_1^2 y_2^2 - \frac{q^2}{2} y_1^2 y_2 y_1^2 - y_2 y_3 y_1 + \frac{q}{2} y_2^2 y_1^2 + y_2 y_1 y_3 + \frac{q}{2} y_1^2 y_3 y_1 - \frac{q}{2} y_1^3 y_3 + \frac{q^2}{4} y_1^4 y_2, \end{aligned} \tag{13}$$
$$\begin{aligned} \Pi_{y_3 y_1 y_2 y_1} = {} & y_3 y_1 y_2 y_1 - y_3 y_1^2 y_2 - \frac{q}{2} y_2 y_1^2 y_2 y_1 - y_1 y_3 y_2 y_1 + y_1 y_3 y_1 y_2 + \frac{q}{2} y_1^2 y_2^2 y_1 \\ & - \frac{q}{2} y_1^2 y_2 y_1 y_2 - y_2 y_1 y_3 y_1 + \frac{q}{2} y_2 y_1 y_2 y_1^2 + y_2 y_1^2 y_3 + y_1 y_2 y_3 y_1 \\ & - \frac{q}{2} y_1 y_2^2 y_1^2 - y_1 y_2 y_1 y_3 + \frac{q}{2} y_1 y_2 y_1^2 y_2. \end{aligned} \tag{14}$$

$$\Sigma_{y_1} = y_1, \tag{15}$$
$$\Sigma_{y_2} = y_2, \tag{16}$$
$$\Sigma_{y_2 y_1} = y_2 y_1 + \frac{q}{2} y_3, \tag{17}$$
$$\Sigma_{y_3 y_1 y_2} = y_3 y_1 y_2 + y_3 y_2 y_1 + q y_3^2 + \frac{q}{2} y_4 y_2 + \frac{q^2}{3} y_6 + \frac{q}{2} y_5 y_1, \tag{18}$$
$$\begin{aligned} \Sigma_{y_3 y_1 y_2 y_1} = {} & 2 y_3 y_2 y_1^2 + q y_3 y_2^2 + y_3 y_1 y_2 y_1 + \frac{3q}{2} y_3^2 y_1 + \frac{q}{2} y_3 y_1 y_3 + \frac{q^2}{2} y_3 y_4 + \frac{q}{2} y_4 y_2 y_1 \\ & + \frac{q^2}{4} y_4 y_3 + q y_5 y_1^2 + \frac{q^2}{2} y_5 y_2 + \frac{q^2}{2} y_6 y_1 + \frac{q^3}{8} y_7. \end{aligned} \tag{19}$$

## 3 Conclusion

Since the pioneering works of Schützenberger and Reutenauer [23, 22], the question of computing bases in duality (maybe at the cost of a more cumbersome procedure, but without inverting a Gram matrix) remained open in the case of cocommutative deformations of the shuffle product. We have given such a procedure, based on the computation of $\log_*(I)$ on the letters which allows a great simplification for an interpolation between shuffle and stuffle products (this interpolation reduces to the shuffle for $q = 0$ and the stuffle for $q = 1$). Our algorithm boils down to the classical one in the case when $q = 0$. In the next framework, this product will be continuously deformed, in the most general way but still commutative (see [12] for examples).

## References


1. J. Berstel, C. Reutenauer.– Rational series and their languages, Monographs on Theorical Computer Sciences, Springer-Verlag (1984).
2. J. Berstel, D. Perrin.– Theory of codes, Academic Press (1985).
3. Boubaki, N. .– N. Bourbaki, Groupes et Algbres de Lie, Chap II-III, Springer (2006)
4. C. Bui.– Master dissertation, Paris (2012).
5. P. Cartier.– Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents, Sém BOURBAKI, $53^{\text{ème}}$ 2000-2001, $n°885$.
6. K.T. Chen, R.H. Fox, R.C. Lyndon.– Free differential calculus, IV. The quotient groups of the lower central series, Ann. of Math. , 68 (1958) pp. 8195.
7. C. Costermans.– *Calcul non nommutative : analyse des constantes d'arbre de fouille*, thèse, Lille, 2008.
8. C. Costermans and Hoang Ngoc Minh.– *Noncommutative algebra, multiple harmonic sums and applications in discrete probability*, Journal of Symbolic Computation (2009), pp. 801-817.
9. G. Duchamp, A. Klyachko, D. Krob, J.-Y. Thibon.–Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras. Discrete Mathematics & Theoretical Computer Science 1(2): 159-216 (1997)
10. V. Drinfel'd.– Quasi-Hopf Algebras, *Len. Math. J.*, 1, 1419-1457, 1990.
11. V. Drinfel'd.– On quasitriangular quasi-hopf algebra and a group closely connected with gal$(\bar{q}/q)$, *Leningrad Math. J.*, 4, 829-860, 1991.
12. J-Y. Enjalbert, Hoang Ngoc Minh,– *Combinatorial study of Hurwitz colored polyzêtas*, dans Discrete Mathematics, 1. 24 no. 312 (2012), p. 3489-3497.
13. T.Q.T. Lê & J. Murakami.– Kontsevich's integral for Kauffman polynomial, Nagoya Math., pp 39-65, 1996.
14. M. Lothaire.– Combinatorics on words, Addison-Wesley Publishing Co., Reading, Mass., coll. Encyclopedia of Mathematics and its Applications 17 (1983).
15. C. Malvenuto & C. Reutenauer.– Duality between quasi-symmetric functions and the Solomon descent algebra, J. of Alg. 177 (1995), pp. 967-982.
16. Hoang Ngoc Minh, Petitot, M., and Van der Hoeven, J..– L'algèbre des polylogarithmes par les séries génératrices, *Proceedings of FPSAC'99*, 1999.
17. Hoang Ngoc Minh.– Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series, *Proceedings of* $4^{th}$ *International Conference on Words*, pp. 232-250, 2003.
18. Hoang Ngoc Minh.– On a conjecture by Pierre Cartier about a group of associators, to appear (2013).
19. Hoffman, M..– Quasi-shuffle products, *J. of Alg. Cominatorics*, 11 (2000), pages 49-68.
20. D.E. Radford.– A natural ring basis for shuffle algebra and an application to group schemes Journal of Algebra, 58, pp. 432-454, 1979.
21. Ree R.,– Lie elements and an algebra associated with shuffles *Ann. Math* **68** 210–220, 1958.
22. Reutenauer, C..– *Free Lie Algebras*, London Math. Soc. Monographs, New Series-7, Oxford Science Publications, 1993.
23. Schützenberger, M.P..– *Sur une propriété combinatoire des algèbres de Lie libres pouvant être utilisée dans un problème de mathématiques appliquées*, Séminaire Dubreil. Alg ebre et théorie des nombres, tome 12, no 1 (1958-1959), exp. num 1,p. 1-23.

    `http://www.numdam.org/item?id=SD_1958-1959__12_1_A1_0`

24. L. Solomon.– A. Mackey formula in the group ring of a Coxeter group J. Algebra, 41 (1976), pp. 255264
25. Viennot, G.– Algèbres de Lie Libres et Monodes Libres, (1978) 691: 94-112, January 01, 1978
26. D. Zagier.– Values of zeta functions and their applications, in "First European Congress of Mathematics", vol. 2, Birkhäuser (1994), pp. 497-512.