On the role of connectivity in Linear Logic proofs

Raffaele Di Donna[1] and Lorenzo Tortora de Falco[2]

[1] Université Paris Cité, Paris, France and Università Roma Tre, Rome, Italy
didonna@irif.fr
[2] Università Roma Tre, Rome, Italy
tortora@uniroma3.it

1 Introduction

Linear logic (LL, [8]) is a refinement of both classical and intuitionistic logic in which a formula is treated as a computational resource, thanks to a decomposition of the connectives for conjunction ($\wedge$) and disjunction ($\vee$) into their multiplicative ($\otimes$ and $\parr$) and additive (& and $\oplus$) variants, and thanks to the exponential modalities that bridge the gap between them and make the structural rules of contraction and weakening explicit in the syntax (see Figure 1).

One of the main contributions of LL to the field of proof theory is the shift from a term-like or tree-like representation of proofs to a more general *graph*-like one, which removes much of the bureaucratic order imposed by sequent calculus and exposes the intrinsic graph structure of proofs. Graphs representing proofs in LL are usually called *proof-nets*, and are special inhabitants of a wider set of *proof-structures* which may not represent proofs. It is clearly an interesting theoretical question to find abstract properties (*correctness criteria*) allowing us to isolate, among proof-structures, those that are images of sequent calculus proofs (*sequentializable* proof-structures).

One of the most famous correctness criteria is the Danos-Regnier ACC property ([3]), relating the logical correctness of a proof in multiplicative LL without the units (MLL^-) with the acyclicity and connectivity of every graph obtained by detaching one premise for each $\parr$ node. However, as soon as one adds the multiplicative units to MLL^- (the fragment thus obtained is denoted by MLL), connectivity is lost: when translating the $\bot$ rule of sequent calculus in proof-nets, the $\bot$ node is isolated from the rest of the graph (Figure 3d). It is common practice ([16, 9, 1], ...) to add *jumps* to proof-structures to recover ACC, but this technique has major downsides (jumps have no canonical position in general, their behavior through cut elimination is a mess, ...).

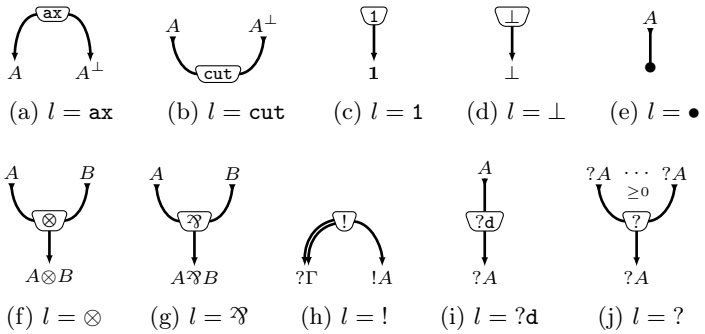
On the other hand, it is quite natural to extend ACC in the presence of the units by relating the number of connected components ($\sharp\mathsf{cc}$) and the number of $\bot$ nodes ($\sharp\mathsf{w}$) through the equality $\sharp\mathsf{cc} = \sharp\mathsf{w} + 1$ ([15]). Although this condition is not sufficient for a proof-structure of MLL to be sequentializable (Figure 4a), its extension to multiplicative exponential LL (MELL), which we denote by $\mathsf{ACC}_{\sharp\mathsf{w}}$, is necessary (Proposition 8) and stable under cut elimination (Theorem 10).

We then reverse the point of view by asking what the logical counterpart of connectivity is. More concretely, we identify a purely geometric restriction on proof-structures that turns $\mathsf{ACC}_{\sharp\mathsf{w}}$ into a sufficient condition for a proof-structure of MELL to be sequentializable (Theorem 12), and then turn this restriction into a fragment of MELL for which $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion in the absence of cuts (Corollary 14). As a byproduct, we obtain that $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion for the classical polarization of MELL in the absence of cuts (Corollary 15), and thus $\mathsf{ACC}_{\sharp\mathsf{w}}$ is equivalent in this setting to the criterion by Laurent ([11, 12, 13]). In the intuitionistic polarization of MELL (IMELL), $\mathsf{ACC}_{\sharp\mathsf{w}}$ is equivalent to the requirement of having exactly one output formula in the conclusion sequent (Proposition 19), which is equivalent to the familiar requirement of having exactly one formula on the right side of the turnstile in intuitionistic linear logic with two-sided sequents ([9]). Finally, we mention a result from a related work ([4]) showing that

$$\frac{}{\vdash A, A^\perp}\ ax \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w$$

Figure 1: Sequent calculus rules.



(a) $l = \mathtt{ax}$ (b) $l = \mathtt{cut}$ (c) $l = \mathtt{1}$ (d) $l = \bot$ (e) $l = \bullet$

(f) $l = \otimes$ (g) $l = \parr$ (h) $l = !$ (i) $l = ?\mathtt{d}$ (j) $l = ?$

Figure 2: The local constraints that a node of a ground-structure with label l has to respect.

$\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion for the fragment of MELL that corresponds to the half-polarized typing system for call-by-push-value in [6, 7], in the absence of multiplicative cuts (Theorem 20).

2 Proof-nets

Formulae of MELL are defined by the grammar:

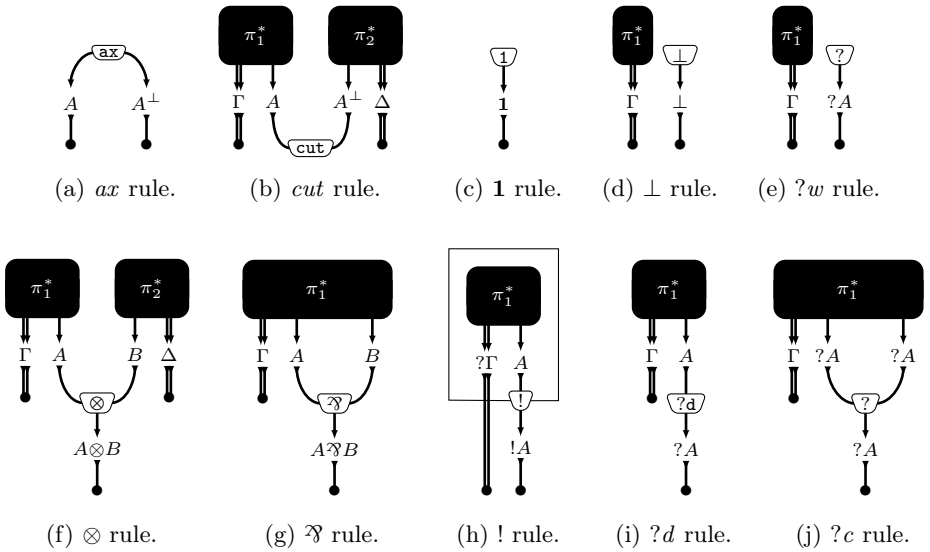
$$A ::= X \mid \mathbf{1} \mid \bot \mid A \otimes A \mid A \parr A \mid !A \mid ?A$$

Definition 1. A *fragment* $\mathcal{F}$ of MELL is a set of formulae of MELL which is closed under sub-formulae, meaning that, for every $A \in \mathcal{F}$ and for every sub-formula B of A, we have $B \in \mathcal{F}$. A *sequent* of a fragment $\mathcal{F}$ of MELL is a finite multiset of formulae of $\mathcal{F}$. A *sequent calculus proof* in $\mathcal{F}$ with conclusion Γ is a finite tree of sequents of $\mathcal{F}$ obtained by using the rules of Figure 1.

Definition 2. A *ground-structure* G *of* MELL is a directed graph with nodes labeled by $\mathtt{ax}$, $\mathtt{cut}$, $\mathbf{1}$, $\bot$, $\otimes$, $\parr$, $!$, $?\mathtt{d}$, $?$ or $\bullet$, arcs labeled by formulae of MELL (called *types*), and such that every node satisfies the local constraints in Figure 2. If a is an arc from n to m, we say that a is a *conclusion* of n and a *premise* of m. A $?\mathtt{w}$ (resp. $?\mathtt{c}$) node is a $?$ node with no (resp. at least two) premises. The number of $\bot$ or $?\mathtt{w}$ nodes of G is denoted by $\sharp\mathsf{w}(G)$. Let $\mathcal{F}$ be a fragment of MELL. We say that G is a ground-structure of $\mathcal{F}$ if its arcs are labeled by formulae of $\mathcal{F}$.

Proof-structures are constructed inductively using ground-structures and ! nodes.

Definition 3. A *proof-structure of* $\mathcal{F}$ is a pair $R = (G_R, \mathsf{box}_R)$ with G_R a ground-structure of $\mathcal{F}$ and box_R a function mapping every ! node n of G_R to a proof-structure of $\mathcal{F}$ (a *box* of R) whose ground-structure has the same number of conclusions as n, and of the same types. Boxes are, inductively, disjoint from the ground-structure G_R and pairwise disjoint. We write $R' \sqsubset R$ when R' is a box of R. A *conclusion* of R is a premise of a $\bullet$ node of G_R.



(a) *ax* rule. (b) *cut* rule. (c) **1** rule. (d) ⊥ rule. (e) ?*w* rule.

(f) ⊗ rule. (g) ⅋ rule. (h) ! rule. (i) ?*d* rule. (j) ?*c* rule.

Figure 3: Desequentialization of sequent calculus proofs in MELL into proof-structures of MELL. A double arrow to a • node represents a finite number of arcs, each targeting a distinct • node.

Definition 4. Let G be a ground-structure of MELL. A *switching* φ of G is a function mapping every ⅋ or ?c node to one of its premises. The *switching graph of G induced by φ*, denoted by G^φ, is the undirected graph obtained from G by adding a fresh • node n_0 for each premise a of a ⅋ or ?c node n such that $a \neq \varphi(n)$, and by setting a to be incident to n_0 rather than n. The number of connected components of G^φ is denoted by $\sharp\mathsf{cc}(G^\varphi)$.

The well-known process of desequentialization maps a sequent calculus proof π in MELL with conclusion $A_1, \ldots, A_k$ to a proof-structure π^* of MELL with conclusions of types $A_1, \ldots, A_k$. The function $\pi \mapsto \pi^*$ is defined by induction on π (see Figure 3). Full details are provided in [1].

Definition 5. Let $\mathcal{F}$ be a fragment of MELL. A proof-structure R of $\mathcal{F}$ is *sequentializable* when there exists a sequent calculus proof π in $\mathcal{F}$ such that $R = \pi^*$.

Definition 6. Let G be a ground-structure and let φ be a switching of G. We write:
$G^\varphi \models \mathsf{AC}$ if G^φ is acyclic; $G^\varphi \models \mathsf{C}$ if $\sharp\mathsf{cc}(G^\varphi) = 1$; $G^\varphi \models \mathsf{C}_{\sharp\mathsf{w}}$ if $\sharp\mathsf{cc}(G^\varphi) = \sharp\mathsf{w}(G) + 1$.

We write $G \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) when, for every switching φ of G, $G^\varphi \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$).

Definition 7. Let R be a proof-structure. We write:

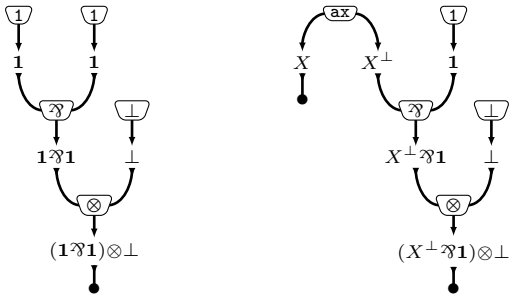
- $R \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) if $G_R \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) and, for $R_0 \sqsubset R$, $R_0 \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$);
- $R \models \mathsf{ACC}$ if $R \models \mathsf{AC}$ and $R \models \mathsf{C}$; $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$ if $R \models \mathsf{AC}$ and $R \models \mathsf{C}_{\sharp\mathsf{w}}$.

Proposition 8. *Let R be a proof-structure of* MELL. *If R is sequentializable, then $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$.*

Remark 9. The converse of Proposition 8 does not hold, because of the example in Figure 4a, due to Laurent Regnier (common knowledge in the community, implicit in [15]).

Cut elimination on proof-structures of MELL is essentially a graph rewriting ([1]). We write $R \to R'$ if R reduces to R' in exactly one cut elimination step. A normal proof-structure of MELL is called *cut-free*. A notable property of $\mathsf{ACC}_{\sharp\mathsf{w}}$ is its stability under cut elimination ([10, 5, 1]):

Theorem 10. *Let R, R' be proof-structures of* MELL. *If $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$, then $R' \models \mathsf{ACC}_{\sharp\mathsf{w}}$.*



(a) Classical (Regnier). (b) Intuitionistic (Section 3).

Figure 4: Two proof-structures of MELL satisfying $\mathsf{ACC}_{\sharp\mathsf{w}}$ which are not sequentializable.

3 Connectivity in general proof-nets

Definition 11. A $\bot$, $⅋$, ?d, ? or $\bullet$ node of a ground-structure G of MELL is *erasing* if its premises are all conclusions of erasing nodes. We say that G is a $(\neg\mathsf{w}\otimes)$-*ground-structure* when no premise of a `cut` or $\otimes$ node is a conclusion of an erasing node. A proof-structure R of MELL is a $(\neg\mathsf{w}\otimes)$-*proof-structure* if G_R is a $(\neg\mathsf{w}\otimes)$-ground-structure and every $R' \sqsubset R$ is a $(\neg\mathsf{w}\otimes)$-proof-structure.

Theorem 12 ([5])**.** *Let R be a $(\neg\mathsf{w}\otimes)$-proof-structure. If $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$, then R is sequentializable.*

Definition 13. The fragment $(\neg?\otimes)$LL of MELL is defined by the following grammar:

$$A ::= X \mid \mathbf{1} \mid A \otimes A \mid A ⅋ A \mid A ⅋ E \mid E ⅋ A \mid !A \mid !E \qquad E ::= \bot \mid E ⅋ E \mid ?A \mid ?E$$

Corollary 14. *For R cut-free proof-structure of $(\neg?\otimes)$LL: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.*

Hence, $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion for the proof-structures of the classical polarization of MELL ([11, 14]) and, in particular, for the image of the call-by-name λ-calculus.

Corollary 15. *Let LL_{pol} be the fragment of* MELL *defined by the following grammar:*

$$P ::= \mathbf{1} \mid !X \mid !N \mid P \otimes P \quad \textit{(positives)} \qquad N ::= \bot \mid ?X \mid ?P \mid N ⅋ N \quad \textit{(negatives)}$$

For every cut-free proof-structure R of LL_{pol}, we have: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.

Remark 16. A refinement of Corollaries 14 and 15 yields canonical positions for jumps ([5]).

4 Connectivity in intuitionistic proof-nets

In this section, the set of atoms is split in two subsets: output atoms (X) and their duals ($X^\bot$).

Definition 17 ([2, 15, 9])**.** The fragment IMELL of MELL is defined by the following grammar:

$$O ::= X \mid \mathbf{1} \mid O \otimes O \mid O ⅋ I \mid I ⅋ O \mid !O \qquad \text{(outputs)}$$
$$I ::= X^\bot \mid \bot \mid I ⅋ I \mid I \otimes O \mid O \otimes I \mid ?I \qquad \text{(inputs)}$$

Proposition 18 ([9])**.** *Every provable sequent of* IMELL *has exactly one output formula.*

Proposition 19 ([5])**.** *Let R be a proof-structure of* IMELL *such that $R \models \mathsf{AC}$. Then $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$ if and only if R has exactly one output conclusion.*

4.1 Further work and further restriction ([4])

Theorem 20. *Let* VIMELL *be the fragment of* MELL *defined by the grammar:*

$$P ::= X \mid \mathbf{1} \mid P \otimes P \mid !O \qquad \textit{(positives)} \qquad N ::= X^{\perp} \mid \perp \mid N \parr N \mid ?I \qquad \textit{(negatives)}$$
$$O ::= P \mid O \parr N \mid N \parr O \qquad \textit{(outputs)} \qquad I ::= N \mid I \otimes P \mid P \otimes I \qquad \textit{(inputs)}$$

For a multiplicative-normal proof-structure R of VIMELL*: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.*

Remark 21. Like in the classical polarization of linear logic, positive formulae carry a structure of !-coalgebra, meaning that they can be "equipped with structural rules", but we still keep the benefits of the intuitionistic polarization: VIMELL contains not just the call-by-name λ-calculus, but also the call-by-value λ-calculus and, more generally, the bang calculus.

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