A story of tensorial logic with two negations

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Abstract

Tensorial logic is a primitive logic of sum, tensor and negation which refines linear logic by relaxing the requirement that linear negation is involutive. The logic is designed in such a way that its formulas [modulo canonical isomorphism] are in a one-to-one correspondence with dialogue games, and that its proofs [modulo cut-elimination] are in a one-to-one correspondence with innocent strategies. In this paper, we explain how to accommodate the exponential modality of linear logic $A \mapsto\, !A$ and extend our functorial and proof-theoretic approach to game semantics beyond the multiplicative and additive fragment of tensorial logic. The extension relies on a notion of relative linear-non-linear adjunction on a dialogue category which refines the traditional notion of linear-non-linear adjunction on a $*$-autonomous category. In this description of full propositional tensorial logic, every exponential modality of linear logic $A \mapsto\, !A$ is attached to a negation of tensorial logic $A \mapsto \neg A$. Two forms of negations thus coexist in the logic: the original non-involutive tensorial negation $A \mapsto \neg A$ and a new backtracking exponential negation $A \mapsto \dot{\neg}\, A$ which stands for the "modality + negation" combinator $A \mapsto\, !\neg A$ and defines a relative comonad on the linear non-involutive negation.

1 Introduction

Tensorial logic is a primitive logic of sum, tensor and negation whose multiplicative additive formulas are described by the grammar below:

$$A, B \quad ::= \quad 0 \mid A \oplus B \mid 1 \mid A \otimes B \mid \neg A$$

In this talk, I will explain how to extend the multiplicative additive fragment with an *exponential* and *backtracking* negation $A \mapsto \dot{\neg}\, A$ in order to obtain full propositional tensorial logic. I use the notation $A \mapsto \dot{\neg}\, A$ because it is both reminiscent of the tensorial negation $A \mapsto \neg A$ and of the exponential modality $A \mapsto\, !A$ of linear logic. This pointed form of negation should be informally understood as $A \mapsto\, !\neg A$ where the exponential modality $!$ is somewhat surprisingly applied after the linear negation instead of before the linear negation, as one would probably expect for an intuitionistic variant of tensorial negation.

The proofs of tensorial logic are defined as *derivation trees* in a sequent calculus where every sequent $A_1, \ldots, A_k \vdash B$ consists of a sequence of formulas A_i as hypothesis, and with conclusion a formula B or the constant $\perp$ expressing that the sequence of hypothesis $A_1, \ldots, A_k$ can be refuted.

$$\text{Axiom} \quad \dfrac{}{A \vdash A} \qquad\qquad \dfrac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\textbf{Shuffle}(\Gamma, \Delta) \vdash B} \quad \text{Cut}$$

where $\textbf{Shuffle}(\Gamma, \Delta)$ is a context obtained by "shuffling" the formulas of Γ and Δ. The left and right introduction rules of the negation:

$$\text{Left } \neg \quad \dfrac{\Gamma, \Delta \vdash A}{\Gamma, \neg A, \Delta \vdash \perp} \qquad\qquad \dfrac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \quad \text{Right } \neg$$

The left and right introduction rules of the tensor product:

$$\text{Left } \otimes \quad \dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C} \qquad\qquad \dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\textbf{Shuffle}(\Gamma, \Delta) \vdash A \otimes B} \quad \text{Right } \otimes$$

$$\text{Left } 1 \quad \dfrac{\Gamma, \Delta \vdash A}{\Gamma, 1, \Delta \vdash A} \qquad\qquad \dfrac{}{\vdash 1} \quad \text{Right } 1$$

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The left and right introduction rules of the sum:

$$
\begin{array}{llll}
\text{Left } \oplus & \dfrac{\Gamma\,,\,A\,,\,\Delta\,\vdash\,C \qquad \Gamma\,,\,B\,,\,\Delta\,\vdash\,C}{\Gamma\,,\,A\oplus B\,,\,\Delta\,\vdash\,C} & \dfrac{\Gamma\,\vdash\,A}{\Gamma\,\vdash\,A\oplus B} & \text{Right } \oplus_L \\[2ex]
 & & \dfrac{\Gamma\,\vdash\,B}{\Gamma\,\vdash\,A\oplus B} & \text{Right } \oplus_R \\[2ex]
\text{Left } 0 & \dfrac{}{\Gamma\,,\,0\,\vdash\,A} & \text{no introduction rule} & \text{Right } 0
\end{array}
$$

My main purpose in this talk will be to explain how to accommodate the exponential modality of linear logic and to integrate it to tensorial logic, in a way consistent with the research programme on *functorial game semantics* initiated in [3] and developed in a series of papers [4, 5, 6, 7, 9, 8].

One simple and straightforward way to extend tensorial logic with an exponential modality $A \mapsto\ !A$ of linear logic is to proceed as done twenty years ago in joint work with Nicolas Tabareau [10, 11] where the grammar of tensorial formulas was extended with the modality itself, in this way:

$$
A, B \quad ::= \quad 0 \mid A \oplus B \mid 1 \mid A \otimes B \mid \neg\, A \mid\ !A
$$

The four introduction rules of linear logic are then added *verbatim* to the multiplicative additive fragment of tensorial logic:

$$
\begin{array}{llll}
\text{Dereliction} & \dfrac{\Gamma\,,\,A\,\vdash\,B}{\Gamma\,,\,!A\,\vdash\,B} & \dfrac{!A_1\,,\,\cdots\,,\,!A_n\,\vdash\,B}{!A_1\,,\,\cdots\,,\,!A_n\,\vdash\,!B} & \text{Promotion} \\[2ex]
\text{Weakening} & \dfrac{\Gamma\,\vdash\,B}{\Gamma\,,\,!A\,\vdash\,B} & \dfrac{\Gamma,\,!A,\,!A\,\vdash\,B}{\Gamma\,,\,!A\,\vdash\,B} & \text{Contraction}
\end{array}
$$

One of the main advantages this immediate extension of is that tensorial logic automatically inherits its sequent calculus and its categorical semantics from full propositional linear logic, after only minor adaptations. In particular, the categorical interpretation of the exponential modality $A \mapsto\ !A$ relies on the existence of a linear-non-linear adjunction

$$
\begin{tikzcd}
\mathcal{M} \arrow[rr, bend left, "Lin"] \arrow[rr, phantom, "\perp"] & & \mathcal{L} \arrow[ll, bend left, "Mult"]
\end{tikzcd}
\tag{1}
$$

defined as a symmetric monoidal adjunction between a cartesian category $(\mathcal{M}, \times, 1)$ and a dialogue category $(\mathcal{L}, \otimes, I)$. The exponential modality $A \mapsto\ !A$ is interpreted in this setting as the comonad $! = Mult \circ Lin$ on the dialogue category $\mathcal{L}$ associated to the adjunction $Lin \dashv Mult$. This proximity with full propositional linear logic explains why the extension appears so natural from both a proof-theoretic and a categorical point of view, see [11] for details.

At the same time, the straightforward and somewhat naive integration of the exponential modality $A \mapsto\ !A$ in tensorial logic does not seem adapted to the functorial account of game semantics based on tensorial logic and dialogue categories [3, 8]. The main obstruction is that there is no apparent way to extend the normal form theorem stating that every formula of the multiplicative additive fragment of tensorial logic is isomorphic to a *sum of tensors of negations*

$$
A \quad = \quad \bigoplus_{i\in I} \bigotimes_{j\in J_i} \neg A_{ij}
$$

Here, I will follow here another exploratory track guided by the desire to uncover and to axiomatize the properties of an exponential form of negation $A \mapsto \dot{\neg}\, A$. The formulas of full propositional tensorial logic are thus defined by the grammar:

$$A, B \quad ::= \quad 0 \mid A \oplus B \mid 1 \mid A \otimes B \mid \neg A \mid \dot\neg\, A$$

Pointed negation comes with four introduction rules:

$$\text{Dereliction} \quad \frac{\Gamma\,,\ \neg A \vdash \bot}{\Gamma\,,\ \dot\neg\, A \vdash \bot} \qquad\qquad \frac{\dot\neg\, A_1\,,\ \ldots\,,\ \dot\neg\, A_n \vdash \neg B}{\dot\neg\, A_1\,,\ \ldots\,,\ \dot\neg\, A_n \vdash \dot\neg\, B} \quad \text{Promotion}$$

$$\text{Weakening} \quad \frac{\Gamma \vdash \bot}{\Gamma\,,\ \dot\neg\, A \vdash \bot} \qquad\qquad \frac{\Gamma\,,\ \dot\neg\, A\,,\ \dot\neg\, A \vdash \bot}{\Gamma\,,\ \dot\neg\, A \vdash \bot} \quad \text{Contraction}$$

From now on, we call full (propositional) tensorial logic the multiplicative additive fragment of tensorial logic extended with the pointed negation $A \mapsto \dot\neg\, A$ and its four introduction rules. It is not difficult to see that every formula A of full tensorial logic is isomorphic to a formula of the form

$$A \quad = \quad \bigoplus_{i\in I} \bigotimes_{j \in J_i} \overset{\varepsilon_{ij}}{\neg}\, A_{ij} \qquad\qquad \text{where} \quad \varepsilon_{ij} \in \{\mathit{tensorial}, \mathit{pointed}\}.$$

I will explain during my talk that full tensorial logic comes with a categorical semantics defined as a *relative* linear-non-linear adjunction

$$\begin{tikzcd} & \mathcal{M} \arrow[dr, "Lin"] & \\ \mathcal{L}^{op} \arrow[ur, "Neg"] \arrow[rr, "\neg"'] & \vdash & \mathcal{L} \end{tikzcd} \tag{2}$$

defined as a $\neg$-relative symmetric monoidal adjunction $Lin \dashv Neg$ between a cartesian category $\mathcal{M}$ and a dialogue category $\mathcal{L}$ with negation functor $A \mapsto \neg A$, see [12, 1] for details. Pointed negation $A \mapsto \dot\neg\, A$ is interpreted in this setting as the relative symmetric monoidal comonad $\dot\neg = Lin \circ Neg$ associated to the relative adjunction (2). One important feature of the relative adjunction (2) is that the definition coincides with the usual definition of a linear-non-linear adjunction (1) when the dialogue category $\mathcal{L}$ happens to be $*$-autonomous. In that case, one recovers the usual modalities $A \mapsto !\, A$ and $A \mapsto ?\, A$ of linear logic by combining the tensorial and the pointed negations of tensorial logic in the following way:

$$!\, A \ := \ \dot\neg\, \neg\, A \qquad\qquad ?\, A \ := \ \neg\, \dot\neg\, A.$$

This demonstrates that full propositional tensorial logic can be seen as a *conservative refinement* of full propositional linear logic, which one recovers when the tensorial negation $A \mapsto \neg A$ is involutive. Recall that we say that tensorial negation is involutive precisely when the canonical map $\eta_A : A \longrightarrow \neg\neg A$ is an isomorphism for all objects A of the dialogue category $\mathcal{L}$. This is equivalent to asking the dialogue category $\mathcal{L}$ is in fact a $*$-autonomous category. In linear logic, one has the Seely isomorphism

$$!\,(A \,\&\, B) \ \cong \ !A \otimes !B \qquad\qquad !\top \ \cong \ 1$$

This family of isomorphisms has a counterpart in tensorial logic, given by the family of isomorphisms

$$\dot\neg\,(A \oplus B) \ \cong \ \dot\neg\, A \otimes \dot\neg\, B \qquad\qquad \dot\neg\, 0 \ \cong \ 1$$

which I suggest to call the *Lamarche-Seely isomorphisms* of tensorial logic.

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A Lamarche-Seely dialogue categories

In this short appendix, we give the counterpart in tensorial logic of the usual notion of Lamarche-Seely category in linear logic, which we suggest to call in this way after calling them Seely categories in [2].

Suppose given a dialogue category $\mathcal{D}$ with finite sums equipped with a relative comonad $A \mapsto \dot{\neg} A$ over the negation functor $A \mapsto \neg A$. In that situation, we construct the category $\mathcal{M}$ whose objects are the objects of $\mathcal{D}$ and whose maps $A \rightsquigarrow B$ are maps $\dot{\neg} A \to \neg B$ in the dialogue category $\mathcal{D}$. The category $\mathcal{M}$ comes equipped with a pair of functors

$$Lin \;:\; \mathcal{M} \longrightarrow \mathcal{D} \qquad\qquad Neg \;:\; \mathcal{D}^{op} \longrightarrow \mathcal{M}$$

making the triangular below commute:

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "Lin"] & & \mathcal{D} \\ & \mathcal{D}^{op} \arrow[ul, "Neg"] \arrow[ur, "\dot{\neg}"'] & \end{tikzcd}$$

The functor $Lin : \mathcal{M} \to \mathcal{D}$ transports every object $A \in \mathcal{M}$ to the object $\dot{\neg} A \in \mathcal{D}$, and every map $f : A \rightsquigarrow B$ to the map $f^{\dagger} : \dot{\neg} A \to \dot{\neg} B$. The functor $Neg : \mathcal{D}^{op} \to \mathcal{M}$ transports every object $A \in \mathcal{D}$ to $A \in \mathcal{M}$, and every map $f : A \to B$ to the map

$$\dot{\neg}\, B \xrightarrow{\varepsilon_B} \neg B \xrightarrow{\neg f} \neg A$$

The category $\mathcal{M}$ is cartesian. We describe the necessary and sufficient conditions which ensure that the category $\mathcal{M}$ defines a linear-non-linear negation on the dialogue category $\mathcal{D}$.

1. the category $\mathcal{D}$ is a dialogue category with finite sums,
2. the functor $A \mapsto \dot{\neg}\, A$ is a relative comonad over the functor $A \mapsto \neg A$,
3. there is a family of isomorphisms, called Lamarche-Seely isomorphisms:

$$m_{A,B} \quad : \quad \dot{\neg}\, A \otimes \dot{\neg}\, B \xrightarrow{iso} \dot{\neg}\, (A \oplus B)$$

4. the diagram below commutes in $\mathcal{D}$

$$\begin{tikzcd}
\dot{\neg}\, A \otimes \dot{\neg}\, B \arrow[r, "m"] \arrow[d, "h_A^{\dagger} \otimes h_B^{\dagger}"'] & \dot{\neg}\, (A \oplus B) \arrow[d, "\langle h_A, h_B \rangle^{\dagger}"] \\
\dot{\neg}\, A' \otimes \dot{\neg}\, B' \arrow[r, "m"] & \dot{\neg}\, (A' \oplus B')
\end{tikzcd}$$

for all maps $h_A : \dot{\neg}\, A \to \neg A'$ and $h_B : \dot{\neg}\, B \to \neg B'$.