Multicategories and representability for substructural intuitionistic modal logics

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Abstract

We have recently constructed sequent calculi for non-commutative linear intuitionistic versions of modal logics K, T, K4 and S4 that we have defined via semantics in terms of monoidal categories with a strong monoidal endofunctor, copointed endofunctor, semicomonad or comonad, optionally well-copointed/idempotent. Here we show that the designs of these sequent calculi are directly justified by equivalent semantics of these logics in terms of representable multicategories for suitable notions of (generalized) multicategory and representability: they are presentations of the free such representable multicategories.

We are interested in systematic design of proof systems with good metatheoretical properties for intuitionistic and substructural intutionistic modal logics. We take a logic to be specified by a semantics in terms of categories with some structure (e.g., Cartesian closed categories for intuitionistic logic with implication only or biclosed categories for the Lambek calculus). Such a semantics provides not just a notion of provability, but also notions of proof and proof equality.

In recent work, we considered substructural intuitionistic versions of the modal logics K, T, K4 and S4 defined by such semantics. The logics have no exchange, weakening or contraction; they have only a small selection of connectives: truth, conjuction and a box modality (and optionally implications). We constructed adequate and well-behaved sequent calculi for these logics. Here we show that these sequent calculi are directly justified by equivalent semantics in terms of representable multicategories for suitable notions of (generalized) multicategory and representability: they are presentations of the free such entities.

The defining semantics of our logics have as a model a monoidal category with an endofunctor on it (interpreting box), justifying inference rule $A \to B$ / $\Box A \to \Box B$, with some structure on the endofunctor. Specifically, we require the endofunctor to be strong monoidal, corresponding to axioms $I \cong \Box I$ and $\Box A \otimes \Box B \cong \Box(A \otimes B)$, for the basic system SIK. Optionally, it may come with additional data and equations that make it a strong monoidal copointed functor, semicomonad or comonad, with axioms $\Box A \to A$ or/and $\Box A \to \Box\Box A$, for systems SIT, SIK4, SIS4.

The free monoidal category with a strong monoidal endofunctor, copointed endofunctor, semicomonad or comonad can be straightforwardly presented as a Hilbert-like system with an equational theory stipulating which proofs we consider equal, corresponding to the equations of monoidal category, strong monoidal endofunctor etc.

Our design of sequent calculus counterparts for these Hilbert systems is based on sequents of the form $\Gamma \to A$ where Γ is a list of formulae with natural-number indices and A is a single formula. An indexed formula A^i intuitively stands for $\Box^i A$ and a list $A_1^{i_1}, \ldots, A_n^{i_n}$ stands for $\Box^{i_1} A_1 \otimes (\ldots \otimes (\Box^{i_n} A_n \otimes I)) \ldots)$, so the index on a formula in the antecedent of a sequent records its box-depth, i.e., the number of boxes that were decomposed to reach it (under the root-first, i.e., backward, reading of the proof).

The following are the inference rules of SIK.

$$\frac{}{A \to A}\ ax \qquad \frac{}{\to I}\ IR \quad \frac{\Delta, \Delta' \to C}{\Delta, I^i, \Delta' \to C}\ IL^i \qquad \frac{\Gamma_0 \to A \quad \Gamma_1 \to B}{\Gamma_0, \Gamma_1 \to A \otimes B}\ \otimes R \quad \frac{\Delta, A^i, B^i, \Delta' \to C}{\Delta, A \otimes B^i, \Delta' \to C}\ \otimes L^i$$

$$\frac{\Gamma \to A}{\Gamma^{1+} \to \Box A}\ \Box R \qquad \frac{\Delta, A^{i+1} \Delta' \to C}{\Delta, \Box A^i, \Delta' \to C}\ \Box L^i$$

(Γ^{i+} is the list of formulae obtained from Γ by incrementing all indices by i.)

The systems SIT, SIK4 and SIS4 additionally have the first, the second or both of the following rules of structural character:

$$\frac{\Delta, A^{i+j}, \Delta' \to C}{\Delta, A^{i+1+j}, \Delta' \to C}\ T^{i,j} \qquad \frac{\Delta, A^{i+2+j}, \Delta' \to C}{\Delta, A^{i+1+j}, \Delta' \to C}\ 4^{i,j}$$

In all these systems, the following cut rule is admissible:

$$\frac{\Gamma \to A \quad \Delta, A^i, \Delta' \to C}{\Delta, \Gamma^{i+}, \Delta' \to C}\ cut$$

The generating equations of the equational theories of proofs consist for all four sequent calculi of a number of η-conversions and a number of permutative conversion equations.

Each Hilbert system is equivalent to its sequent calculus counterpart in the following sense. For any Γ and A, the proofs of $\Gamma \to A$ in the sequent calculus are in an effective bijection with the proofs of $\langle\Gamma\rangle \to A$ in the Hilbert system (there are translations in both directions). (Here $\langle\Gamma\rangle$ is the translation of the list Γ of formulae with indices into a formula.) Moreover the bijection (the translations) respects the proof equality relations of the Hilbert system and the sequent calculus.

Because of the equivalence of the Hilbert systems and the sequent calculi not just the former but also the latter are sound and complete, in particular, also in the equational sense, wrt. the categorical semantics.

Strengthening the copointed endofunctor, semicomonad or comonad to a well-copointed endofunctor, "quasi-idempotent" semicomonad or idempotent comonad corresponds to identifying the two inequal proofs of $\Box\Box A \to \Box A$ (the outer or the inner box on the left can be deleted by axiom T). This semantics defines logics SIT*, SIK4* and SIS4* which have the same provability as SIT, SIK4, SIS4, but a coarser proof equality relation (identifying more proofs).

In terms of sequent calculus, this means that rules $T^{i,j}$, $T^{k,\ell}$ have to be equated for $i + j = k + \ell$ and ditto for $4^{i,j}$, $4^{k,\ell}$. Therefore, we can collapse them into rules T^{i+j} resp. 4^{i+j} only recording the sum of i,j.

System SIS4* further admits a simplification where the indices are just 0, 1 and + gets replaced with $\vee$ (maximum). This makes it close to Kavvos' [1] and Miranda Perea et al.'s [2] natural deduction and sequent calculi with 2-compartment antecedents for unboxed and boxed formulae in the setting with exchange (intuitionistic S4).

Each of the sequent calculi is a presentation of the free representable multicategory for some notion of multicategory and some notion of representability. We explain this for SIK.

A multicategory is in the case of SIK given by a set of objects $|\mathcal{C}|$ and, for every list Γ of objects weighted by natural numbers and an object A, a set of multimaps $\mathcal{C}(\Gamma, A)$. There is an identity multimap for every object, typed like the axiom rule, and a composition operation on multimaps, typed like the cut rule. The identity maps and composition are subject to equations of unitality and associativity and commutation of independent compositions.

A multicategory of this kind is precisely a generalized multicategory for a monad structure on the endofunctor of lists weighted by natural numbers, namely the monad arising from the canonical distributive law of the writer monad for the additive monoid of natural numbers over the list monad.

The condition of representability is most systematically stated as the requirements that the multicategory comes with operations typed like the rules IL, $\otimes L$ and $\Box L^i$, appropriately (multi)natural, and that these operations are invertible.

The rules IR, $\otimes R$ and $\Box R$ are interderivable with the inverses of IL, $\otimes L$ and $\Box L^i$ with the help of the axiom and cut.

The 2-category of monoidal categories with a strong monoidal endofunctor is 2-equivalent to the 2-category of representable multicategories for these notions of multicategory and representability.

For the logics SIT, SIK4, SI4, SIT*, SIK4*, SI4*, a multicategory must come with operations corresponding to rules $T^{i,j}$ and $4^{i,j}$ subject to their equations.

For logics where box is modelled via an endofunctor that need not be strongly monoidal a sequent form is needed where the antecedents are trees of a particular kind. The corresponding notion of generalized multicategory is defined by a monad structure on an endofunctor of trees. Moortgat [3] constructed sequent calculi for a number of nonassociative logics using a sequent form like this, but did not consider equality of proofs.

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References

[1] G. A. Kavvos (2020): *Dual-Context Calculi for Modal Logic. Log. Methods Comput. Sci.* 16(3), pp. 10:1–10:66, doi:10.23638/lmcs-16(3:10)2020.

[2] Favio Ezequiel Miranda-Perea, Lourdes del Carmen González Huesca & Pilar Selene Linares Arévalo (2022): *A Dual-Context Sequent Calculus for the Constructive Modal Logic S4. Math. Struct. Comput. Sci.* 32(9), pp. 1205–1233, doi:10.1017/s0960129522000378.

[3] Michael Moortgat (1997): *Categorial Type Logics.* In Johan van Benthem and Alice ter Meulen, editors: *Handbook of Logic and Language*, Elsevier, pp. 93–177, doi:10.1016/b978-044481714-3/50005-9. [6]