Classical Linear Logic is not Conservative over System F

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In memory of Harold Schellinx (1956–2025)

Abstract

We exhibit a formula in the language of system F which is provable in classical linear logic (when implication is interpreted as a linear one) but not in system F.

A system $\mathbb{S}_2$ is *conservative* over a system $\mathbb{S}_1$ when any formula, written in the language of $\mathbb{S}_1$ and provable in $\mathbb{S}_2$, is also provable in $\mathbb{S}_1$. This notion is commonly used when $\mathbb{S}_1$ is a sub-system of $\mathbb{S}_2$. However it can be generalized to the notion of *conservative translation* τ from $\mathbb{S}_1$ to $\mathbb{S}_2$: τ is conservative if, for any formula A of $\mathbb{S}_1$, $\tau(A)$ provable in $\mathbb{S}_2$ entails A provable in $\mathbb{S}_1$. One can also say, in this case, that τ is *complete* (or also *faithful*) as it is the converse implication with respect to the correctness of τ.

From H. Schellinx's work [Sch94, Chapter 2, Corollary 3.4], we know that Girard's translation defined by $\sigma(\forall X\, A) = \forall X\, \sigma(A)$ and $\sigma(A \to B) = ?\sigma(A)^\perp \parr \sigma(B)$ is correct and conservative from Girard's system F [Gir71] to Girard's classical linear logic [Gir87] (and this holds as well with the variant $\sigma(A \to B) = !\sigma(A) \multimap \sigma(B)$ targeting intuitionistic linear logic [GL87]). We consider here the translation $\tau(A \to B) = \tau(A)^\perp \parr \tau(B)$ (resp. $\tau(A \to B) = \tau(A) \multimap \tau(B)$) which is clearly *not* correct. We prove it is not conservative either, which implies that the embedding of intuitionistic linear logic into classical linear logic mapping $A \multimap B$ to $A^\perp \parr B$ is not conservative in the second-order case (even when restricted to the language containing only $\multimap$ and $\forall$).

In what follows, we implicitly consider $\multimap$ as part of the language of classical linear logic through $A \multimap B := A^\perp \parr B$. The uses of the τ embedding of system F into classical linear logic are also made implicit.

The question of the conservativity of *propositional* classical linear logic (LL) over *propositional* intuitionistic linear logic (ILL) has been widely studied [Sch91, Laf99, Lau18, Wu18]. While some sufficient conditions prove that almost any intuitionistic linear formula provable in LL is also provable in ILL, there are known counterexamples (which, as indicated by the sufficient conditions, require a simultaneous use of $\multimap$ and 0):

- $((X \multimap Y) \multimap 0) \multimap (X \multimap (0 \multimap Y') \multimap Z) \multimap Z$ (due to H. Schellinx [Sch91]);
- $((X \multimap Y) \multimap 0) \multimap (X \otimes \top)$ (due to Y. Lafont [Laf99]);
- $(((X \otimes \top) \mathbin{\&} (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y'))$ (due to O. Laurent [Lau18]);
- $(((\top \multimap 1) \multimap 0) \multimap 0) \multimap 1$ (due to J.-H. Wu [Wu18]);
- $((((0 \multimap 1) \multimap 1) \multimap 0) \multimap 0) \multimap 1$ (due to J.-H. Wu).

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We address here the case of second-order quantifiers by considering the formula Φ:

$$\boxed{\forall X \forall Y \forall Z \forall U \, (X \multimap ((Z \multimap Y) \multimap \forall A \, A) \multimap (\forall B \, (B \multimap Z \multimap U)) \multimap U)}$$

with the following properties:

- Φ is provable in second-order LL (thus in second-order Gentzen's LK): Theorem 1;
- Φ is not provable in system F nor Gentzen's LJ when $\multimap$ is interpreted as $\to$ (which entails that Φ itself is not provable in ILL): Theorem 2.

Thanks to the reversibility of the connectives $\multimap$ and $\forall$, the provability of Φ in all the considered systems (LL, ILL, system F, LJ, LK) is equivalent to the provability of:

$$X, (Z \multimap Y) \multimap \forall A \, A, \forall B \, (B \multimap Z \multimap U) \vdash U \, .$$

A linear proof of Φ requires to instantiate the $\forall A$ and $\forall B$ quantifiers thus contains a substitution of A and B by some LL formulas.

Theorem 1 (Provability in LL)
Φ is provable in LL *with proofs relying on any of the following five substitutions:*

$$\left[{}^{0}/_{A}, {}^{\top}/_{B}\right] \quad \left[{}^{X^\perp}/_{A}, {}^{Y^\perp}/_{B}\right] \quad \left[{}^{X^\perp}/_{A}, {}^{\top}/_{B}\right] \quad \left[{}^{0}/_{A}, {}^{Y^\perp}/_{B}\right] \quad \left[{}^{\perp}/_{A}, {}^{\top}/_{B}\right] .$$

Other substitutions are also possible.

Proof. To highlight the points where the intuitionistic constraint is broken, we use a two-sided presentation of LL proofs with appropriate rules for $\multimap$.

We consider two families of proofs of $X, (Z \multimap Y) \multimap \forall A \, A, \forall B \, (B \multimap Z \multimap U) \vdash U$:

$$\dfrac{\dfrac{\dfrac{\vdash B, Y \qquad \dfrac{\dfrac{}{Z \vdash Z}\, ax \quad \dfrac{}{U \vdash U}\, ax}{Z \multimap U, Z \vdash U} \multimap L}{B \multimap Z \multimap U, Z \vdash Y, U} \multimap L}{B \multimap Z \multimap U \vdash Z \multimap Y, U} \multimap R \qquad \dfrac{X, A \vdash}{X, \forall A \, A \vdash} \forall L}{X, (Z \multimap Y) \multimap \forall A \, A, B \multimap Z \multimap U \vdash U} \multimap L}{X, (Z \multimap Y) \multimap \forall A \, A, \forall B \, (B \multimap Z \multimap U) \vdash U} \forall L$$

and

$$\dfrac{\dfrac{\dfrac{X \vdash B, Y \qquad \dfrac{\dfrac{}{Z \vdash Z}\, ax \quad \dfrac{}{U \vdash U}\, ax}{Z \multimap U, Z \vdash U} \multimap L}{X, B \multimap Z \multimap U, Z \vdash Y, U} \multimap L}{X, B \multimap Z \multimap U \vdash Z \multimap Y, U} \multimap R \qquad \dfrac{A \vdash}{\forall A \, A \vdash} \forall L}{X, (Z \multimap Y) \multimap \forall A \, A, B \multimap Z \multimap U \vdash U} \multimap L}{X, (Z \multimap Y) \multimap \forall A \, A, \forall B \, (B \multimap Z \multimap U) \vdash U} \forall L$$

which differ on the position of X in the contexts.

The substitutions $\left[{}^{0}/_{A}, {}^{\top}/_{B}\right]$, $\left[{}^{X^\perp}/_{A}, {}^{Y^\perp}/_{B}\right]$, $\left[{}^{X^\perp}/_{A}, {}^{\top}/_{B}\right]$ and $\left[{}^{0}/_{A}, {}^{Y^\perp}/_{B}\right]$ allow us to close the first one while $\left[{}^{0}/_{A}, {}^{\top}/_{B}\right]$ and $\left[{}^{\perp}/_{A}, {}^{\top}/_{B}\right]$ allow us to close the second one. □

These proofs are also valid in LK.

While the substitution $\left[{}^{0}/_{A}, {}^{\top}/_{B}\right]$ relies on a propositional formula which happens to be a counterexample to the conservativity of propositional LL over propositional ILL, the alternative substitution $\left[{}^{X^\perp}/_{A}, {}^{Y^\perp}/_{B}\right]$ reveals a different behaviour by avoiding the use of the units $\top$ and 0 (remember 0 is necessary for propositional counterexamples).

Lemma 1
$X, (Z \to Y) \to \forall A\, A, Z \to U \vdash U$ is not provable in system F.

Proof. We look for a typing derivation for a term s in η-long normal form:

$$x : X, w : (Z \to Y) \to \forall A\, A, v : Z \to U \vdash s : U$$

There are two possibilities: $s = w\ (\lambda z^Z.r^Y)\ \{U\}$ and $s = v\ (w\ (\lambda z^Z.r^Y)\ \{Z\})$.

We thus want to prove that there is no η-long normal form r such that:

$$x : X, w : (Z \to Y) \to \forall A\, A, v : Z \to U, z : Z \vdash r : Y$$

If such an r exists, we assume selected one of smallest size, and we are going to exhibit a contradiction. We must have $r = w\ (\lambda z^Z.q^Y)\ \{Y\}$. But we then would have:

$$x : X, w : (Z \to Y) \to \forall A\, A, v : Z \to U, z : Z \vdash q : Y$$

with q strictly smaller than r, a contradiction. □

Theorem 2 (Non-Provability in System F)
Φ (with $\multimap$ is interpreted as $\to$) is not provable in system F.

Proof. Assuming we have a term t of type Φ in system F, we can build a typing derivation of system F with conclusion:

$$x : X, w : (Z \to Y) \to \forall A\, A, v : Z \to U \vdash t\ \{X\}\ \{Y\}\ \{Z\}\ \{U\}\ x\ w\ (\Lambda B.\lambda b^B.v) : U$$

This contradicts Lemma 1. □

Since all propositional connectives of LJ (as well as second-order $\exists$) are definable (with respect to provability) in the language of system F, Φ is not provable in LJ nor ILL.

To conclude with a focus on linear logic, while propositional LL happens to be "almost" conservative over propositional ILL, the counterexample Φ for the second-order setting seems to kill any hope for sufficient conditions ensuring conservativity as it uses the connectives $\forall$ and $\multimap$ only (and an LL proof based on second-order instantiations with negated atoms rather than any other connective).

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