From Phase Semantics to Base-extension Semantics (and back)

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Linear logic [14] admits a wide range of semantic presentations, reflecting the fact that its notion of consequence is sensitive not only to formula formation but also to the structural behaviour of contexts. Unlike classical and intuitionistic systems, where semantic clauses can often be stated over sets of valuations or Kripke-style structures with relatively familiar closure conditions, the semantics of linear logic must account for the controlled use of assumptions. This has led to several distinct semantic frameworks, each capturing different aspects of proofs and provability in the logic. Such semantic interpretations include, but are not limited to, coherent spaces [14], relational semantics [9, 11], Kripke-style semantics [4, 16], categorical models [1, 10] and game semantics [2, 3].

Among the standard semantics for linear logic is *phase semantics* [8, 12, 14]. In this semantics, formulas are interpreted over a *phase space* consisting of a commutative monoid (whose elements are called *phases*) and a fixed subset, with respect to which an orthogonality relation is defined. Each atom is interpreted as a *fact* – a set of phases that is equal to its closure induced by orthogonality. The interpretation is then inductively extended to formulas via the semantic clauses that mirror the resource-sensitive behaviour of the logical connectives. Phase semantics yields the standard soundness-and-completeness theorem for provability in linear logic via a canonical phase model built from contexts (i.e. multisets of formulas) [15, 19]. Phase semantics has, for instance, been used to prove decidability [7, 17, 18] and cut admissibility results [19, 20].

A rather different and much more recent approach is given by *base-extension semantics* [21, 22, 24, 25]. Base-extension semantics (B-eS) is a strand of *proof-theoretic semantics* [26, 27] that provides an alternative approach to the meaning of logical operators. It is based on inferentialism – a view according to which the meaning of expressions arises from their use in the system of inference and is defined by the rules of said system. The meaning is thus expressed in terms of proofs and provability: the concept of truth inherent to model-theoretic semantics is substituted with that of *proofs*.

In base-extension semantics, the starting point is a fixed atomic system – a set of atomic propositions together with a collection of inference rules over them – called a *base*. The characterisation of consequence in B-eS is given via a judgement called *support* (denoted as $\Vdash_{\mathcal{B}}$ for some base $\mathcal{B}$), which is inductively defined on the structure of formulas, with the base case (i.e. support of atoms) determined by *provability in a base*. In this sense, base-extension semantics replaces the usual assignment-based starting point of model-theoretic semantics by an inferential base, while retaining a compositional notion of semantic validity. For linear logic, however, this approach is not immediate. Because the logic is substructural, the shape of contexts cannot be treated as a secondary matter: multiplicity and structural behaviour of exponentials

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must already be reflected at the level of the semantic clauses, requiring B-eS to be capable of tracking the same structural distinctions as the proof system itself [5, 6, 13].

In this work, we establish an equivalence between these two semantics. At first sight, phase semantics and base-extension semantics are formulated in rather different terms. Nonetheless, both are intended to capture the same underlying proof-theoretic behaviour of linear consequence. In fact, the two exhibit certain similarities beyond both being provability semantics. For instance, in both of them, the set of operational units of the semantics has a structure to it: algebraic in the case of phase semantics – elements of the monoid in a phase space have a binary operation defined over them, and proof-theoretic in the case of B-eS – atoms in a base are related by inference rules. Furthermore, both begin with fixing the interpretation of $\perp$ – a chosen subset of the monoid for phase semantics, and a chosen atom for base-extension semantics. This makes it natural to ask whether they can be compared directly, and in particular whether one can establish a precise correspondence between them.

Such a comparison is useful for several reasons. First, an equivalence result shows that the two semantics not only define the same notion of validity, but are also intertranslatable on a structural level. This gives a correctness check on both frameworks: a semantics formulated by induction over proofs and a semantics formulated over phase spaces are able to express exactly the same inferences. Second, once a correspondence is established, results obtained in one setting can be transferred to the other. Completeness arguments, countermodel constructions, and structural properties that are transparent in phase semantics may then be reformulated in base-extension terms, while inductive arguments native to base-extension semantics may clarify the proof-theoretic content of phase model constructions. Lastly, it places base-extension semantics within the established semantic theory of linear logic, and thereby makes it possible to compare it systematically with other standard frameworks.

As the first step of constructing the equivalence, we observe that there is a systematic way of converting a phase space into a base and, similarly, constructing a phase space from a given base. To do so, we define the following maps, where $\mathbb{P}$ is the set of all phase spaces and $\mathbb{B}$ is the set of all bases:

$$(-)_{\mathbb{B}} : \mathbb{P} \rightleftarrows \mathbb{B} : (-)_{\mathbb{P}}$$

We proceed to give examples of the effect of these maps on a base and a phase space.

Example 1. Let $\mathcal{B}$ be a base over atoms $\{p, q, \perp\}$ with the following set of rules (where L is a multiset of atoms):

$$\left\{ \ \frac{}{\vdash p} \ \right\} \cup \left\{ \ \frac{L \vdash q}{L \vdash p} \ \right\}_L$$

Then $\mathcal{B}_{\mathbb{P}}$ is the monoid of multisets of linear logic formulas inductively defined on $\{p, q, \perp\}$ with a fixed subset given by $\perp_{\mathbb{P}} = \{\Gamma \mid \Gamma \Vdash_{\mathcal{B}} \perp\}$. Every fact of $\mathcal{B}_{\mathbb{P}}$ is of shape $\{\Gamma \mid \Gamma \Vdash_{\mathcal{B}} \phi\}$ for some formula ϕ. Below we list some examples of facts:

- $\{\Gamma \mid \Gamma \Vdash_{\mathcal{B}} p\} = \{\varnothing, \{p\}, \{q\}, \{p \mathbin{\&} \perp\}, \dots\}$
- $\{\Gamma \mid \Gamma \Vdash_{\mathcal{B}} q\} = \{\{q\}, \{p, p \multimap q\}, \dots\}$
- $\{\Gamma \mid \Gamma \Vdash_{\mathcal{B}} p \otimes q\} = \{\{q\}, \{p, q\}, \{p \otimes q\}, \dots\}$

Example 2. Consider $(\mathbb{N}, 0, +)$ as an additive monoid over natural numbers, and fix $\perp = \{0, 1\}$. It forms a phase space P whose set of facts is $D_P = \{\varnothing, \{0\}, \{0, 1\}, \mathbb{N}\}$. Hence, $P_{\mathbb{B}}$ is a base whose set of atoms is D_P. Below are some examples of the rules in $P_{\mathbb{B}}$ (where $\bot = \{0,1\}$ and L, L_1, L_2 and K are multisets of atoms) and the support statements they result in:

$$\frac{L \vdash \varnothing \qquad K;\{0\} \vdash \bot}{L, K \vdash \bot} \qquad\qquad \varnothing \Vdash_{P_{\mathbb{B}}} \{0\}$$

$$\frac{L_1 \vdash \bot \qquad L_2 \vdash \mathbb{N} \qquad K;\mathbb{N} \vdash \bot}{L_1, L_2, K \vdash \bot} \qquad\qquad \bot \otimes \mathbb{N} \Vdash_{P_{\mathbb{B}}} \mathbb{N}$$

$$\frac{L \vdash \{0\} \qquad L \vdash \bot \qquad K;\{0\} \vdash \bot}{L, K \vdash \bot} \qquad\qquad \{0\} \mathbin{\&} \bot \Vdash_{P_{\mathbb{B}}} \{0\}$$

$$\frac{L;\varnothing \vdash \bot}{L \vdash \mathbb{N}} \qquad\qquad \varnothing \multimap \bot \Vdash_{P_{\mathbb{B}}} \mathbb{N}$$

After defining the bidirectional maps, we introduce the structure-preserving maps between bases – *base morphisms* – and phase spaces – *phase space morphisms*. We construct a phase space morphism between P and $(P_{\mathbb{B}})_{\mathbb{P}}$ and show that it is an isomorphism. Next, we construct a base morphism between $\mathcal{B}$ and $(\mathcal{B}_{\mathbb{P}})_{\mathbb{B}}$ and show that it is an isomorphism up to *intersupport* (i.e. support in both directions). Formally, this amounts to the following two theorems:

Theorem 3. Every phase space P is isomorphic to $(P_{\mathbb{B}})_{\mathbb{P}}$.

Theorem 4. Every base $\mathcal{B}$ is isomorphic to $(\mathcal{B}_{\mathbb{P}})_{\mathbb{B}}$ up to intersupport.

Time permitting, we will discuss how the definitions of base and phase morphisms that allow us to prove the equivalence are very similar in spirit to the completeness proofs (w.r.t. linear logic) of both phase and base-extension semantics.

Further details of this work can be found in [23], including the discussion on extending the result to the equivalence of categories in the final section.

References

[1] Samson Abramsky. Computational interpretations of linear logic. *Theor. Comput. Sci.*, 111(1&2):3–57, 1993. doi:10.1016/0304-3975(93)90181-R.

[2] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *J. Symb. Log.*, 59(2):543–574, 1994. doi:10.2307/2275407.

[3] Samson Abramsky and Paul-André Melliès. Concurrent games and full completeness. In *14th Annual IEEE Symposium on Logic in Computer Science, Trento, Italy, July 2-5, 1999*, pages 431–442. IEEE Computer Society, 1999. doi:10.1109/LICS.1999.782638.

[4] Gerard Allwein and J. Michael Dunn. Kripke models for linear logic. *J. Symb. Log.*, 58(2):514–545, 1993. doi:10.2307/2275217.

[5] Victor Barroso-Nascimento, Ekaterina Piotrovskaya, and Elaine Pimentel. A proof-theoretic approach to the semantics of classical linear logic. In Clemens Kupke and Stefan Milius, editors,

Proceedings of the 41st Conference on the Mathematical Foundations of Programming Semantics, MFPS XLI, University of Strathclyde, Glasgow, UK, June 16-21, 2025, Electronic Notes in Theoretical Informatics and Computer Science. EpiSciences, April 2025. doi:10.46298/ENTICS.16888.

[6] Yll Buzoku. A proof-theoretic semantics for intuitionistic linear logic. *CoRR*, abs/2402.01982, 2024. doi:10.48550/ARXIV.2402.01982.

[7] Ugo Dal Lago and Simone Martini. Phase semantics and decidability of elementary affine logic. *Theoretical Computer Science*, 318(3):409–433, 2004.

[8] Abhishek De, Farzad Jafarrahmani, and Alexis Saurin. Phase semantics for linear logic with least and greatest fixed points. In Anuj Dawar and Venkatesan Guruswami, editors, *42nd IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2022, IIT Madras, Chennai, India, December 18-20, 2022*, LIPIcs, pages 35:1–35:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICS.FSTTCS.2022.35.

[9] Lorenzo Tortora de Falco. Obsessional experiments for linear logic proof-nets. *Math. Struct. Comput. Sci.*, 13(6):799–855, 2003. doi:10.1017/S0960129503003967.

[10] Thomas Ehrhard. Hypercoherences: A strongly stable model of linear logic. *Math. Struct. Comput. Sci.*, 3(4):365–385, 1993. doi:10.1017/S0960129500000281.

[11] Thomas Ehrhard. The scott model of linear logic is the extensional collapse of its relational model. *Theor. Comput. Sci.*, 424:20–45, 2012. doi:10.1016/J.TCS.2011.11.027.

[12] François Fages, Paul Ruet, and Sylvain Soliman. Linear concurrent constraint programming: Operational and phase semantics. *Inf. Comput.*, 165(1):14–41, 2001. doi:10.1006/INCO.2000.3002.

[13] Alexander V. Gheorghiu, Tao Gu, and David J. Pym. Proof-theoretic semantics for intuitionistic multiplicative linear logic. In Revantha Ramanayake and Josef Urban, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 32nd International Conference, TABLEAUX 2023, Prague, Czech Republic, September 18-21, 2023, Proceedings*, volume 14278 of *Lecture Notes in Computer Science*, pages 367–385. Springer, 2023. doi:10.1007/978-3-031-43513-3_20.

[14] Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987. doi:10.1016/0304-3975(87)90045-4.

[15] Jean-Yves Girard. On the unity of logic. *Ann. Pure Appl. Logic*, 59(3):201–217, 1993. doi:10.1016/0168-0072(93)90093-S.

[16] Joshua S. Hodas and Dale Miller. Logic programming in a fragment of intuitionistic linear logic. *Inf. Comput.*, 110(2):327–365, 1994. doi:10.1006/INCO.1994.1036.

[17] Yves Lafont. The undecidability of second order linear logic without exponentials. *The Journal of Symbolic Logic*, 61(2):541–548, 1996.

[18] Yves Lafont. The finite model property for various fragments of linear logic. *The Journal of Symbolic Logic*, 62(4):1202–1208, 1997.

[19] Mitsuhiro Okada. An introduction to linear logic: Expressiveness and phase semantics. In *Theories of types and proofs*, volume 2, pages 255–296. Mathematical Society of Japan, 1998.

[20] Mitsuhiro Okada. Phase semantic cut-elimination and normalization proofs of first-and higher-order linear logic. *Theoretical Computer Science*, 227(1-2):333–396, 1999.

[21] Thomas Piecha. Completeness in proof-theoretic semantics. *Advances in proof-theoretic semantics*, pages 231–251, 2016.

[22] Thomas Piecha, Wagner de Campos Sanz, and Peter Schroeder-Heister. Failure of completeness in proof-theoretic semantics. *J. Philos. Log.*, 44(3):321–335, 2015. doi:10.1007/

S10992-014-9322-X.

[23] Ekaterina Piotrovskaya. From phase semantics to base-extension semantics (and back), 2026. URL: https://arxiv.org/abs/2606.13855, arXiv:2606.13855.

[24] Tor Sandqvist. Base-extension semantics for intuitionistic sentential logic. *Log. J. IGPL*, 23(5):719–731, 2015. doi:10.1093/JIGPAL/JZV021.

[25] Peter Schroeder-Heister. Validity concepts in proof-theoretic semantics. *Synth.*, 148(3):525–571, 2006. doi:10.1007/S11229-004-6296-1.

[26] Peter Schroeder-Heister. Proof-Theoretic Semantics. In Edward N. Zalta and Uri Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2022 edition, 2022. URL: https://plato.stanford.edu/archives/win2022/entries/proof-theoretic-semantics/.

[27] Heinrich Wansing. The idea of a proof-theoretic semantics and the meaning of the logical operations. *Studia Logica*, 64(1):3–20, 2000.