The λ-calculus as a 2-dimensional clone

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In the recent decades, the multicategorical approach to term calculi and proof systems has been a topic of considerable interest. In this talk, we adopt this multicategorical approach to better understand the untyped λ-calculus presented as the uni-typed λ-calculus. It has the (multicategorical) structure of an *abstract clone*. Abstract clones are also referred to as *algebraic theories* in universal algebra or more simply, *clones*. Exhibiting the untyped λ-calculus as an algebraic theory was done by Hyland [4, 5]. More precisely, Hyland defines λ-*theories*, the algebraic framework for the semantics of λ-calculus, as *semi-closed* algebraic theories. Compared to cartesian closed categories, algebraic theories have a stronger connection with the syntax of the calculus. In particular, they can be presented without the need of an explicit product type, granting a more refined treatment of free variables and substitution. More crucially, this framework allows for a semantics of the untyped λ-calculus that does not require a 'typed universe', as it is instead the case for semantics *via* reflexive objects. In this setting, Hyland gave an elegant characterization of the untyped λ-calculus (and in particular, of terms considered equal up to β-reduction) as the *initial* λ-theory. In this way, the *denotational semantics* itself gets a definition by *universal property*: given a λ-theory $\mathcal{L}$, the *interpretation* of the λ-calculus (Λ) in $\mathcal{L}$ is given by the unique morphism of λ-theories:

$$\Lambda \xrightarrow{[\![-]\!]} \mathcal{L}$$

In this talk, we discuss ongoing work about both a refinement and a generalization of Hyland's approach:

- The refinement: we show how the equivalence between cartesian operads and clones is the categorical counterpart of the equivalence between the *multiplicative* and *additive* presentations of the untyped λ-calculus.
- The generalization: we describe the 2-dimensional structure of the untyped λ-calculus.

Building on the 2-operadic approach to λ-calculi of [7, 6], we propose a definition of 2-dimensional (extensional) λ-theory, as a closed 2-algebraic theory. Our main goal will then be to prove that the λ-calculus, equipped with its operational semantics (containing β-reduction and η-expansion), presents the *initial* 2-dimensional λ-theory. In this way, we hope to reach an *algebraic* definition of λ-term rewriting, given by universal property, bridging Hilken's 2-dimensional approach to term rewriting [2] with Hyland's proposal.

Ordinals and sequences We denote by $\mathbb{F}$ the skeleton of the category of finite sets whose objects are all the sets $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and whose morphisms are all the functions between these sets. We recall that $\mathbb{F}$ is cocartesian monoidal and that in particular, its tensor product is given by the addition: $[n] \otimes [m] := [n + m]$.

Clones and cartesian operads An *operad* $\mathcal{O}$ consists of

- For each $n \in \mathbb{N}$, a set $\mathcal{O}(n)$ of *multimorphisms*.
- For each $n, k_1, \ldots, k_n \in \mathbb{N}$, an associative *composition*

$$\begin{aligned} \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) &\to \mathcal{O}(\Sigma_{i=1}^{n} k_i) \\ (f, g_1, \ldots, g_n) &\mapsto f(g_1, \ldots, g_n) \end{aligned}$$

- An *identity* $\mathsf{id} \in \mathcal{O}(1)$ satisfying identity laws.

There is a notion of *morphism of operads*, making operads and morphisms of operads into a category Operad.

Let $n \in \mathbb{N}$. We denote the symmetric group of degree n by S_n. A *symmetric* operad is then given by:

- An operad $\mathcal{O}$.
- For each $n \in \mathbb{N}$ and each $\sigma \in S_n$, a map

$$\sigma \cdot - : \mathcal{O}(n) \to \mathcal{O}(n)$$

  satisfying axioms about its interaction with the composition of $\mathcal{O}$.

From the definition of a symmetric operad, one can get that of a cartesian operad by replacing permutations by morphisms of $\mathbb{F}$. In other words, a *cartesian* operad is given by:

- An operad $\mathcal{O}$.
- For each $n, m \in \mathbb{N}$ and each $f \in \mathbb{F}([n], [m])$ a map

$$f \cdot - : \mathcal{O}(n) \to \mathcal{O}(m)$$

  satisfying axioms which describe its interaction with the composition of $\mathcal{O}$.

A *morphism of cartesian operads* is a morphism of operads that preserves the cartesian structure. Cartesian operads and morphisms of cartesian operads form a category $\mathsf{COperad}$.

A *clone* [1, 5] is given by:

- For each $n \in \mathbb{N}$, a set $\mathcal{T}(n)$ of *multimorphisms*.
- For each $n \in \mathbb{N}^*$ and each $1 \leq k \leq n$, a *projection* $\mathbf{pr}_k^n \in \mathcal{T}(n)$ satisfying projection laws.
- For each $n, k \in \mathbb{N}$, an associative *composition*

$$\mathcal{T}(n) \times \mathcal{T}(k)^n \to \mathcal{T}(k)$$

There is a notion of *morphism of clones* which preserves projections and the composition. Clones and morphisms of clones form a category Clone. We now state a well-known fact that is relevant for what follows: the categories $\mathsf{COperad}$ and Clone are isomorphic.

We also remark that the structure of a clone $\mathcal{T}$ induces a map $f \cdot - : \mathcal{T}(n) \to \mathcal{T}(m)$, for any $f \in \mathbb{F}([n], [m])$. Given $h_1 \in \mathcal{T}(2)$ and $h_2 \in \mathcal{T}(n)$, we write $h_1 \circ (h_2, \mathsf{id})$ for the composite $h_1 \circ (\mathsf{i}_n^{n+1} \cdot h_2, \mathbf{pr}_{n+1}^{n+1})$, where $\mathsf{i}_n^{n+1} : [n] \to [n+1]$ is the canonical inclusion.

We use the term *algebraic theory* to denote both cartesian operads and clones. A *closed* algebraic theory is an algebraic theory equipped with a multimorphism $\mathsf{ev} \in \mathcal{T}(2)$, called the *evaluation* or *application*, such that the following map induced by precomposition:

$$\mathcal{T}(n) \xrightarrow{\mathsf{ev}\ \circ\ (-,\mathsf{id})} \mathcal{T}(n+1)$$

is invertible. A *morphism of closed algebraic theories* is a morphism of algebraic theories that preserves the structure.

An (extensional) λ-*theory* is then defined as a closed algebraic theory. We observe that we have two isomorphic categories of λ-theories, one arising from cartesian operads and the other from clones.

Multiplicative	**Additive**
$\dfrac{}{x \vdash_{\mathsf{m}} x}\,\mathsf{var}^{l}$	$\dfrac{1 \leq i \leq n+1}{x_1, \ldots, x_{n+1} \vdash_{\mathsf{a}} x_i}\,\mathsf{var}$
$\dfrac{\Gamma, x \vdash_{\mathsf{m}} M}{\Gamma \vdash_{\mathsf{m}} \lambda x.M}\,\mathsf{abs}^{l}$	$\dfrac{\Gamma, x \vdash_{\mathsf{a}} M}{\Gamma \vdash_{\mathsf{a}} \lambda x.M}\,\mathsf{abs}$
$\dfrac{\Gamma \vdash_{\mathsf{m}} M \qquad \Delta \vdash_{\mathsf{m}} N}{\Gamma, \Delta \vdash_{\mathsf{m}} MN}\,\mathsf{app}^{l}$	$\dfrac{\Gamma \vdash_{\mathsf{a}} M \qquad \Gamma \vdash_{\mathsf{a}} N}{\Gamma \vdash_{\mathsf{a}} MN}\,\mathsf{app}$
$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{m}} M \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash_{\mathsf{m}} f \cdot M}\,\mathsf{struct}$	$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{a}} M \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash_{\mathsf{a}} f \cdot M}\,\mathsf{struct}^{*}$
$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{m}} M \qquad (\Delta_i \vdash_{\mathsf{m}} N_i)_{i=1}^{n}}{\Delta_1, \ldots, \Delta_n \vdash_{\mathsf{m}} M\{N_1, \ldots, N_n / x_1, \ldots, x_n\}}\,\mathsf{sub}^{*}$	$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{a}} M \qquad (\Gamma \vdash_{\mathsf{a}} N_i)_{i=1}^{n}}{\Gamma \vdash_{\mathsf{a}} M\{N_1, \ldots, N_n / x_1, \ldots, x_n\}}\,\mathsf{sub}^{*}$

Figure 1: Multiplicative and additive arity assignment systems

The λ-calculus as a cartesian operad and as a clone One of our starting points is the traditional presentation of the untyped λ-calculus recalled in the following paragraph.

Let $\mathcal{V}$ be a countably infinite set of *variables*. *Terms* are defined by

$$\Lambda \ni M ::= x \in \mathcal{V} \mid \lambda x.M \mid MM$$

We consider terms up to renaming of bound variables. We denote as $M\{N_1, \ldots, N_n/x_1 \ldots, x_n\}$ the *n-ary substitution* operation on terms. Let $f \in \mathbb{F}([n],[m])$, let $z_1, \ldots, z_m$ be an arity context and $x_1, \ldots, x_n \vdash M$, we denote by $f \cdot M = M\{z_{f(1)}, \ldots, z_{f(n)}/x_1, \ldots, x_n\}$.

In fact, terms form an algebraic theory but this isn't clear from the traditional presentation of the untyped λ-calculus. This is why we extend this presentation in two different ways. This results in the *multiplicative* presentation and in the *additive* one.

To give these two presentations, we first need to define *arity contexts*: they are repetition-free, finite sequences of variables. They are denoted by $\Gamma, \Delta, \ldots$ We denote as Γ, Δ the concatenation of two contexts (that are assumed to be disjoint).

Secondly, we need to assign arity contexts to terms via a *multiplicative arity assignment system* and alternatively, via an *additive arity assignment system*. In Figure 1, we present the two arity assignment systems. We write $\Gamma \vdash M$ whenever we do not want to specify the particular assignment system we are referring to. If $\Gamma \vdash M$, we say that M is of *arity* Γ.

We can now refer to the multiplicative presentation which is the one including the $\vdash_{\mathsf{m}}$ relation. It *decomposes* the untyped λ-calculus in a linear logic fashion: the rules of the arity assignment are divided into a *linear fragment* (i.e. rules with l as a superscript), on the top of which we have a *structural rule* (i.e. struct which gives all weakened, contracted and exchanged proofs). The additive presentation (i.e. the one including the $\vdash_{\mathsf{a}}$ relation) has only three rules: var, abs and app. The structural rule is admissible which is why we include it in the additive arity assignment system of Figure 1 and tag it with the superscript $*$ like all the other admissible rules.

Given these two presentations (multiplicative and additive), the categorical structure of the untyped λ-calculus becomes clearer: derived terms from the multiplicative presentation have the structure of a cartesian closed operad (it is even an *initial* cartesian closed operad according to [5]) while derived terms from the additive presentation have that of a clone. We will explain how one can get the idea to define a cartesian operad or a clone from the derived terms. In fact, these algebraic theories are even closed.

What are the cartesian operad and the clone which we can build from the untyped λ-calculus? We start by showing that the linear fragment of the multiplicative presentation has the structure of a closed

Multiplicative	**Additive**
$\dfrac{\Gamma, x \vdash_{\mathsf{m}} M \qquad \Delta \vdash_{\mathsf{m}} N}{\Gamma, \Delta \vdash_{\mathsf{m}} \beta_{M,N} : (\lambda x.M)N \to M\{N/x\}}\ \beta$	$\dfrac{\Gamma, x \vdash_{\mathsf{a}} M \qquad \Gamma \vdash_{\mathsf{a}} N}{\Gamma \vdash_{\mathsf{a}} \beta_{M,N} : (\lambda x.M)N \to M\{N/x\}}\ \beta$
$\dfrac{\Gamma \vdash_{\mathsf{m}} M}{\Gamma \vdash_{\mathsf{m}} \eta_M : M \to \lambda x.Mx}\ \eta$	$\dfrac{\Gamma \vdash_{\mathsf{a}} M}{\Gamma \vdash_{\mathsf{a}} \eta_M : M \to \lambda x.Mx}\ \eta$
$\dfrac{\Gamma, x \vdash_{\mathsf{m}} \alpha : M \to N}{\Gamma \vdash_{\mathsf{m}} \lambda x.\alpha : \lambda x.M \to \lambda x.N}\ \mathsf{abs}^l$	$\dfrac{\Gamma, x \vdash_{\mathsf{a}} \alpha : M \to N}{\Gamma \vdash_{\mathsf{a}} \lambda x.\alpha : \lambda x.M \to \lambda x.N}\ \mathsf{abs}$
$\dfrac{\Gamma \vdash_{\mathsf{m}} \alpha : M \to M' \qquad \Delta \vdash_{\mathsf{m}} N}{\Gamma, \Delta \vdash_{\mathsf{m}} \alpha N : MN \to M'N}\ \mathsf{app}_1^l$	$\dfrac{\Gamma \vdash_{\mathsf{a}} \alpha : M \to M' \qquad \Gamma \vdash_{\mathsf{a}} N}{\Gamma \vdash_{\mathsf{a}} \alpha N : MN \to M'N}\ \mathsf{app}_1$
$\dfrac{\Gamma \vdash_{\mathsf{m}} M \qquad \Delta \vdash_{\mathsf{m}} \alpha : N \to N'}{\Gamma, \Delta \vdash_{\mathsf{m}} M\alpha : MN \to MN'}\ \mathsf{app}_2^l$	$\dfrac{\Gamma \vdash_{\mathsf{a}} M \qquad \Gamma \vdash_{\mathsf{a}} \alpha : N \to N'}{\Gamma \vdash_{\mathsf{a}} M\alpha : MN \to MN'}\ \mathsf{app}_2$
$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{m}} \alpha : M \to N \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash_{\mathsf{m}} f \cdot \alpha : f \cdot M \to f \cdot N}\ \mathsf{struct}$	$\dfrac{x_1, \ldots, x_n \vdash_{\mathsf{a}} \alpha : M \to N \qquad f : [n] \to [m]}{z_1, \ldots, z_m \vdash_{\mathsf{a}} f \cdot \alpha : f \cdot M \to f \cdot N}\ \mathsf{struct}^*$
$\dfrac{\Gamma \vdash_{\mathsf{m}} M}{\Gamma \vdash_{\mathsf{m}} \mathsf{id}_M : M \to M}\ \mathsf{id}$	$\dfrac{\Gamma \vdash_{\mathsf{a}} M}{\Gamma \vdash_{\mathsf{a}} \mathsf{id}_M : M \to M}\ \mathsf{id}$
$\dfrac{\Gamma \vdash_{\mathsf{m}} \alpha_1 : M \to P \qquad \Delta \vdash_{\mathsf{m}} \alpha_2 : P \to Q}{\Gamma, \Delta \vdash_{\mathsf{m}} \alpha_2 \circ \alpha_1 : M \to Q}\ \mathsf{comp}$	$\dfrac{\Gamma \vdash_{\mathsf{a}} \alpha_1 : M \to P \qquad \Gamma \vdash_{\mathsf{a}} \alpha_2 : P \to Q}{\Gamma \vdash_{\mathsf{a}} \alpha_2 \circ \alpha_1 : M \to Q}\ \mathsf{comp}$

Figure 2: Multiplicative and additive presentations of term rewritings.

cartesian operad Λ_m. We set $\Lambda_m(n) := \{M \in \Lambda \mid x_1, \ldots, x_n \vdash_{\mathsf{m}} M\}$. The canonicity of proof trees yields the composition of Λ_m. When terms are taken up to β-reduction and η-expansion, one can show that Λ_m is closed: pick

$$\dfrac{\dfrac{}{x \vdash_{\mathsf{m}} x}\ \mathsf{var}^l \qquad \dfrac{}{y \vdash_{\mathsf{m}} y}\ \mathsf{var}^l}{x, y \vdash_{\mathsf{m}} xy}\ \mathsf{app}^l$$

as $\mathrm{ev} \in \Lambda_m(2)$.

The additive presentation of the untyped λ-calculus has the structure of a closed clone Λ_a. We set $\Lambda_a(n) := \{M \in \Lambda \mid x_1, \ldots, x_n \vdash_{\mathsf{a}} M\}$. Fix an enumeration of variables $(x_i)_{i \in \mathbb{N}}$ and define the projections as all the proof trees

$$\dfrac{1 \leq i \leq n+1}{x_1, \ldots, x_{n+1} \vdash_{\mathsf{a}} x_i}\ \mathsf{var}$$

As previously, the composition of Λ_a is given by the admissible sub^* rule. Again, when terms are taken up to β-reduction and η-expansion, one can show that Λ_a is closed. By taking terms only up to β-reduction, we would only have that the bijection from $\Lambda_a(n)$ to $\Lambda_a(n+1)$ is a retraction. That is the condition for a clone to be *semi-closed* [4]. Lifting this condition to a 2-dimensional setting would mean asking for a *lax retraction* in Cat. Instead, we ask for a stronger notion: an adjunction. This is because it induces a richer equational theory on rewrites.

Towards the second dimension Our goal is to generalize Hyland's approach to the context of 2-dimensional category theory. More precisely, we want to prove that the λ-calculus, equipped by its rewriting theory, gives rise to an *initial* closed 2-dimensional algebraic theory. First, we need to perform a *change of enrichment base*, passing from Set-enriched algebraic theories to Cat-enriched algebraic theories. This means that homsets are replaced by homcategories and then, as customary in 2-dimensional category theory, we get two notions of composition: a *vertical* one and an *horizontal* one. In a Cat-algebraic theory, we have an additional dimension of structure: besides objects (called 0-cells) and morphisms (called 1-cells), we also have morphisms between morphisms (called 2-cells). It is well-established that 2-dimensional categorical structures are the appropriate mathematical setting to get a non-trivial semantics for program execution [9, 2, 3]. In particular, both Cat-operads and Cat-clones are already been used for the semantics of λ-calculi [6, 8]. A *morphism of Cat-algebraic theory* is then a morphism that preserves the structure *strictly*. In restricting to strict morphisms we follow [8]. This choice is more convenient when we want to present term calculi as appropriate free constructions.

We extend the definition of λ-theories to the second dimension. We say that a *closed* Cat-algebraic theory is a Cat-algebraic theory $\mathcal{L}$ equipped with a 1-cell $\mathsf{ev} \in \mathcal{L}(2)$, called *evaluation* or *application*, such that the functor induced by precomposition:

$$\mathcal{L}(n) \xrightarrow{\mathsf{ev}\ \circ\ (-,\mathsf{id})} \mathcal{L}(n+1)$$

has a right adjoint, denoted as $\lambda(-)$. A λ-*rewriting* theory is then a closed Cat-algebraic theory. As for the standard definition of λ-theory, we have that the two adjoint maps correspond, respectively, to application with a variable Mx and to abstraction $\lambda x.M$. The *unit* of the adjunction gives rise to the η-expansion, while the counit to β-reduction. Hence, β and η equalities are now replaced by directed rewritings.

A morphism of λ-rewriting theories consists of a morphism of Cat-algebraic theories that preserves the closed structure strictly.

We extend the notion of *arity* assignment to reduction paths (that we call *rewrites*), some of the rules are shown in Figure 2. We can extend the notion of substitution to rewrites and it is to this substitution we refer to when we write $f \cdot \alpha$: more precisely, it is defined as $\alpha\{z_{f(1)}, \dots, z_{f(n)}/x_1, \dots, x_n\}$. In this way, we can equip both $\Lambda_{\mathsf{m}}(n)$ and $\Lambda_{\mathsf{a}}(n)$ with the structure of a category: objects are terms of arity $x_1, \dots, x_n$ while morphisms are rewrites between those terms (β-reduction and η-expansion are morphisms). In particular, we need to take composite rewritings up to associativity and identity laws for vertical composition.

Equational theory by tridimensional rewriting What are the equations that rewrites have to satisfy in order to equip the λ-calculus with the structure of λ-rewriting theory? As already pointed out in the literature [3, 6], in order to define a calculus as a 2-dimensional structure, it is necessary to prove that *substitution* behaves well wrt composition of rewrites. The standard way to do so is to impose *permutation equivalence*. This notion is part of the classic theory of λ-calculus, as it is known that a rewrite between two terms is permutationally equivalent to a *standard* rewrite. The intuition being that standard rewrites play the role of normal forms. *Standardization* is then the theorem stating that if two terms are related by a rewrite, then there exist a *standard* rewrite relating them.

However, there is more than just permutation equivalence. the closed structure give rise to additional equations. If we spell out what it means for a Cat-algebraic theory to be closed, we get the following natural isomorphism of hom-categories:

$$\begin{tikzcd} \Lambda(n)(M, \lambda x.N) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \Lambda(n+1)(Mx, N) \arrow[ll, bend left] \end{tikzcd}$$

We wrote the isomorphism as an adjunction because we want to suggest the idea of using it to build a rewriting system on rewrites themselves. The unit of the adjunction would give rise to the extensional rule, while the counit to the conversion rule. Hence, our goal is to investigate further this idea, by defining and studying the properties of this rewriting on rewrites. In doing so, we hope to achieve the final goal to prove that the λ-calculus together with is operational semantics, up to the equational theory induced by this rewriting, present the initial 2-dimensional λ-theory.

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