AN ALGEBRAIC VIEW ON THE LINEAR LAMBDA CALCULUS

ARTURO DE FAVERI

In his pioneering papers, Alonzo Church introduced the λ-calculus together with a restriction that forbids erasure, known as the λI-calculus, in which every abstraction must bind at least one occurrence of a variable. Another well-known and extensively studied fragment is the linear λ-calculus, where each abstraction is required to bind *exactly* one occurrence of a variable. The linear λ-calculus, along with its planar and braided variants [5, 6], has inspired a number of works, for example in connection with geometric combinatorics [12, 11]. Furthermore, the class of linear λ-terms known as finite hereditary permutations plays a fundamental role in the study of invertible λ-terms modulo λ-theories [4, 2]. The aim of the present note is to report on ongoing work to investigate the linear fragment of the λ-calculus using the tools of categorical universal algebra, adapting the techniques and ideas developed by Hyland in his papers [8, 7], to the linear setting of operads and monoidal categories. We start by investigating algebraic models of the linear λ-calculus through the notion of *semiclosed operad* and we prove that each of such models gives rise to a *linear reflexive object* in a closed monoidal category. Inspired by the notion of λ-algebra, we define a category of *linear* λ-algebras, and we prove that this category is equivalent to that of semiclosed operads. Therefore, combining the two results, we have that:

(1) every linear λ-algebra gives rise to a linear reflexive object in a closed monoidal category;
(2) conversely, a linear reflexive in a closed monoidal category corresponds to a semiclosed operad and hence to a linear λ-algebra.

1. Linear lambda calculus

A λ-term M is said to be *linear* if every variable, free or bound, has exactly one occurrence in t. The following are examples of linear combinators:

$$\mathbf{I} = \lambda x.x \quad \mathbf{B} = \lambda fgx.f(gx) \quad \mathbf{C} = \lambda fxy.fyx \quad \mathbf{1} = \lambda xy.xy.$$

Definition 1.1 (cf. [1, Def. 9.3.1])**.** A **BCI**-*algebra* is an applicative structure with three constants $\mathbf{b}, \mathbf{c}, \mathbf{i} \in A$ satisfying the following identities:

(1) $\mathbf{b}xyz = x(yz)$;
(2) $\mathbf{c}xyz = xzy$;
(3) $\mathbf{i}x = x$.

We will denote by **BCI** the category of **BCI**-algebras and **BCI**-algebras homomorphisms.

We introduce the following notion whose meaning will be clearer later. It is inspired by the notion of λ-algebra due to Curry [1, Cor. 7.3.15]. An axiomatisation equivalent to Curry's is given by Selinger [10, Sect. 2].

Definition 1.2. A *linear λ-algebra* is a **BCI**-algebra that satisfies the following additional identities:

L1: $\mathbf{bii} = \mathbf{i}$;
L2: $\mathbf{bi}(\mathbf{c}xy) = \mathbf{c}xy$;
L3: $\mathbf{bi}(\mathbf{b}xy) = \mathbf{b}xy$;
L4: $\mathbf{c}(\mathbf{c}(\mathbf{bb}x)y)z = \mathbf{c}x(yz)$;
L5: $\mathbf{c}(\mathbf{b}(\mathbf{b}x)y)z = \mathbf{bc}x(\mathbf{c}yz)$;
L6: $\mathbf{b}(\mathbf{b}xy)z = \mathbf{b}x(\mathbf{b}yz)$;
L7: $\mathbf{c}(\mathbf{c}(\mathbf{bc}x)y)z = \mathbf{c}(\mathbf{c}xz)y$;
L8: $\mathbf{c}(\mathbf{b}(\mathbf{c}x)y)z = \mathbf{b}(xz)y$;
L9: $\mathbf{b}(\mathbf{c}xy)z = \mathbf{c}(\mathbf{b}xz)y$.

We will denote by $\lambda_{lin}\mathbf{Alg}$ the category of linear λ-algebras and linear λ-algebras homomorphisms.

2. Operads

We refer the reader to [9] on operads and their algebras.

Definition 2.1. An *operad* P is a sequence of sets $\{P(n) : n \in \mathbb{N}\}$ together with an element $1 \in P(1)$ called the *identity* and *composition* functions

$$\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n),\ (f, g_1, \ldots, g_n) \mapsto f \circ (g_1, \ldots, g_n),$$

satisfying, writing $f(g_1, \ldots, g_n)$ in place of $f \circ (g_1, \ldots, g_n)$, the following identities:

(1) $f(1, \ldots, 1) = f$;
(2) $1(f) = f$;
(3) $f(g_1(h^1_1, \ldots, h^1_{k_1}), \ldots g_n(h^n_1, \ldots, h^n_{k_n})) = f(g_1, \ldots, g_n) \circ (h^1_1, \ldots h^1_{k_1}, \ldots, h^n_1, \ldots, h^n_{k_n})$.

Moreover, each $P(n)$ is endowed with a right action of S_n, the symmetric group on n elements, $(\sigma, f) \mapsto f\sigma$, satisfying some compatibility conditions ([9, Section I.1.2]).

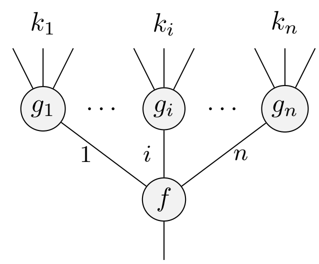

Figure 1. Operadic composition: $f \in P(n)$ receives in input $g_1 \in P(k_1), \ldots, g_n \in P(k_n)$.

Example 2.2. Let $\mathcal{C}$ be a symmetric monoidal category, and let x be an object of $\mathcal{C}$. Let $x^{\otimes n}$ denote the n-fold monoidal product $x \otimes \cdots \otimes x$. Then $P(n) := \mathcal{C}(x^{\otimes n}, x)$ is an operad called *endomorphism operad* of x.

Given a group G that acts on the right on a set X and on the left on a set Y we denote by $X \times_G Y$ the quotient $X \times Y/\sim$ such that $(xg, y) \sim (x, gy)$ for every $x \in X, y \in Y$ and $g \in G$.

Remark 2.3. Every operad P gives rise to a category $\mathcal{P}$ in the following way. The objects of $\mathcal{P}$ are the natural numbers in $\mathbb{N}$. For every $n, m \in \mathbb{N}$, we define (as illustrated in Figure 2)

$$\mathcal{P}(n, m) := \coprod_{k_1 + \cdots + k_m = n} (P(k_1) \times \cdots \times P(k_m)) \times_{(S_{k_1} \times \cdots \times S_{k_m})} S_n \tag{1}$$

where we observe that $S_{k_1} \times \cdots \times S_{k_m}$ acts on S_n on the left by multiplication by the image of the natural inclusion of $S_{k_1} \times \cdots \times S_{k_m}$ into S_n. With products given by $n \otimes m := n + m$ for every $n, m \in \mathbb{N}$, and with unit 0, $\mathcal{P}$ becomes a symmetric monoidal category. This category is an example of a PROP (*pro*duct and *p*ermutation category). See [9, Section I.1.7].

Definition 2.4. Let P be an operad. A P-*algebra* is a set X together with maps $\cdot : P(n) \times X^n \to X$ for all $n \in \mathbb{N}$ that satisfy:

(1) $f(g_1, \ldots, g_n) \cdot (x^1_1, \ldots x^1_{k_1}, \ldots, x^n_1, \ldots, x^n_{k_n}) = f \cdot (g_1 \cdot (x^1_1, \ldots, h^1_{k_1}), \ldots g_n \cdot (x^n_1, \ldots, x^n_{k_n}))$;
(2) $1 \cdot x = x$;

and a compatibility condition for the action of the symmetric group. Given an operad P, we write $\mathrm{Alg}(P)$ for the category of algebras of P.

Remark 2.5. Let P be an operad and let $\mathcal{P}$ be the corresponding monoidal category. The category of P-algebras X is equivalent to the category of functors $F : \mathcal{P} \to \mathbf{Set}$ preserving the finite monoidal product strictly. Given such F, the underlying set of the corresponding P-algebra X is given by $F(1)$.

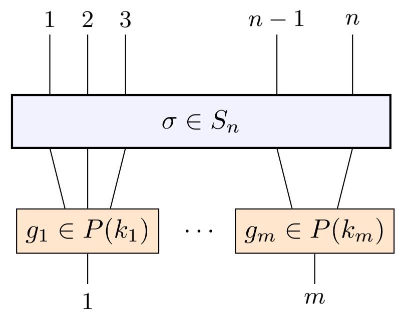

FIGURE 2. An element of $\mathcal{P}(n, m)$.

3. The operad of linear lambda terms

For every $n \in \mathbb{N}$, let $L(n)$ be the set of linear λ-terms modulo β-equivalence with n free variables. We think of an element $M \in L(n)$ as a typing judgment $x_1, \ldots, x_n \vdash M$.

Lemma 3.1. *The sequence of sets $\{L(n) : n \in \mathbb{N}\}$ is an operad with identity (the equivalence class of) $x \vdash x \in L(1)$ and composition given by substitution.*

Proposition 3.2. *There is a fully faithful functor $G : \mathrm{Alg}(L) \to \mathbf{BCI}$.*

The idea is that every L-algebra A can be given a **BCI**-algebra structure GA as follows:
- $ab := xy \cdot (a, b)$, where $xy \in L(2)$ and $(a, b) \in A^2$;
- $\mathbf{b} := \mathbf{B} \cdot ()$;
- $\mathbf{c} := \mathbf{C} \cdot ()$;
- $\mathbf{i} := \mathbf{I} \cdot ()$, where $\mathbf{B}, \mathbf{C}, \mathbf{I} \in L(0)$ are as defined above and $()$ is the empty sequence.

We now characterise the essential image of the functor G. This allows us to see L-algebras as a subcategory of **BCI**. For a similar construction in the case of the full λ-calculus see [10]. Given a linear λ-algebra A, we can consider the *linear* terms $A_{lin}[x]$ generated by the grammar $p, q ::= x \mid a \mid pq$ with x variable and $a \in A$. Let p be a term in $A_{lin}[x]$ with free variable x. We can encode the λ-abstraction in A through a function $\lambda^* x.(-) : A_{lin}[x] \to A, t \mapsto \lambda^* x.t$, defined by induction on the structure of t:
- $\lambda^* x.x := \mathbf{i}$;
- $\lambda^* x.pq := \mathbf{c}(\lambda^* x.p)q$, if x is free in p;
- $\lambda^* x.pq := \mathbf{b}p(\lambda^* x.q)$, if x is free in q.

Inductively, we have thus a way to interpret every linear λ-term $M \in L(n)$ in $A_{lin}[x_1, \ldots, x_n]$. We denote this interpretation by $\bar{M}$. We now endow A with a structure of L-algebra FA as follows. For every $M = M(x_1, \ldots, x_n) \in L(n)$, we define $M \cdot (a_1, \ldots, a_n)$ as the interpretation in A of the closed term $\bar{M}[a_1/x_1, \ldots, a_n/x_n]$ in the syntax defined above. This definition does not depend on the β-equivalence class of $M \in L(n)$. The base step is that if $(\lambda x.M)N \to_\beta M[N/x]$ then $(\lambda x.M)N \cdot (a_1, \ldots a_n) = M[N/x] \cdot (a_1, \ldots a_n)$; this follows from the axioms of **BCI**-algebra. Moreover, axioms (L1)-(L9) guarantee that the action is invariant under λ-abstraction.

Proposition 3.3. *The functors $F : \lambda_{lin}\mathbf{Alg} \to \mathrm{Alg}(L)$ and $G : \mathrm{Alg}(L) \to \lambda_{lin}\mathbf{Alg}$ are the left and right adjoint, respectively, of an adjoint equivalence between the category of linear λ-algebras and the category $\mathrm{Alg}(L)$ of algebras for the operad L.*

4. Monoidal closed structure on presheaves

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. The category of presheaves $\mathrm{PSh}(\mathcal{C})$ on $\mathcal{C}$ canonically inherits a monoidal category structure via *Day's convolution product* [3]: given $F, G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ we define

$$(F \otimes G)(c) := \int^{x,y \in \mathcal{C}} \mathcal{C}(c, x \otimes y) \times F(x) \times G(y) \tag{2}$$

for every $c \in \mathcal{C}$. The unit of this product is given by the representable presheaf $J(c) := \mathcal{C}(c, I)$ for every $c \in \mathcal{C}$. This product makes the Yoneda embedding $y : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ strong monoidal, i.e., $\mathcal{C}(-, a) \otimes \mathcal{C}(-, b) \simeq \mathcal{C}(-, a \otimes b)$ for every $a, b \in \mathcal{C}$. Finally, this structure is actually monoidal *closed* with internal exponentials given by

$$(F \multimap G)(c) = \int_{x \in \mathcal{C}} \mathbf{Set}(F(x), G(c \otimes x)) \tag{3}$$

for every $c \in \mathcal{C}$.

We recall that a symmetric monoidal category $\mathcal{C}$ is closed if for all $b \in \mathcal{C}$ the tensor product functor $- \otimes b : \mathcal{C} \to \mathcal{C}$ has a right adjoint $b \multimap - : \mathcal{C} \to \mathcal{C}$: for all $a, b, c \in \mathcal{C}$ there is a natural isomorphism

$$\mathcal{C}(a \otimes b, c) \simeq \mathcal{C}(a, b \multimap c).$$

If we take $F = \mathcal{C}(-, a)$, we obtain $(\mathcal{C}(-, a) \multimap G)(c) = G(c \otimes a)$. Taking in addition $G = \mathcal{C}(-, b)$ we get

$$(\mathcal{C}(-, a) \multimap \mathcal{C}(-, b))(c) = \mathcal{C}(c \otimes a, b). \tag{4}$$

Now, let P be an operad, and let $\mathcal{P}$ be the associated PROP. As $(\mathcal{P}, \otimes, I)$ is symmetric monoidal, $(\mathrm{PSh}(\mathcal{P}), \otimes, \multimap, J)$ is a symmetric closed monoidal category.

A simple yet fundamental result, consequence of the Yoneda lemma and the fact that the Yoneda embedding is strong monoidal, is the following.

Lemma 4.1. *Let P be an operad and $\mathcal{P}$ be the associated* PROP*. Then P is isomorphic to the operad of endomorphisms of the presheaf $\mathcal{P}(-, 1)$.*

5. Semiclosed operads

Definition 5.1 (cf. [8, Def. 3.1])**.** We say that an operad P is *semiclosed* if for every $n \in \mathbb{N}$ there is a retraction $P(n+1) \triangleleft P(n)$ natural in n and compatible with the compositions.

Lemma 5.2. *The operad L defined above is semiclosed and is in fact the initial semiclosed operad.*

Definition 5.3. Let $(\mathcal{C}, \otimes, \multimap, I)$ be a symmetric closed monoidal category. An object $x \in \mathcal{C}$ is a *linear reflexive object* if there are morphisms $s : (x \multimap x) \to x$ and $r : x \to (x \multimap x)$ such that $r \circ s = \mathrm{id}_{x \multimap x}$.

Remark 5.4. Let $(\mathcal{C}, \otimes, \multimap, I)$ be a symmetric closed monoidal category. Let $x \in \mathcal{C}$ be a linear reflexive object. Then the endomorphism operad P on x is semiclosed:

$$P(n+1) = \mathcal{C}(x^{\otimes n} \otimes x, x) \simeq \mathcal{C}(x^{\otimes n}, x \multimap x) \ \triangleleft\ \mathcal{C}(x^{\otimes n}, x) = P(n).$$

Specialising (4) to $\mathcal{C} = \mathrm{PSh}(\mathcal{P})$, $a = b = 1$ and $c = n$ we get the following.

Proposition 5.5. *Let P be a semiclosed operad and let $\mathcal{P}$ be its associated* PROP*. Then $\mathcal{P}(-, 1)$ is linear reflexive in the closed monoidal category* $\mathrm{PSh}(\mathcal{P})$*.*

Putting everything together, we obtain the following.

Theorem 5.6. *Let P be an operad, with associated* PROP *$\mathcal{P}$. Let U be the presheaf $\mathcal{P}(-, 1)$. Then P is isomorphic to the endomorphism operad of U in the closed monoidal category* $\mathrm{PSh}(\mathcal{P})$*. Moreover, P is semiclosed iff U is linear reflexive in* $\mathrm{PSh}(\mathcal{P})$*.*

6. The main result

In this last section we show the most important result of this note: every linear λ-algebra corresponds to a linear reflexive object in a closed monoidal category. To this end, we first show that the category of semiclosed operads is equivalent to the category of algebras for the initial semiclosed operad L.

If P is a semiclosed operad, then $P(0)$ is a L-algebra as follows. Firstly, $P(0)$ is a P-algebra with action $\cdot : P(n) \times P(0)^n \to P(0)$ given by composition. Next, as L is initial, we have a homomorphism of operads $L \to P$, that, specialised to $L(n) \to P(n)$, gives an action $L(n) \times P(0)^n \to P(0)$, i.e., a L-algebra structure on $P(0)$.

Proposition 6.1. *The map $P \mapsto P(0)$ is a functor from the category of semiclosed operads to the category of L-algebras.*

Conversely, every linear λ-algebra A gives rise to a semiclosed operad. For every $n \in \mathbb{N}$, let

$$\mathbf{1}_n := \lambda^* x x_1 \ldots x_n . x x_1 \cdots x_n \in A$$

and let $P_A(n) := \{a \in A : \mathbf{1}_n a = a\}$. Note that for $n = 0$, $P_A(0) = A$. Let the identity of the operad be $\mathbf{i} \in P_A(1)$ and define the composition maps as follows:

$$\begin{aligned} \circ : P_A(n) \times P_A(k_1) \times \cdots \times P_A(k_n) &\to P_A(k_1 + \cdots + k_n) \\ (a, b_1, \ldots, b_n) &\mapsto \lambda^* x_1^1 \ldots x_{k_n}^n . a(b_1 x_1^1 \cdots x_{k_1}^1) \cdots (b_n x_1^n \cdots x_{k_n}^n) \end{aligned}$$

The above definition gives indeed an operad $\{P_A(n) : n \in \mathbb{N}\}$. Moreover, the operad P_A is semiclosed, as we now show. Fix $n \in \mathbb{N}$. First of all $P_A(n+1) \subseteq P_A(n)$ as $\mathbf{1}_n(\mathbf{1}_{n+1} a) = a$ for every $a \in P_A(n+1)$. Let $r_n : P_A(n) \to P_A(n+1)$ be given by $r_n(a) = \mathbf{1}_{n+1} a$. This map is well-defined since $\mathbf{1}_{n+1}$ is idempotent; moreover, for every $a \in P_A(n+1)$, $r_n(a) = \mathbf{1}_{n+1} a = a$ as desired.

Proposition 6.2. *The map $A \mapsto P_A$ is a functor from the category of linear λ-algebras to the category of semiclosed operads.*

The two last propositions are part of a stronger result, that has to be read modulo the equivalence of Proposition 3.3.

Theorem 6.3. *The pair of functors $A \mapsto P_A$ and $P \mapsto P(0)$ are the left and right adjoint, respectively, of an adjoint equivalence between the category of L-algebras and the category of semiclosed operads.*

A linear λ-algebra A corresponds to a semiclosed operad P_A by Theorem 6.3. Up to considering the associated PROP $\mathcal{P}_A$, this operad gives a closed monoidal category $\mathrm{PSh}(\mathcal{P}_A)$ and a linear reflexive object $\mathcal{P}_A(-,1)$ in it by Theorem 5.6. We therefore state the main result.

Theorem 6.4. *Let A be a linear λ-algebra. Then P_A is a semiclosed operad and $\mathcal{P}_A(-,1)$ is a linear reflexive object in $\mathrm{PSh}(\mathcal{P}_A)$. Conversely, if $\mathcal{C}$ is a closed monoidal category and x a linear reflexive object in $\mathcal{C}$, then $P(n) := \mathcal{C}(x^{\otimes n}, x)$ is a semiclosed operad and $P(0)$ a L-algebra, or, equivalently, a linear λ-algebra.*

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CNRS, Université Paris Cité, IRIF, 8 Place Aurélie Nemours, 75013 Paris, France