The Reasonable Effectiveness of Linear Logic in Quantitative Methods

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Linear Logic borrows heavily from arithmetic intuition, most conspicuously reflected in Girard's terminology for connectives—*additives* and *multiplicatives*. The reason is quite simple, and it is suggested by the shape of the rules characterizing the two pairs, specifically in the introduction rules for additives, which 'distribute' the context Γ equally across the two premises:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \tag{$\wedge_R$}$$

However, (real) numbers with addition and multiplication do not literally form an algebraic model of linear logic, since additive connectives turn out to be necessarily *cartesian*. We then wonder—*what does it take for additives to actually be addition?*

Upon closer look, we realize the culprit hides in plain sight, and yet completely invisible. The rules for additives in sequent calculus merely internalize meta-level connectives—the implicit connective hiding in the empty space between sequents:

$$\frac{\Gamma \vdash A \qquad \text{and} \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \tag{1}$$

This suggests that for additives 'to be addition' the empty space in $\wedge_R$ should be thought as a sum itself:

$$\frac{\Gamma \vdash A \qquad + \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \tag{2}$$

Doing so necessarily promotes sequents too to have an intended *quantitative* meaning, rather than their usual *qualitative* ('boolean') one.

This leaves us with an 'enriched' notion of sequent calculus which is up to the task of capturing the logical structure of the reals.

The reals as an alethic theory. We call *alethic theory* the meta-level object we formulate our calculi with. Usually, this object is $(\mathbf{Prop}, \wedge)$, that is truth values with the sole algebraic structure of conjunction. Instead, for *quantitative calculi* we consider $[0, \infty]$ with the algebraic structure from Table 1.

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	polarity	additive	multiplicative
duality $a^* := 1/a$	**positive**	$\bot := 0$ $a + b$	$\mathbf{1} := 1$ $a \otimes b$ $\infty \otimes 0 := 0$
	negative	$\top := \infty$ $a +^* b$	$\mathbf{1} := 1$ $a \otimes^* b$ $\infty \otimes^* 0 := \infty$

Table 1: *Algebraic structure on* $[0, \infty]$*. Red and blue backgrounds denote, respectively, disjunctive and conjunctive operations, based on the behaviour of their restriction to* $\{0, +\infty\}$*. Positive and negative polarities relate to each other via De Morgan duality.*

Such structure is itself organized like linear logic organizes its connectives. The 'positive' operations are *addition* $+$ and *(conjunctive) multiplication* $\otimes$, where $0 \otimes \infty = 0$. By introducing the involution $a^* = 1/a$, we double up the roster of operations by De Morgan duality. The resulting operations are *harmonic addition* $+^*$ and *(disjunctive) multiplication* $\otimes^*$, where $0 \otimes^* \infty = \infty$.

Quantitative Linear Logic. We can then 'enrich' the sequent calculus for MALL in this structure, by simply decorating each additive rule with additive connectives and each multiplicative rule with multiplicative ones (though we are also free of making other, less sensible, choices). Similarly, we decorate axioms with *validities*, with additive rules getting either ∞ or 0 (the additive scalars) while multiplicative ones get 1. We obtain the calculus in Figure 1.

Derivations are defined in the usual way, and a closed derivation is one whose leaves contain no sequents, just scalars. One can then traverse a proof tree and obtain an expression which evaluates in the reals, which we call the *validity* of the proof. For instance, the following proof has validity $|1 +^* 1| = 1/2$:

$$\dfrac{\dfrac{1}{A \vdash A}\,\mathrm{AX} \quad +^* \quad \dfrac{1}{A \vdash A}\,\mathrm{AX}}{A \vee A \vdash A}\,\vee_{\mathrm{L}} \tag{3}$$

The provability of a sequent is similarly enriched, defined as the supremum of the validity of its proofs:

$$|\Gamma \vdash \Delta| := \bigvee \left\{ |\pi| \;\middle|\; \begin{array}{c} \vdots\,\pi \\ \Gamma \vdash \Delta \end{array} \text{closed} \right\} \tag{provability}$$

From a proof-theoretic perspective, QLL is completely analogous to MALL: in fact a sequent is provable in MALL if and only if it has strictly positive provability in QLL. Therefore one can conceive of QLL as a further analysis of MALL's proof theory, in which proofs are evaluated based on their 'efficiency'.

The properties of this calculus are broken down in [CAGK26]: it admits cut-elimination, thus it is consistent, decidable and enjoys the subformula property—thus the supremum defining the provability of a sequent is in fact a maximum. It also admits a sound and complete semantics in a variety of preorders enriched in $[0, \infty]$ equipped with suitable algebraic structure.

(a) *Structural Rules*

$$\frac{1}{A \vdash A}\ \text{AX} \qquad \frac{1}{\vdash}\ \text{EMP} \qquad \frac{0}{\Gamma \vdash \Delta}\ \text{EFQ}$$

$$\frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{CUT} \qquad \frac{\Gamma \vdash \Delta \quad \otimes \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{MIX}$$

(b) *Multiplicative Rules*

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes_{\text{L}} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ \parr_{\text{R}}$$

$$\frac{\Gamma, A \vdash \Delta \quad \otimes \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ \parr_{\text{L}} \qquad \frac{\Gamma \vdash A, \Delta \quad \otimes \quad \Gamma', \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}\ \otimes_{\text{R}}$$

$$\frac{1}{\mathbf{1} \vdash}\ \mathbf{1}_{\text{L}} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\ (-)^{\perp}_{\text{L}} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ (-)^{\perp}_{\text{R}} \qquad \frac{1}{\vdash \mathbf{1}}\ \mathbf{1}_{\text{R}}$$

(c) *Additive Rules*

$$\frac{\Gamma, A \vdash \Delta \quad +^{*} \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_{\text{L}} \qquad \frac{\Gamma \vdash A, \Delta \quad +^{*} \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_{\text{R}}$$

$$\frac{\Gamma, A \vdash \Delta \quad + \quad \Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_{\text{L}} \qquad \frac{\Gamma \vdash A, \Delta \quad + \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_{\text{R}}$$

$$\frac{\infty}{\Gamma, \bot \vdash \Delta}\ \bot_{\text{L}} \qquad \frac{\infty}{\Gamma \vdash \top, \Delta}\ \top_{\text{R}}$$

Figure 1: *Two-sided calculus for (1-)Quantitative Linear Logic (1QLL).*

Hence QLL gives a logical interpretation to the arithmetic structure of the reals without abandoning the philosophical justification of rules provided by Martin-Löf [Mar96]. This also realizes the Lambek side of an early vision of Lawvere, namely that enriched categories still have a logical significance [Law73].

Practical significance. Aside from a philosophical standpoint, QLL aspires to be a foundation to bridge structural methods, such as logic and category theory, and quantitative sciences, such as statistics or probability theory and thus machine learning.

We provide two pieces of evidence in the way of this claim. First is the fact one can readily encode elementary Bayesian probability theory in QLL: indeed the positive fragment of the logic (cf. Table 1) is precisely what Bayesian probability rests on.

In fact, fix a finite set Ω of *outcomes*. An event $A \subseteq \Omega$ corresponds to a formula with propositional variables in the set Ω, namely $\sum_{a\in A} a$, which we denote[1] again by A. Every sequent $A \vdash B$ where A, B are such propositions correspond then to the relative probability of B compared to A, i.e. the *odds* of B to A. In particular the judgment $A \vdash A \cap B$ can be derived by the following proof schema:

$$\dfrac{\dfrac{\dfrac{\dfrac{\bigoplus_{b\in A\cap B}\bigoplus^*_{a\in A}\ \vdash a^* \otimes \vdash b}{\bigoplus_{b\in A\cap B}\bigoplus^*_{a\in A}\ (a\vdash \otimes \vdash b)}\ \bigoplus_b\bigoplus^*_a ((-)^\perp_{\mathrm{L}}\otimes \mathrm{id})}{\bigoplus_{b\in A\cap B}\bigoplus^*_{a\in A}\ a\vdash b}\ \bigoplus_b\bigoplus^*_a \mathrm{MIX}}{\bigoplus_{b\in A\cap B}\ A\vdash b}\ \bigoplus_b \vee_{\mathrm{L}}}{A\vdash A\cap B}\ \vee_{\mathrm{R}} \tag{4}$$

Then a probability density $p : \Omega \to [0, 1]$ induces a valuation of the propositional variables, giving the above sequent the semantic validity expected by the Bayesianist:

$$\frac{\bigoplus_{b\in A\cap B} p(b)}{\bigoplus_{a\in A} p(a)} = \frac{p(A\cap B)}{p(A)} = p(B \mid A). \tag{5}$$

The second, related, fact is that QLL appears to be very effective at *property-driven training* of neural networks, as empirically demonstrated in [FKCM26]. This refers to the technique of augmenting the training objective of a gradient-based model so as to include the semantics of a logical constraint, suitably translated to a differentiable function of the network's outputs and training data.

Such translation is effected by means of an alternative semantics of QLL, obtained by transporting the operations of Table 1 along the isomorphism

$$-\log : [0,\infty]_\otimes \xrightarrow{\sim} [-\infty,+\infty]_\oplus : 1/\exp\,. \tag{Napier}$$

This isomorphism is put to good use in statistics, where it is habitually employed to translate probabilities or likelihoods (which are *multiplicative* quantities) into

[1]This encoding is not perfect, since A is considered a multiset in this way. A first-order encoding would be more satisfactory, but we don't have the facilities to explain it here.

logits or log-likelihoods (which are *additive* quantities)[2]. In fact, loss functions for prediction tasks are most commonly cross-entropies, whose 'type' is that of logits.

Aside from this 'type correctness' property, the additive semantics of QLL has analytical properties which prove remarkably adapted for gradient-based optimisation.

Take for instance additive conjunction in $[-\infty, +\infty]_\oplus$, given by $x \cap y = \log(\mathrm{e}^x + \mathrm{e}^y)$. Its gradient is

$$\partial_x(x \cap y) = \frac{\mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^y} = 1/\exp(x) \multimap (1/\exp(x) +^* 1/\exp(y)). \tag{6}$$

which has an enticing logical interpretation, supported by the very well-known fact that $[0, \infty]_\otimes$ is the Lie algebra of the Lie group $[-\infty, +\infty]_\oplus$.

Concluding remarks. A manifest goal of this talk is to intrigue the linear logic community into looking at such quantitative phenomena with a logical eye. Some outstanding research directions are particularly intriguing. What are exponential modalities in this setting? What the link with *differential* linear logic? Can we account for Napier's isomorphism and the relationship between the additive and multiplicative reals through it?

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[2]Note 'multiplicative' and 'additive' here are used in a different way than in LL: they both refer to the semantics of multiplicatives in QLL, thus to the operation utilized to combine quantities pertaining independent or non-interacting objects, e.g. entropy is additive because the joint entropy of two independent random variables is the sum of the respective entropies.