Adapted Pairs and Logical Relations in Linear Logic

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Abstract

It is folklore that there is a tight link between biorthogonal-closure constructions, familiar to practitioners of linear logic (LL), and logical relations, as they appear in, for instance reducibility-candidate flavoured proofs of normalisation for typed lambda calculi. One manifestation of this is in the technique of adapted pairs, due to Krivine, which weakens the extensionality condition defining a logical relation to facilitate the required induction over types. In this technique, the full logical-relation condition appears as a kind of fixed-point of the adapted-pair condition. We aim to reformulate the duality underlying the definition of adapted pairs using linear logic, and connect this duality to techniques for proving denotational completeness (or definability) in models of LL.

1 Background

A logical relation (sometimes logical predicate, in the unary case we consider) constructs sets of terms based on their extensional behaviour. For example, if $\mathcal{S}, \mathcal{R} \subset \Lambda$ are sets of untyped λ-terms, then their *Kleene implication* is defined as

$$\mathcal{S} \to \mathcal{R} \doteq \{t \mid \forall s \in \mathcal{S}, ts \in \mathcal{R}\}.$$

This definition appears, broadly speaking, in two contexts. In *realisability*, it gives precise content to the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic: following Kleene [Kle45], a realiser of the formula $A \to B$, is a term that takes, via function application, realisers of A to realisers of B. On the other side, arguments using *reducibility candidates*, or *computability predicates* [Tai67, Gal89], first prove the required property for all terms matching an extensional specification — defined using a logical relation — then prove that the class of terms in question match an appropriate specification.

In the context of "reducibility-style" proofs, the extensional specification is provided by types, and the induction over types is sometimes facilitated by the technique of *adapted pairs*, due to Krivine [Kri93]. As a concrete example, consider the hierarchy $\mathbb{T}$ of simple types over base types $\mathbb{B}$. Then, having fixed $R_o \subset \Lambda$ for $o \in \mathbb{B}$, we can define the "reducible terms" or "realisers" of every type $\sigma \in \mathbb{T}$ by induction:

$$R_{\sigma\to\tau} \doteq R_\sigma \to R_\tau. \tag{1}$$

Let us call a pair (M_σ, N_σ) of $\mathbb{T}$-indexed families of subsets of Λ *adapted* if

$$M_{\sigma\to\tau} \subset N_\sigma \to M_\tau \subset M_\sigma \to N_\tau \subset N_{\sigma\to\tau}. \tag{2}$$

This definition is designed so that if $M_o \subset R_o \subset N_o$ at base types o, then $M_\sigma \subset R_\sigma \subset N_\sigma$ for every type. Instantiating M_σ as a collection *neutral* terms $xt_1 \cdots t_n$ and N_σ as the collection of strongly normalising terms, (M, N) is adapted and one can use this technique to provide (yet another) proof of strong normalisation for the simply-typed lambda calculus.

It is evident from this presentation that eq. (2) is a weakening of the definition eq. (1): that a family R_σ defines a logical predicate if and only if (R_σ, R_σ) is an adapted pair. With the biorthogonal constructions of linear logic (LL) in mind, we might expect to find a duality for which logical predicates R_σ are fixed points. Indeed this perspective is implicit in presentations of reducibility candidates for linear logic [Gir87, Acc13], and has been made explicit for the lambda calculus in [MM11]. In a realisability setting, a similar perspective is put forward in [RSdF24].

2 Kripke Logical Predicates

Logical relations also arise as a technique for demonstrating definability results: in a slightly generalised form they were used in [JT93] to characterise those morphisms in a model of the lambda calculus which arise as denotations of terms, and this technique was extended to linear logic in [Str99].

To this end we must consider open terms, so that the logical predicates will be indexed by a type σ and a context Γ, in order to characterise morphisms definable by terms t typed as $\Gamma \vdash t : \sigma$. This leads to the definition of *Kripke logical relations*. Let Tm be the category of $\mathbb{T}$-terms — the free cartesian-closed category (CCC) over the set $\mathbb{B}$ — and Ctx the wide subcategory of structural rules. Now fixing a CCC C and a function $\mathbb{B} \to \mathsf{C}$, we have an interpretation $i : \mathsf{Tm} \to \mathsf{C}$. Kripke logical relations live in the Artin-Wraith gluing [AGV06, Wra74] G of C to the category $\hat{\mathsf{Ctx}}$ of presheaves on Ctx, along the extended Yoneda functor $\text{よ}^i : \mathsf{C} \to \hat{\mathsf{Ctx}}$, defined by

$$\text{よ}^i A \doteq \mathsf{C}(i-, A).$$

More explicitly:

Definition 2.1. A *Kripke predicate* on $A \in \mathsf{C}$ is a map R from contexts Γ to subsets of $\mathsf{C}(i\Gamma, A)$ so that, if $w : \Gamma' \to \Gamma$ is a morphism in Ctx, then

$$w^*(R\Gamma) \doteq \{f \circ iw \mid f \in R\Gamma\} \subset R\Gamma'.$$

Kripke predicates (for varying A) form a cartesian-closed category G, whose morphisms $(A, R) \to (B, S)$ are those C-maps $R \to S$ which preserve the predicate. The product is pointwise, and the internal hom recovers the Kleene arrow: given relations R on A and S on B and a context Γ, the predicate $R \to S$ on $A \to B$ is

$$(R \to S)\Gamma = \{f \in \mathsf{C}(i\Gamma, A \to B) \mid \forall w : \Gamma' \to \Gamma, a \in R\Gamma', \mathrm{ev}(f \circ iw, a) \in S\Gamma'\}.$$

This mimics our earlier definition, taking account of the fact that the "test" argument a may be defined only in a larger context Γ' — it is worth noting however that this fact is a consequence of the cartesian-closed structure on G, originally defined in topos theory.

The prototypical example of a Kripke predicate is the family of definable morphisms D_σ into a type σ, viz:

$$D\Gamma \doteq \{f \in \mathsf{C}(i\Gamma, i\sigma) \mid \exists t \in \mathsf{Tm}(\Gamma, \sigma),\ it = f\}.$$

Indeed, Kripke logical predicates characterise definable morphisms: the result of [JT93] amounts to showing that D_σ is logical, in the sense that the syntactic and semantic constructions of the arrow type agree:

$$D_{\sigma \to \tau} = D_\sigma \to D_\tau.$$

In the guise of *full completeness* [AJ94], this is also presented in [Tay99, Ali95].

As observed (at least) by Fiore [Fio22], the technique of adapted pairs scales up to the presheaf setting. Indeed, for A_σ, B_σ indexed families of Kripke predicates on $i\sigma$, we can upgrade the definition of eq. (2) to inclusions of presheaves

$$A_{\sigma\to\tau} \longhookrightarrow (B_\sigma \to A_\tau) \longhookrightarrow (A_\sigma \to B_\tau) \longhookrightarrow B_{\sigma\to\tau},$$

and precisely the same argument applies: if R_o is a Kripke predicate on each base type o, and $A_o \hookrightarrow R_o \hookrightarrow B_o$, then the (unique) extension of R to $\mathbb{T}$ satisfies $A_\sigma \hookrightarrow R_\sigma \hookrightarrow B_\sigma$ for all σ. This becomes particularly interesting if we generalise the inclusions to morphisms: for instance replacing A and B with presheaves M, N of neutral and *normal* (rather than normalising) terms, constructing a morphism $R_\sigma \to N_\sigma$ in an appropriate category provides an algorithm implementing normalisation [AHS95, Fio22].

3 Linear Logic

A similar technique is applied by Streicher to characterise which morphisms in a model C of (classical) linear logic (LL) arise as denotations of proofs [Str99]. Here the situation is not quite as simple: the glued category G may not be $*$-autonomous, that is, the interpretation in presheaves may not respect the duality of linear logic.

Suppose that we interpret the type $\perp$ as a Kripke predicate

$$\mathcal{B} \longhookrightarrow \mathsf{C}(i-, i\perp).$$

Then if we have interpreted σ and $\sigma^\perp$ as presheaves R and $R^\perp$, for any $a \in R\Gamma$ and $b \in R^\perp\Gamma$ we must have $\mathrm{cut}(a, b) \in \mathcal{B}(\Gamma \otimes \Delta)$. If we take this requirement as a definition, and ensure at every step that the interpretation R_σ of any type σ satisfies $R_\sigma^{\perp\perp} = R_\sigma$, we obtain the following.

Definition 3.1 ([Str99] Definition 2.5)**.** Fixing R_o for $o \in \mathbb{B}$, and an interpretation $\mathcal{B} = R_\perp$, R_σ is defined inductively by

$$R_{\sigma^\perp}\Delta = \{b \in \mathcal{C}(i\Delta, A^\perp) \mid \forall \Gamma, a \in R\Gamma,\ \mathrm{cut}(a, b) \in \mathcal{B}(\Gamma \otimes \Delta), \}, \tag{3}$$

$$R_{\sigma\otimes\tau}\Gamma = \left\{(a \otimes b) \circ iw \mid a \in R_\sigma\Delta, b \in R_\tau\Delta', w \in \mathsf{Ctx}(\Gamma, \Delta \otimes \Delta')\right\}^{\perp\perp}. \tag{4}$$

As for conventional logical predicates, Streicher's definition characterises the "proof objects" in C. The denotation of any proof satisfies all Kripke predicates arising from this inductive construction, and if we set

$$\mathcal{B}\Gamma = D_\perp\Gamma = \{f \in \mathsf{C}(i\Gamma, i\perp) \mid \exists t \in \mathsf{Tm}(\Gamma, \perp),\ it = f\},$$

and $R_o = D_o$, R_σ contains exactly the denotations of proofs. However, the correspondence between syntax and semantics is not as close: G is symmetric-monoidal-closed, but the correct tensor product is that of eq. (4) *without* the double-orthogonal (the Day convolution product [Day07]) — that is, the interpretation of types (formulas) as Kripke predicates no longer provides a model of LL[1]. Interestingly, the convolution product recovers the "splitting of the context" necessary to proving a tensor in linear logic [Has99].

To obtain a bona-fide model of linear logic, one solution due to Hyland and Tan [Tan97, HS03] is to define for each $A \in \mathsf{C}$ two predicates, one "covariant" and one "contravariant", so

[1] Hasegawa observes that it can in fact be seen as such, by moving to the category of algebras for a double-dualisation monad [Has99, Example 3.9]

that the duality swaps the two. Many models of LL arise as variations of this construction, including coherence and finiteness spaces [Ehr05, Gir87], and the construction as a whole can also interpreted in the style of Gödel's Dialectica interpretation [AF95] and classical realisability [Kri04], where the covariant part represents proofs of A, and the contravariant part counter-proofs [Tan97, §1.2].

4 Adapted Pairs in the Glued Category

We aim, in ongoing work, to connect the ways that categories of logical relations are made self-dual, in order to be models of linear logic, with Krivine's technique of adapted pairs. The latter is well-established and understood, and, as sketched at the end of Section 2, has been generalised to provide normalisation-by-evaluation results.

At present we suffice to show how Hyland and Tan's construction sandwiches self-dual Kripke predicates between a kind of adapted pair. Let C be $*$-autonomous, and fix a sub-presheaf $\mathcal{O} \hookrightarrow \mathsf{C}(i-, \perp)$ in $\hat{\mathsf{Ctx}}$. Following [HS03, §5] this defines an *orthogonality* on maps $f : i\Gamma \to A$ and $g : \Delta \otimes A \to \perp$ by

$$f \perp_{\mathcal{O}} g \iff g \circ (1 \otimes f) \in \mathcal{O}(\Delta \otimes \Gamma).$$

For the tensor, evidently

$$R_\sigma \otimes R_\tau \longhookrightarrow R_{\sigma\otimes\tau} = (R_\sigma \otimes R_\tau)^{\perp_{\mathcal{O}}\perp_{\mathcal{O}}}.$$

On the other hand, $\sigma \otimes \tau$ is dual to both $\sigma \multimap \tau^\perp$ and $\tau \multimap \sigma^\perp$. If $a \in R_\sigma\Delta$ and $b \in R_\tau\Delta'$, then for any $f \in (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}})\Gamma$, the cut of f against $a \otimes b$ is

$$\Gamma \otimes \Delta \otimes \Delta' \xrightarrow{f\otimes a\otimes 1} (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}}) \otimes R_\sigma \otimes \Delta' \xrightarrow{\mathrm{ev}\otimes b} R_\tau^{\perp_{\mathcal{O}}} \otimes R_\tau.$$

(Of course, the argument swapping the roles of σ and τ works just as well.) As such $R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}} \hookrightarrow (R_\sigma \otimes R_\tau)^{\perp_{\mathcal{O}}}$, so finally

$$R_\sigma \otimes R_\tau \longhookrightarrow R_{\sigma\otimes\tau} \longhookrightarrow (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}})^{\perp_{\mathcal{O}}}.$$

This is essentially the slack orthogonality subcategory of Hyland and Schalk's double gluing construction, generalised to the presheaf setting. This argument suggests that the lower part of a classical adapted pair comes from the (non-self-dual) interpretation of LL in the glued category G, and the upper part is the dual (with respect to the orthogonality in G), of the interpretation of the dual formula. As suggested above, the self-dual *definable elements* live in between, as fixed points of the duality on G.

The construction of a generalised adapted pair we have sketched here connects to the classical version if we set $\mathsf{C} = \mathsf{Tm}$, the syntactic model, and $\mathcal{O}$ to be the presheaf of strongly normalising terms, that is

$$\mathcal{O}\Gamma = \{\vdash t : \Gamma^\perp \mid t \text{ is strongly normalising}\}.$$

Then we recover the upper component N of the pair in eq. (2) (the presheaf of strongly normalising terms) in the LL setting as (cf: [Str99, Lemma 4.1])

$$\begin{aligned}\{\Gamma \vdash \pi : A \mid \pi \text{ strongly normalising}\} &= \{\Gamma \vdash \pi : A \mid \mathrm{cut}(\pi, \mathrm{ax}_{A^\perp}) \text{ strongly normalising}\} \\ &= \{\mathrm{ax}_{A^\perp}\}^{\perp_{\mathcal{O}}}.\end{aligned}$$

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