VERY SMALL DIALECTICA CATEGORIES

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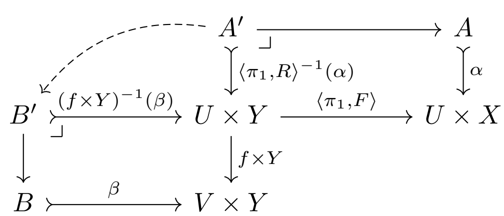
Abstract. Categorification of a mathematical structure often identifies a common abstraction that connects formerly unrelated structures and offers a bridge between disparate disciplines. In the case of de Paiva's Dialectica categories, we find that a specialization of the categorification of Gödel's Dialectica construction to partial orders produces (functorial) embeddings of Heyting algebras into residuated lattices that appear to have been overlooked. For the non-categorical audience, we present this specialization and take care to reproduce the original proofs in the algebraic setting. Along the way, we prove new results about the specialization and use it to better understand the linear logics associated to and properties of the original Dialectica categories.

1. Introduction

The Dialectica categories of Valeria de Paiva [deP91] generalize the translation of Intuitionistic Heyting Arithmetic into Gödel's System T. Gödel's original intent was to provide a relative consistency proof of Peano arithmetic. One of the original motivations for de Paiva's categorification was to extract from the interpretation its categorical structure, which then provided one of the first non-collapsed categorical models of linear logic.

Let $\mathbf{C}$ be a category with finite limits. The **Dialectica Category** $\mathrm{D}(\mathbf{C})$ associated to $\mathbf{C}$, has as objects triples consisting of a pair of objects $U, X \in \mathrm{Ob}(\mathbf{C})$ and a monomorphism $\alpha\colon A \rightarrowtail U \times X$. We write such an object as (α, U, X).

A morphism from (α, U, X) to (β, V, Y) consists of a pair of morphisms of $\mathbf{C}$, (f, F), $f\colon U \to V$, $F\colon U \times Y \to X$ such that a non-trivial condition is satisfied: pulling α back along $\langle \pi_1, F\rangle$ and β along $f \times A$, the first subobject is smaller than the second as in the following diagram



(1)

In the category **Set**, we can identify subobjects with subsets, and the pullback as the preimage operation. Then the condition becomes: $(f, F)\colon \alpha \to \beta$ is a morphism iff whenever $(u, F(u, y)) \in \alpha$, then $(f(u), y) \in \beta$. That is,

$$\langle \pi_1, F\rangle^{-1}(\alpha) \le (f \times Y)^{-1}(\beta).$$

Composition of morphisms is defined in Dialectica categories, and it is shown that $\mathrm{D}(\mathbf{C})$ is a category. If $\mathbf{C}$ is Cartesian closed (with well-behaved coproducts), then $\mathrm{D}(\mathbf{C})$ has sufficient structure, so that the associated logic satisfies all the rules of ILL . This is no easy task, especially since to capture the proof theory of ILL in D($\mathbf{C}$), the additional operations on D($\mathbf{C}$) need to be natural.

Since Dialectica categories have strong connections to logic and not every logician is a categorical one, an original motivation for this note was to provide a more gentle introduction to Dialectica categories via small models. Just as it is helpful to learn about the finite cyclic groups, dihedral groups, the quaternions, etc. when first learning group theory, we ask: Are there compelling finite Dialectica categories that can improve our intuitions?

The Dialectica construction D($\mathbf{C}$) requires a category $\mathbf{C}$ with at least finite limits, and a category with products and two parallel morphisms necessarily has infinitely many objects and so, we consider partial-order categories, i.e. those where there is at most one morphism between objects, and isomorphic objects are identified. When $\mathbf{C}$ is a poset, the additional operations defined on D($\mathbf{C}$) remain compelling, and we no longer need to concern ourselves with the naturality of such operations, since every diagram trivially commutes in a poset category.

There is another reason for considering such categories: In (proof-theoretic) categorical logic, a category $\mathbf{C}$ provides semantics for a logic as follows: Each atomic formula A of the logic is assigned to an object of $\mathbf{C}$, and the connectives of the logic are interpreted as natural operations on the category which inductively assign an object $[\phi]$ of $\mathbf{C}$ to each formula ϕ. As a simple example, if your logic of interest has meets, then your category $\mathbf{C}$ providing semantics needs binary products, and $[\phi \wedge \psi]$ is interpreted as $[\phi] \times [\psi]$.

Then $\phi \vdash \psi$ is said to hold if and only if, for every such assignment of atomic formulas to objects of $\mathbf{C}$, there is a morphism from $[\phi]$ to $[\psi]$. The naturality of the operations interpreting the connectives ensures that when we take the poset reflection[1] of $\mathbf{C}$, $P(\mathbf{C})$, the associated ordered algebra provides semantics for the same logic. The difference is, morphisms of the category are intended to represent proofs, and the algebra captures derivability (or provability), i.e., only the existence of a proof.

Although we lose a lot of information, ordered-algebraic semantics provides a powerful and greatly simplified view into the associated logic. In what follows, we study the order-theoretic specialization of the Dialectica construction, its associated logics, and connections to the general construction. Special attention is paid to D($\mathbf{Set}$) not only for its historical significance but also since it is the jumping off point for the categorification.

2. Dialectica Over Meet-Semilattices

Recall that the Dialectica construction D($\mathbf{C}$) over a category $\mathbf{C}$ ([Pai89]) requires that the category $\mathbf{C}$ has binary products and pullbacks. If $\mathbf{C}$ is a poset, then the categorical product $a \times b$ is necessarily the meet $a \wedge b$. Moreover, if $a \leq c$ (i.e. $\exists f\colon a \to c$) and $b \leq c$ (i.e. $\exists g : b \to c$), then the pullback of f, g is simply $a \wedge b$. Applying these observations to the Dialectica morphism shown in Diagram 1, we have:

[1] The poset reflection is a functor $P\colon \mathbf{Cat} \to \mathbf{Pos}$, which maps each category $\mathbf{C}$ to a poset $P(\mathbf{C})$ obtained first by identifying parallel morphisms and then identifying isomorphic objects. One can see that functors $F\colon \mathbf{C} \to \mathbf{D}$ respect the resulting equivalence classes, and so $P(F)$ is the result of applying F to representatives of the classes.

Definition 1. The Dialectica category over a meet-semilattice $\mathbf{M} = (M, \leq, \wedge, 1)$, written as $\mathrm{D}(\mathbf{M})$, is a preorder whose objects are triples (a, u, x) where a, u, x are elements of $\mathbf{M}$ and $a \leq u \wedge x$. In $\mathrm{D}(\mathbf{M})$, $(a, u, x) \leq (b, v, y)$ if and only if the following three inequalities hold:

(i) $u \leq v$
(ii) $u \wedge y \leq x$ and
(iii) $a \wedge (u \wedge y) \leq b \wedge (u \wedge y)$

Since algebraists prefer to work with ordered, instead of pre-ordered algebras we define the *Dialectica Algebra*, $D'(\mathbf{M})$, to be the sub-order of $D(\mathbf{M})$ defined as

$$\mathrm{D}'(\mathbf{M}) := \{(a, u, x) \in M : a \leq x \leq u\}.$$

Proposition 2. *For $\mathbf{M}$ a meet-semilattice, $P\,\mathrm{D}(\mathbf{M})$ is order-isomorphic to $D'(\mathbf{M})$. And so, as categories, $\mathrm{D}'(\mathbf{M})$ is equivalent to $\mathrm{D}(\mathbf{M})$.*

In the sequel, we will take care to define operations on $\mathrm{D}(\mathbf{M})$ so that they restrict to operations on $\mathrm{D}'(\mathbf{M})$, allowing us to work with $\mathrm{D}'(\mathbf{M})$ instead of $P\,\mathrm{D}(\mathbf{M})$.

If $\mathbf{M}$ has a smallest element 0 then $\mathbf{1} := (0, 1, 0)$ is the top of $\mathrm{D}(\mathbf{M})$ and $\mathbf{0} := (0, 0, 0)$ is the bottom. Moreover, let $i := (1, 1, 1)$ and for $\alpha = (a, u, x)$, and $\beta = (b, v, y)$, define $\alpha \otimes \beta := (a \wedge b, u \wedge v, x \wedge y)$. Observe that $(\mathrm{D}(\mathbf{M}), \otimes, i)$ is an ordered commutative monoid or in categorical language, $(\otimes, i)$ a symmetric monoidal tensor product.

Moreover, since the tensor product $\otimes$ is defined on objects via coordinate-wise meets, it is immediate that the associated logic satisfies the following inference rules:

$$\frac{A, \Gamma \vdash B}{A, A, \Gamma \vdash B}\ \text{dupl} \qquad \frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ \text{cont}$$

The first rule is a restricted form of weakening we call duplication, and the second rule is contraction. Just as important are the rules the logic doesn't satisfy. We will see for the two element Boolean algebra $\mathbf{2}$, $\mathrm{D}(\mathbf{2})$ shown in Figure 1 fails to satisfy the following more general rule of weakening:

$$\frac{A, \Gamma \vdash B}{A, C, \Gamma \vdash B}\ \text{weak}$$

And so, the logics associated to classes of $\mathrm{D}'(\mathbf{M})$ algebras seem to be relevance logics: That is, antecedents cannot be exhausted, but they need to play a role in the derivation of the consequent.

We would now like a linear implication $\multimap$ in the Dialectica preorder. In the categorical semantics, what is needed is for the tensor $\otimes$ on $\mathrm{D}(\mathbf{C})$ to be closed, and in [deP91], it is shown this holds whenever $\mathbf{C}$ is Cartesian closed. The order theoretic version of a Cartesian closed category is a residuated meet-semilattice $\mathbf{R}$. That is, $\mathbf{R}$ is a meet-semilattice such that for all $x, y, z \in R$ the following condition is satisfied:

$$x \wedge y \leq z \iff y \leq x \to z.$$

Proposition 3. *The operation $\multimap\colon (\mathrm{D}\,\mathbf{R})^{op} \times \mathrm{D}\,\mathbf{R} \to \mathrm{D}\,\mathbf{R}$, defined for $\beta = (b, v, y)$ and $\gamma = (c, w, z)$ by*

$$\beta \multimap \gamma := ((b \to c) \wedge s \wedge t, s, s \wedge t)$$

where $s = (v \to w) \wedge ((v \wedge z) \to y)$ and $t = v \wedge z$ is the internal-hom in $\mathrm{D}(\mathbf{R})$. I.e., each $\alpha \otimes (\cdot)$ in $\mathrm{D}(\mathbf{R})$ is residuated with residual $\alpha \multimap (\cdot)$.

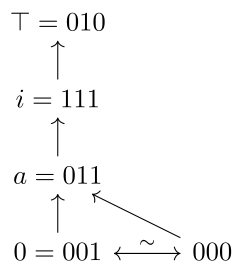


FIGURE 1. The residuated meet-semilattice $P\,\mathrm{D}(\mathbf{2})$.

Consider $\mathbf{R} = \mathbf{2}$. For simplicity, we write aux for (a, u, x). The preorder of $\mathrm{D}(\mathbf{2})$ is shown in Figure 1: If we identify isomorphic objects and let $0 = 000$, $a = 011$, $i = 111$ and $1 = 010$, we have the following operation table for $P\,\mathrm{D}(2) = \mathrm{D}'(2)$:

$\otimes$	0	a	i	1
0	0	0	0	0
a	0	a	a	1
i	0	a	i	1
1	0	1	1	1

$\multimap$	0	a	i	1
0	1	1	1	1
a	0	i	i	1
i	0	a	i	1
1	0	0	0	1

(2)

The result is a four-element commutative, idempotent, involutive, residuated lattice, which is not integral, known as the Sugihara algebra S_4. [Sug55]

Moreover, assuming choice, $\mathrm{D}'(\mathbf{2})$ has the following striking connection to $\mathrm{D}(\mathbf{Set})$, the motivating example for the original categorification:

Theorem 4. *Over* ZF, $\mathrm{D}'(2) = P\,\mathrm{D}(\mathbf{Set})$ *is equivalent to the axiom of choice.*

Corollary 5. *Assuming choice, the logic associated to* $\mathrm{D}(\mathbf{Set})$ *is the logic associated* S_4*. As a consequence, the logic* $\vdash_{\mathrm{D}(\mathbf{Set})}$ *is strictly stronger than* ILL*.*

Let $\alpha = (a, u, x)$, $\beta = (b, v, y)$ and $w = u \wedge v$. If $\mathbf{R} = \mathbf{H}$ is a Heyting algebra then $\mathrm{D}(\mathbf{H})$ has finite meets, where the binary meet is defined as

$$\alpha \,\&\, \beta := (((w \wedge x) \to a) \wedge ((w \wedge y) \to b) \wedge w \wedge (x \vee y), w, w \wedge (x \vee y)).$$

Now we would like to search for a suitable operation $! \colon \mathrm{D}(\mathbf{H}) \to \mathrm{D}(\mathbf{H})$ to interpret the "of course" modality of intuitionistic linear logic. Informally, in linear logic, $!A$ means that we can "use" the argument A as many times as we want, including none. If we were allowed infinitary formulas, we could express this as

$$I \,\&\, A \,\&(A \otimes A) \,\&(A \otimes A \otimes A) \,\& \ldots$$

However, in $\mathrm{D}(\mathbf{H})$, we have contraction, and so the above expression is reducible to $I \,\&\, A$. Using the definitions of I and $\&$ above we define

$$!(a, u, x) := i \,\&(a, u, x) = (((u \wedge x) \to a) \wedge u, u, u) = ((x \to a) \wedge u, u, u).$$

Proposition 6. *The* $!$ *operation defines an interior operator (comonad) on* $\mathrm{D}(\mathbf{H})$ *satisfying* $!(a, u, x) \otimes !(b, v, y) \leq !((a, u, x) \otimes (b, v, y))$*.*

All together we have the following result.

Theorem 7. *If* $\mathbf{H}$ *is a Heyting algebra, then the associated logic* $\vDash_{\mathrm{D}(\mathbf{H})}$ *satisfies the rules of Intuitionistic Linear logic with the "of course" modality.*

3. Open Questions/Future Work

In investigating the algebraic case, we have converged on Heyting algebras, as suitable structures over which the Dialectica construction should occur. However, in [deP91] one considers not bi-Cartesian closed categories, but categories $\mathbf{C}$ which are finitely complete, locally Cartesian closed, and with stable and disjoint coproducts. There is unfinished work to reconcile the different assumption sets. Moreover, we have barely scratched the surface of understanding the algebras $\mathrm{D}'(H)$, and it will be interesting to see what results and algebraic approaches can lift to the general Dialectica construction.

In algebraic logic, one often considers a logic associated to a class of algebras. For example, the variety of Heyting algebras $\mathcal{H}$ provides semantics for intuitionistic logic in the following sense: We say $\phi \vDash_{\mathcal{H}} \psi$ if and only if, for each Heyting algebra $\mathbf{H} \in \mathcal{H}$, $\phi \vDash_{\mathbf{H}} \psi$. By definition, $\vDash_{\mathcal{H}}$ is the infimum of the logics defined by Heyting algebras, and it is the logic $\vDash_{\mathbf{H}}$ where $\mathbf{H}$ is the free Heyting algebra over the propositional atoms. Let $\mathcal{C}$ be the class of all (small) finitely complete locally Cartesian closed categories with stable and disjoint products and $\mathrm{D}(\mathcal{C})$ all associated Dialectical Categories. From [deP91], $\vDash_{\mathrm{D}(\mathcal{C})}$ satisfies all the rules of ILL but is $\vDash_{\mathrm{D}(\mathcal{C})}$ equal to ILL? Moreover, is there a $\mathbf{C} \in \mathcal{C}$, such that $\vDash_{\mathrm{D}(\mathbf{C})} = \vDash_{\mathrm{D}(\mathcal{C})}$?

Let $\mathcal{C}$ be the class of all bi-Cartesian closed categories. Is $\vDash_{\mathrm{D}(\mathcal{C})} = \vDash_{\mathrm{D}(\mathcal{H})}$? Note that a $\mathbf{C} \in \mathcal{C}$ such that $\vDash_{\mathrm{D}(\mathbf{C})}$ does not satisfy contraction would disprove this claim. While such $\mathbf{C}$ are thought to exist, we do not currently have a witness. A model of ZF that doesn't satisfy choice may seem like a good candidate but the function

$$F(u, x_1, x_2) = \begin{cases} x_1 & \text{if } (u, x_1) \notin \alpha \\ x_2 & \text{otherwise.} \end{cases}$$

defined for a Dialectica object $\alpha \subseteq U \times X$ is first-order definable over the signature $\{\in\}$ and so can be defined in ZF via the Axiom of Specification. And F along with the Diagonal function $f\colon U \to U \times U$ determines a morphism from α to $\alpha \otimes \alpha$.

From Theorem 4, if $\mathbf{Set}$ is a model of ZFC, then $P\,\mathrm{D}\,P(\mathbf{Set}) = P\,\mathrm{D}(\mathbf{Set})$, and one may wonder if over $\mathcal{C}$, $P\,\mathrm{D}\,P(\cdot) = P\,\mathrm{D}(\cdot)$? From Theorem 4 again this cannot hold in general since there are models of ZF that do not satisfy choice. It would be interesting to characterize the subclass of $\mathcal{C}$ over which this result holds.

Finally, we did not discuss the other construction $\mathrm{G}(\mathbf{C})$ from [deP91] which provides semantics for classical linear logic. It is identical on objects to $\mathrm{D}(\mathbf{C})$ and morphisms are the same except the backwards map $F\colon U \times Y \to X$ is instead from Y to X. The order theoretic analogue is already a partial order, and $\mathrm{G}(\mathbf{2})$ is a pentagon where 000 is incomparable to 111.

References

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