Proof Nets for First-order MALL

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Introduced by Girard in [3], proof nets are a graphical syntax for linear logic that represent proofs directly, abstracting away from inessential syntactic information found in sequent calculus derivations. In the sequent calculus, the order of rule applications must be fixed even when rules operate on independent parts of a sequent. This constraint introduces unnecessary redundancy, commonly referred to as *bureaucracy* [3], which complicates proof transformations such as cut elimination, where rules need to be permuted to expose redexes.

For MLL, proof nets support polynomial-time correctness, sequentialization, and translation [3]. However, when units are added to MLL, the standard correctness criterion fails [7], requiring substantial revision of the framework.

For propositional MALL, several syntaxes have been developed. Box nets, introduced by Girard [3], handle the additive conjunction & using explicit boxes, allow for polynomial-time correctness and translation but are not canonical with respect to permutations involving &, not even locally. Monomial nets for MALL [6], aim for more abstraction but still lack a polynomial correctness criterion and do not improve over box nets in terms of generality [11]. To address these issues, slice nets for MALL were proposed by Hughes and Van Glabbeek [13, 14]. Slice nets identify derivations modulo all independent rule permutations (see Figure 2), including non-local ones, while keeping a polynomial-time correctness checking. However, they do not have polynomial-time translation, as a single slice net can correspond to an exponentially large set of derivations. This complexity arises especially from permutations between rules like $\otimes$ and &, which can require duplication of entire subproofs. Conflict nets for MALL, introduced by Hughes and Heijltjes [11, 10], address this by restricting attention to local permutations only. Conflict nets structure axiom links using multiplicative concordance and additive conflict, and support polynomial-time correctness, translation, and sequentialization. This comes at the cost of excluding some non-local permutations.

In the first-order setting, similar design challenges appear. Girard introduced witness nets for first-order MLL_1 [4, 5], where quantifier witnesses are included explicitly in the proof. This leads to non-canonical derivations: logically equivalent proofs can differ depending on the witness chosen. Moreover, cut elimination in this setting becomes exponential and requires non-local rewriting [9]. To overcome these limitations, unification nets were introduced by Hughes for

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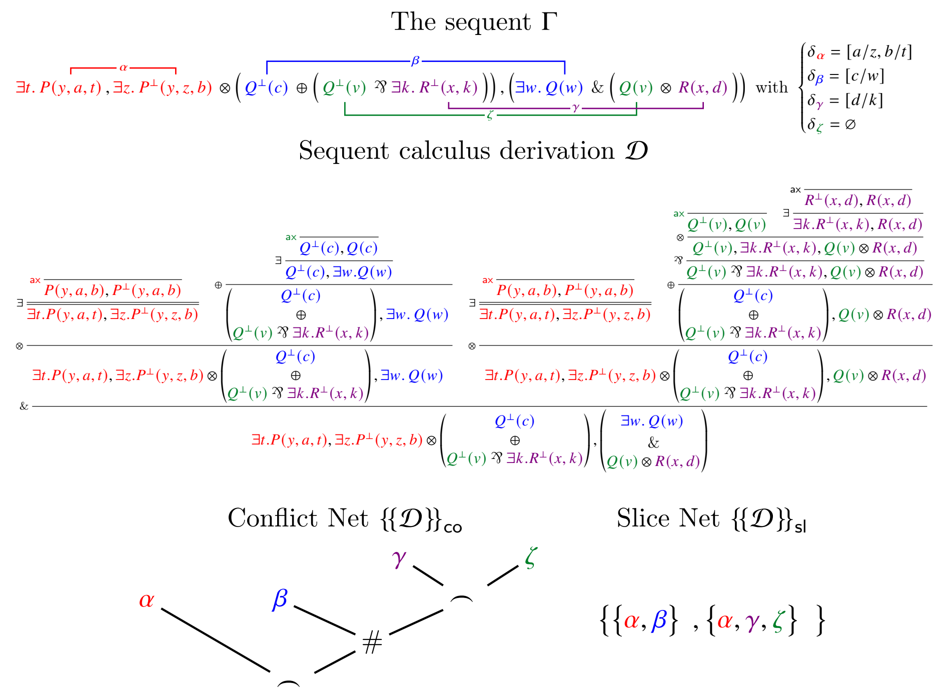

Figure 1: The sequent Γ, where we marked links and dualizers, a derivation for Γ, and the corresponding conflict and slice nets.

first-order MLL_1 [9]. These proof nets abstract away witness terms and identify derivations that differ only by quantifier instantiations, following the principle of *generality* [15]. The methodology of unification nets have since been extended to first-order ALL_1 [8] and classical logic [12].

To our knowledge, no existing framework defines proof nets that handle both additives and quantifiers in first-order (MALL_1). This work fills that gap.

Conflict nets and slice nets for MALL_1 To extend the definitions of conflict nets and slice nets to first-order MALL_1, we enrich the axiom links with witness information for quantifiers, represented in the form of substitution maps, called *dualizers*.

Definition 1. *A **link** α on a sequent $\vdash \Gamma$ is a **sub-sequent** of $\vdash \Gamma$, that is, a sequent $\vdash \Gamma'$ such that Γ' is an induced sub-forest of the forest of Γ. A link is **axiomatic** if it is a sub-sequent made of a single occurrence of a pair of dual atoms. An **axiomatic linking** on $\vdash \Gamma$ is a set of links (resp. axiomatic links) on $\vdash \Gamma$.*

*An **axiomatic linking with witnesses** $\langle \Lambda, \delta^\Lambda \rangle$ on $\vdash \Gamma$ is a set of axiomatic*

Local rule permutations

$$\otimes\,\frac{\vdash \Gamma_1, \Delta_1 \qquad \otimes\,\dfrac{\vdash \Gamma_2, \Delta_2, \Delta_3 \quad \vdash \Gamma_3, \Delta_4}{\vdash \Gamma_2, \Gamma_3, \Delta_2, \Theta_2}}{\vdash \Gamma_1, \Gamma_2, \Gamma_3, \Theta_1, \Theta_2} \approx \otimes\,\frac{\otimes\,\dfrac{\vdash \Gamma_1, \Delta_1 \quad \vdash \Gamma_2, \Delta_2, \Delta_3}{\vdash \Gamma_1, \Gamma_2, \Theta_1, \Delta_2} \qquad \vdash \Gamma_3, \Delta_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3, \Theta_1, \Theta_2}$$

$$\alpha_2\,\frac{\alpha_1\,\dfrac{\vdash \Gamma, \Delta_1, \Delta_2}{\vdash \Gamma, \Theta_1, \Delta_2}}{\vdash \Gamma, \Theta_1, \Theta_2} \approx \alpha_1\,\frac{\alpha_2\,\dfrac{\vdash \Gamma, \Delta_1, \Delta_2}{\vdash \Gamma, \Delta_1, \Theta_2}}{\vdash \Gamma, \Theta_1, \Theta_2} \qquad \alpha\,\frac{\otimes\,\dfrac{\vdash \Gamma_1, \Delta_1, \Delta_2 \quad \vdash \Gamma_2, \Delta_3}{\vdash \Gamma_1, \Gamma_2, \Delta_1, \Theta_2}}{\vdash \Gamma_1, \Gamma_2, \Theta_1, \Theta_2} \approx \otimes\,\frac{\alpha\,\dfrac{\vdash \Gamma_1, \Delta_1, \Delta_2}{\vdash \Gamma, \Theta_1, \Delta_2} \quad \vdash \Gamma_2, \Delta_3}{\vdash \Gamma_1, \Gamma_2, \Theta_1, \Theta_2}$$

$$\&\,\frac{\&\,\dfrac{\vdash \Gamma, A, C \quad \vdash \Gamma, A, D}{\vdash \Gamma, A, C \,\&\, D} \qquad \&\,\dfrac{\vdash \Gamma, B, C \quad \vdash \Gamma, B, D}{\vdash \Gamma, B, C \,\&\, D}}{\vdash \Gamma, A \,\&\, B, C \,\&\, D} \approx \&\,\frac{\&\,\dfrac{\vdash \Gamma, B, C \quad \vdash \Gamma, A, C}{\vdash \Gamma, A \,\&\, B, C,} \qquad \&\,\dfrac{\vdash \Gamma, B, D \quad \vdash \Gamma, A, D}{\vdash \Gamma, A \,\&\, B, D,}}{\vdash \Gamma, A \,\&\, B, C \,\&\, D}$$

$$\alpha\,\frac{\&\,\dfrac{\vdash \Gamma, B, \Delta \quad \vdash \Gamma, A, \Delta}{\vdash \Gamma, A \,\&\, B, \Delta}}{\vdash \Gamma, A \,\&\, B, \Theta} \approx \&\,\frac{\alpha\,\dfrac{\vdash \Gamma, A, \Delta}{\vdash \Gamma, A, \Theta} \quad \alpha\,\dfrac{\vdash \Gamma, B, \Delta}{\vdash \Gamma, B, \Theta}}{\vdash \Gamma, A \,\&\, B, \Theta}$$

Non-local rule permutations

$$\otimes\,\frac{\begin{matrix}\mathcal{D}\,\Vert \\ \vdash \Gamma_1\end{matrix} \qquad \&\,\dfrac{\vdash \Gamma_2, C \quad \vdash \Gamma_2, D}{\vdash \Gamma_2, C \,\&\, D}}{\vdash \Gamma, C \,\&\, D} \simeq \&\,\frac{\otimes\,\dfrac{\begin{matrix}\mathcal{D}\,\Vert \\ \vdash \Gamma_1\end{matrix} \quad \vdash \Gamma_2, C}{\vdash \Gamma, C,} \qquad \otimes\,\dfrac{\begin{matrix}\mathcal{D}\,\Vert \\ \vdash \Gamma_1\end{matrix} \quad \vdash \Gamma_2, D}{\vdash \Gamma, D,}}{\vdash \Gamma, C \,\&\, D}$$

$\alpha, \alpha_1, \alpha_2 \in \{\parr, \oplus, \exists, \forall\}$

Figure 2: Rule permutations in MALL_1.

links on $\vdash \Gamma$ provided with a ***witness map*** δ^Λ *associating to each link $\alpha \in \Lambda$ a (possibly empty)* ***dualizer*** δ_α*, that is, a substitution with domain variables occurring bound by an existential quantifier ($\exists$).*

Conflict nets for MALL_1 are trees alternating *concord* ($\frown$) and *conflict* ($\#$) nodes, having the elements of an axiomatic linking with witnesses as leaves, and satisfying a correctness criterion called *coalescence*. ***Slice nets*** for MALL_1 are sets of axiomatic linkings satisfying a correctness criterion called *erasing steps*. We denote by $\{\!\{\mathcal{D}\}\!\}_{\mathsf{co}}$ (resp. $\{\!\{\mathcal{D}\}\!\}_{\mathsf{sl}}$) the conflict net (resp. slice net) encoding a derivation $\mathcal{D}$ in MALL_1. Details on the criteria[1] criterion and the translations from derivations to proof nets can be found in [1].

We define the following notions of proof equivalence for derivations in MALL_1.

Definition 2. *We call the variable introduced during the proof search by a quantifier rule the* ***active variable*** *of that (occurrence of) rule. The active variable of an existential (resp. universal) quantifier may also be called its* ***witness*** *(resp.* ***eigenvariable****). Two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in MALL_1 are* ***equivalent modulo****:*

- ***active variables renaming*** *(denoted $\mathcal{D}_1 \sim_{\mathsf{w}} \mathcal{D}_2$) if it is possible to transform $\mathcal{D}_1$ into $\mathcal{D}_2$ by changing the active variables of the quantifier rules*

[1] Both criteria are a form of contractibiity, whose intuition is similar to the one of Danos'[2]

(and propagating the changes upwards in the derivation);

- ***rule permutations*** *(denoted $\mathcal{D}_1 \simeq \mathcal{D}_2$) if it is possible to transform $\mathcal{D}_1$ into $\mathcal{D}_2$ using all transformations in Figure 2;*
- ***local rule permutations*** *(denoted $\mathcal{D}_1 \approx \mathcal{D}_2$) if it is possible to transform $\mathcal{D}_1$ into $\mathcal{D}_2$ using only the local rule permutations in Figure 2.*

Moreover, we write $\mathcal{D}_1 \simeq_{\mathsf{w}} \mathcal{D}_2$ (resp. $\mathcal{D}_1 \approx_{\mathsf{w}} \mathcal{D}_2$) if they are equivalent modulo rule permutations (resp. local rule permutations) and active variables renaming, that is, if there are derivations $\mathcal{D}'_1$ and $\mathcal{D}''_2$ such that $\mathcal{D}_1 \sim_{\mathsf{w}} \mathcal{D}'_1 \simeq \mathcal{D}'_2 \sim_{\mathsf{w}} \mathcal{D}_2$ (resp. $\mathcal{D}_1 \sim_{\mathsf{w}} \mathcal{D}'_1 \approx \mathcal{D}'_2 \sim_{\mathsf{w}} \mathcal{D}_2$).

Remark 1. *In Equation* (1) *we show two existential quantifier rules select two distinct witnesses x and z, but the pair of atoms linked by an axiom rule is the same.*

$$\exists\,\frac{\mathsf{ax}\,\dfrac{}{\vdash \langle x!a\rangle, (x?a)}}{\vdash \exists x.\langle x!a\rangle, \exists y.(y?a)} \quad \sim_{\mathsf{w}} \quad \exists\,\frac{\mathsf{ax}\,\dfrac{}{\vdash P(y,a,t), (y?a)}}{\vdash \exists x.\langle x!a\rangle, \exists y.(y?a)} \tag{1}$$

We could argue that these two derivations should be not identified because the choice of the witness is part of the information of the proof. In a boarder sense, it may be useful to not identify a proof using a very elementary witness with a proof using a quite complex one. Note that the two sub-derivations of the $\sim_{\mathsf{w}}$-equivalent derivations in Equation (1) *made only of the* ax*-rules are not $\sim_{\mathsf{w}}$-equivalent. That is* $\mathsf{ax}\,\dfrac{}{\vdash \langle x!a\rangle, (x?a)} \not\sim_{\mathsf{w}} \mathsf{ax}\,\dfrac{}{\vdash P(y,a,t), (y?a)}$.

However, the choice of active variables may change the pair of atoms linked by the ax*-rules. For an example, see Equation* (2) *below, where we show two non $\sim_{\mathsf{w}}$-equivalent derivations in which we trace the occurrences of atoms in the derivation to show how the choice of the active variables changes the pairs of atoms linked by the* ax*-rules.*

$$\exists\,\frac{\mathsf{mix}\,\dfrac{\mathsf{ax}\,\dfrac{}{\vdash \langle x!a\rangle, (x?a)} \quad \mathsf{ax}\,\dfrac{}{\vdash \langle x!b\rangle, (x?b)}}{\vdash \langle x!a\rangle, \langle x!b\rangle, (x?a), (x?b)}}{\vdash \langle x!a\rangle, \langle x!b\rangle, \exists z.(x?z), \exists z.(x?z)} \quad \not\sim_{\mathsf{w}} \quad \exists\,\frac{\mathsf{mix}\,\dfrac{\mathsf{ax}\,\dfrac{}{\vdash \langle x!a\rangle, (x?a)} \quad \mathsf{ax}\,\dfrac{}{\vdash \langle x!b\rangle, (x?b)}}{\vdash \langle x!a\rangle, \langle x!b\rangle, (x?b), (x?a)}}{\vdash \langle x!a\rangle, \langle x!b\rangle, \exists z.(x?z), \exists z.(x?z)} \tag{2}$$

We prove the following canonicity results.

Theorem 1 ([1])**.** *Let $\mathcal{D}$ and $\mathcal{D}'$ be derivations in MALL_1. The following hold:*

1. $\mathcal{D} \approx_{\mathsf{w}} \mathcal{D}'$ *iff* $\{\!\{\mathcal{D}\}\!\}_{\mathsf{co}} = \{\!\{\mathcal{D}'\}\!\}_{\mathsf{co}}$.
2. $\mathcal{D} \simeq_{\mathsf{w}} \mathcal{D}'$ *iff* $\{\!\{\mathcal{D}\}\!\}_{\mathsf{sl}} = \{\!\{\mathcal{D}'\}\!\}_{\mathsf{sl}}$.

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