On Correctness, Sequentialization and Interpolation

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Abstract. In this abstract, we show, following the methodology of proof-relevant interpolation [6], that Maehara Interpolation can be lifted to proof-nets and, moreover, that it can be directly achieved by enriching the process of checking correctness of proof structures with interpolation information. We explain this idea by considering Guerrini & Masini's Parsing criterion and will end outlining how this methodology can be extended to the Contractibility criterion.[4]

1 Introduction: on the Logical Content of Correctness Criteria

While correctness criteria for proof structures are a cornerstone of linear logic proof theory, they have most often been considered with a viewpoint aiming at disconnecting correctness from its logical meaning, in order to emphasize combinatorial, graphical, geometrical or topological invariants that underline the correctness of proof structures[5]. A contrario, it is of interest, if only for pedagogical reasons, to understand which logical principles are at work in some correctness criteria, that is to understand the logical content of a given correctness criterion. While this is often folklore (as a significant part of linear logic proof-theoretical studies of the "golden age'...'), we think that it is often overlooked and not wide-spread enough. For instance, if it is a well-known fact that the Parsing condition corresponds to a distributed, top-down, process sequentializing proof-structures (which, of course, is successful when the proof structure happens to be a proof-net), it is less known that Danos-Regnier switchings corresponds to tests, that is counter-proofs, and that Danos-Regnier expressed purely in terms of correction graphs, is equivalent to Béchet's criterion [1] and that it simply amounts to the fact that a proof structure is correct if, and only if, all its interactions with sequent counter-(para-)proofs (with Daimons) are successful (that is, if no cut-elimination gets stuck). Even less known probably is the fact that the Contractibility criterion can be viewed as a sequentialization process generalizing that of the Parsing condition where the top-down discipline has been relaxed.

With the present abstract, we hope to convince the readers and participants of TLLA that there is interesting logical content to extract from correctness conditions, by focusing on the problem of interpolating proof structures, in the sense of Craig's interpolation and by refining Maehara's method which constitutes the standard proof-theoretical method for proving interpolation in the last 60 years. Our slogan is that the Parsing reduction (and, more generally, contractibility) can be enriched in a more informative rewriting, performing sequentialization, interpolation as cut-introduction, or both, as depicted in Figure 1 which depicts the enrichment over the basic correctness criterion.

[4] While the work on Interpolating as Parsing is done, a significant part on capturing interpolation as proof-structure contraction is still ongoing work.

[5] This was often motivated by some complexity considerations, thus the need to relate the correctness problem with well-studied algorithmic problems and to make them free from inessential details...

		Sequentialization	
		NO	YES
Interpolation	NO	Decision Problem	Sequentializing
	YES	Interpolating Proof Nets	Interpolating Sequent Proofs

Fig. 1. The two dimensions of Parsing enrichment

2 Background on Interpolation, Maehara's method and proof-relevant interpolation

Craig's Interpolation [2] theorem essentially ensures that a logical entailment in classical first-order logic $A \models B$ can be factored through a formula I, the interpolant, which is built from the common vocabulary (that is, the predicate and possibly function symbols) of A and B, that is $\mathsf{voc}(I) \subseteq \mathsf{voc}(A) \cap \mathsf{voc}(B)$, $A \models I$ and $I \models B$.

Soon after Craig's interpolation theorem, Maehara [5] came up with a proof-theoretical approach to interpolation relying on Gentzen's Hauptsatz and exploiting in a crucial way the subformula property of LK. In order to prove the result, the inductive statement had to be overloaded by requiring that not only for logical judgments $A \vdash B$ do we have interpolation, but more generally for any sequent $\Gamma \vdash \Delta$ and any splitting of Γ (resp. Δ) into Γ' and Γ'' (resp. into Δ' and Δ''), there exists a interpolating formula I such that (i) $\mathsf{voc} I \subseteq \mathsf{voc}(\Gamma', \Delta') \cap \mathsf{voc}(\Gamma'', \Delta'')$, (ii) $\Gamma' \vdash I, \Delta'$ and (iii) $\Gamma'', I \vdash \Delta''$. Both Craig's result and Maehara's method were extended to various logics, while Lyndon's showed [4] that in some cases, interpolation could also be done in a polarised way[6].

Recently, Saurin observed [6] that Maehara's method for interpolating could be enriched into a proof-relevant version of interpolation which is compatible with cut-elimination, that is which preserves the denotation of proofs, which can be stated in the following way for the one-sided sequent calculus of LL (with Lyndon's refinement) and holds not only on fo LL, but also LJ and LK:

Theorem 1. *Let Γ, Δ be lists of LL formulas and $\pi \vdash \Gamma, \Delta$ cut-free. There exists a LL formula I such that $\mathsf{voc}^+(I) \subseteq \mathsf{voc}^-(\Gamma) \cap \mathsf{voc}^+(\Delta)$ and $\mathsf{voc}^-(I) \subseteq \mathsf{voc}^+(\Gamma) \cap \mathsf{voc}^-(\Delta)$ and two cut-free proofs π_1, π_2 of $\vdash \Gamma, I$ and $\vdash I^\perp, \Delta$ respectively such that*
$$\dfrac{\overset{\pi_1}{\vdash \Gamma, I} \quad \overset{\pi_2}{\vdash I^\perp, \Delta}}{\vdash \Gamma, \Delta}\ cut \ \longrightarrow^\star_{\mathsf{cut}} \ \pi.$$

In the following, we shall adopt a convenient color-code to represent splittings of contexts, by assuming that formulas in sequents, proofs and proof nets may be uncolored (the usual case) or colored in red or blue in order to express the fact that they are members of the left or right component of the splitting. Note also that an important point for interpolation to hold in form of the statements we presented above is that the logic that we consider comes equipped with units, we shall comment on this below as we come to proof-nets.

3 Interpolation as Parsing

In this section, we show that a proof-relevant Maehara-like interpolation for proof-nets can be obtained from adding more structure on the labelling of the parsing reduction considered by Guerrini and Masini. Our results hold in MELL but we shall only detail the multiplicative case below, adopting

[6] In such a case, the vocabulary is refined into positive and negative vocabularies, $\mathsf{voc}^+(\cdot)$ and $\mathsf{voc}^-(\cdot)$.

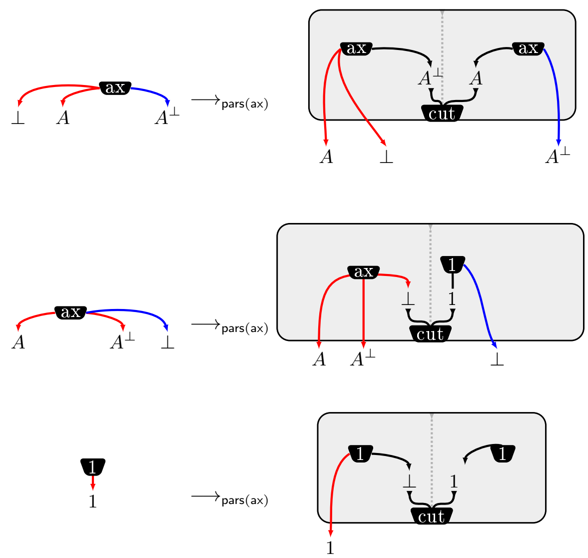


Fig. 2. Interpolating–parsing rewriting for para-proof nets: Axiom cases

the proof-nets used by Guerrini and Masini which contains units with jumps from $\bot$ nodes to an axiom and we will work with η expanded proofs.

Remark 1. Having the ability to consider units is indeed crucial in Maehara's interpolation as mentioned above, when interpolating formulas with an empty common language. In our linear setting it requires to use the units 1 and $\bot$. In MELL (Multiplicative Exponential Linear Logic), those units could be easily encoded by taking a fixed variable X and defining $1 :=!(X \parr X^\bot)$ and $\bot :=?(X \otimes X^\bot)$, which would do the job. On the other side, the Parsing criterion for $MELL$ is more complicated, the crucial point being the way to treat weakenings: in [3], Guerrini and Masini deal with this difficulty by introducing a slight variant of MELL, MELL^{axw}, in which there are jumps from weakening nodes to axioms and a modified version of the promotion rule is introduced, so that $\bot$ is allowed to pass through exponential boxes just like why not formulas. In this abstract, we want to outline the essential of the mechanism of interpolation by parsing and we will then focus on MLL, thus avoiding the difficulties of exponentials. In this setting, we do need to have units. Nonetheless, observe that confluence is not required for the cut-introduction procedure we describe.

Definition 1. *An* **interpolation box** *of conclusion Γ, Δ is the data (T_1, T_2, A) of two colored cut-free paraproof structures T_1, T_2 and a formula A such that: (i) T_1 is completely red with conclusions Γ, A (ii) T_2 is completely blue with conclusions $A^\bot, \Delta$ and (iii) $\mathsf{voc}(A) \subseteq \mathsf{voc}(\Gamma) \cap \mathsf{voc}(\Delta)$.*

Theorem 2. *Suppose we are given a cut-free paraproof structure S. Then, S is DR-correct if and only if, for any splitting Γ, Δ of the conclusions of S, there exists an interpolation box (R_1, R_2, I)*

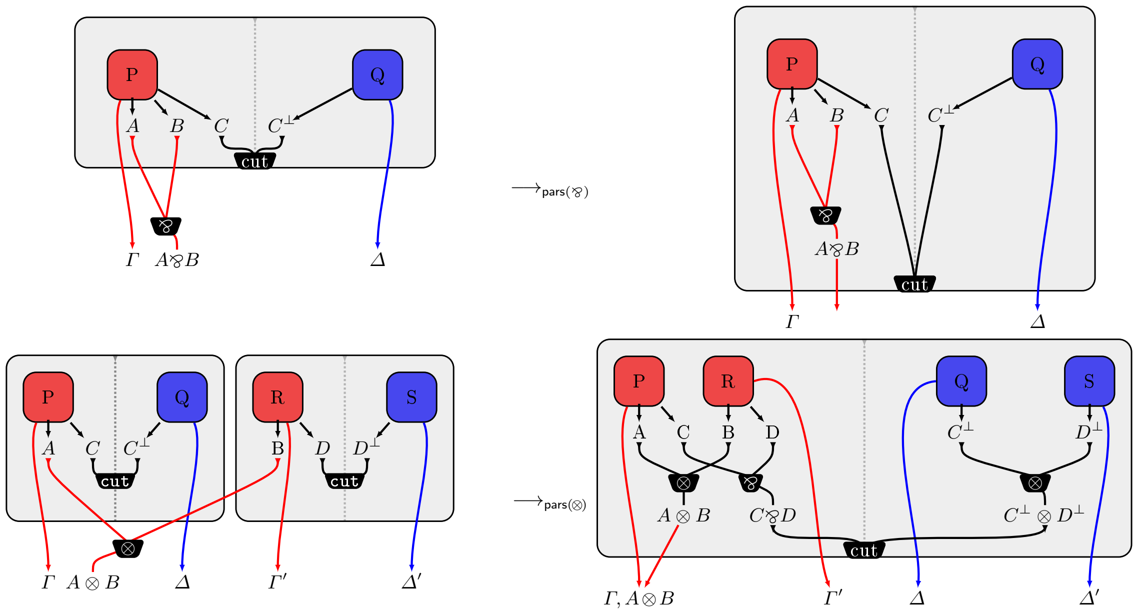
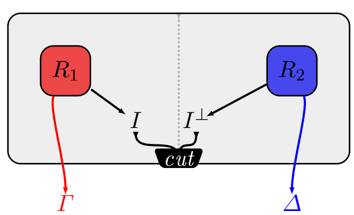


Fig. 3. Interpolating–parsing rewriting for para-proof nets: multiplicative cases

such that R_1, R_2 are DR-correct paraproof structures and



(1)

is equal to S ***up to cut elimination****.*

Theorem 1 is proved by giving an algorithm that, given a cut-free para proof structure, produces the desired interpolation box. This procedure amounts essentially to an enriched labeling of the rewriting used by Guerrini and Masini [3]. This labeling works in two steps:

1. We use the rules $\longrightarrow_{\mathsf{pars(ax)}}$ in Figure 2 to replace every axioms (possibly, with $\perp$ attached) with an interpolation box. This can be seen as a form of cut introduction: if we eliminate the cut between the interpolating formula and its dual, we obtain the replaced axioms. If there are ✠ boxes, we replace them with interpolation boxes by using similar rules as those for axioms.
2. We apply $\longrightarrow_{\mathsf{pars}(⅋)}$ and $\longrightarrow_{\mathsf{pars}(\otimes)}$ of Figure 3 to merge interpolation boxes.

Since this is a labeling of Guerrini and Masini rewriting, if the paraproof structure is correct, we will be able to perform the parsing until a single interpolation box remains, which has the same conclusions as the original paraproof structure. If the original structure did not contain any ✠ box, so does the final interpolation box: this procedure produces interpolating proof nets when applied to proof nets.

4 Conclusion: Towards Interpolation as Proof-Structure Contraction

Maehara's method of interpolation relies on two assumptions, besides admissibility of cuts: (i) the fact that proof are presented as a sequential structure and (ii) the fact that they are inductively defined starting from axioms. In this abstract, we showed that we can perform interpolation directly on proof nets, thus overcoming the requirement of having a sequential structure. It also exemplifies that one can view the parsing rewriting as a way to collect information on the proof-structure which may be in the most basic situation, the correctness of the proof structure, but which can also be seen as providing an actual sequentialization of the proof structure and, as we illustrate here, which can actually compute an interpolant together with interpolating proofs, both dimensions being of course compatible as illustrated in the table from the introduction.

Our approach is not restricted to the Parsing criterion and its rewriting relation. Danos' Contractibility criterion, of which the Parsing criterion can be seen as a particular case, is also amenable to such an analysis: in the course of a contraction path, one can label the nodes of a paired graph used to perform contractibility with additional information which may not only be used to achieve a sequentialization of the proof structure but that can also compute an interpolation. This more general result will show that interpolation can be performed not only when weakening the sequential nature of the sequent proofs, but also when we weaken its inductive nature (indeed, a parsing reduction corresponds to the inductive structure of the sequent proof). Contrarily to the interpolation induced by the parsing which starts in the axioms and is propagated top-down, following a dynamics close to that of Maehara, rewriting for the Contractibility criterion can indeed start from any node in the proof structure. If performed top-down from the axioms, the procedure of Interpolation by parsing just described is found again but one has access to many more contracting reductions.

Another direction that we plan to explore in the future is that doing proof-relevant Maehara's interpolation in proof-nets brings an interesting phenomenon to light: while a given sequent proof has only one Maehara's interpolant our various paths of parsing, or contraction, generate distinct interpolants: we plan to analyze such sets of interpolants of a given proof-nets to understand which structure they share.

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