Intuitionistic BV

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The logic BV was introduced by Guglielmi in [5, 6] in the attempt of providing cut-free deduction system for Retoré's pomset logic [9, 10].[1] To this end, in the same paper Guglielmi developed the deep inference formalism to overcome the design limitations of the traditional proof systems based on Gentzen's work (sequent calculi and natural deduction). In fact, as shown in [12], no cut-free sequent system for BV is possible.

In this talk we discuss the results in [2], where we provide a deep inference system for an intuitionistic version of BV, and a cut-free sequent calculus for a sub-logic in which the connective $\triangleleft$ is non-associative.

$$\begin{tikzcd}
\mathsf{MLL}_{\mathsf{mix}} \arrow[r] & \mathsf{NML} \arrow[r] & \mathsf{BV} \\
\mathsf{IMLL} \arrow[u] \arrow[r] & \boxed{\mathsf{INML}} \arrow[u] \arrow[r] & \boxed{\mathsf{IBV}} \arrow[u]
\end{tikzcd}$$

Intuitionistic BV. In classical BV the triple $\langle \otimes, \parr, \mathbb{I} \rangle$ forms an *isomix category* [4], and the non-commutative connective ***seq***, denoted $\triangleleft$, is a *degenerate linear functor* (in the sense of [3]), that is, it validates the following implication.

$$((A \triangleleft B) \otimes (C \triangleleft D)) \multimap ((A \otimes C) \triangleleft (B \otimes D)) \tag{1}$$

In particular, because the unit and the seq have to be self-dual, they cannot be polarized, and therefore it was assumed that there cannot be an intuitionistic version of BV. ***Intuitionistic*** BV (IBV) is defined by extending intuitionistic multiplicative linear logic (IMLL), where the triple $\langle \otimes, \multimap, \mathbb{I} \rangle$ forms a symmetric monoidal closed structure, with a non-commutative connective $\triangleleft$ validating Equation (1) and the unit laws $A \multimap (\mathbb{I} \triangleleft A)$ and $A \multimap (A \triangleleft \mathbb{I})$. [2] We prove the deduction theorem and cut-elimination for the system IBV (see Figure 1).

Theorem 1 (Cut-elimination)**.** *The rule* $\mathsf{cut}\,\dfrac{A \multimap A}{\mathbb{I}}\bullet$ *is admissible* IBV*.*

Then, we prove that IBV is, indeed, the intuitionistic version of BV by proving that it is a conservative extension of IMLL, which can be extended (conservatively) to BV.

[1] The inclusion of BV in pomset has been known since the introduction of BV [11]. However, that this inclusion is strict has only been proven recently [8, 7].

[2] In classical BV the condition on $\triangleleft$ of being a degenerate linear functor causes the unit of the $\otimes$ and $\parr$, to also be the (left and right) unit for $\triangleleft$. However, if $(\mathbb{I} \triangleleft A) \multimap A$ and $(A \triangleleft \mathbb{I}) \multimap A$ were both valid in IBV, then the connectives $\otimes$ and $\triangleleft$ would collapse.

Formulas

$$A, B \coloneqq a \mid \mathbb{I} \mid A \otimes B \mid A \multimap B \mid A \triangleleft B \qquad\qquad a \in \mathcal{A}$$

Rules

$$\mathsf{ai}^{\circ}_{\downarrow}\,\frac{\mathbb{I}}{a \multimap a}\circ \qquad \mathsf{u}^{\triangleleft}_{\downarrow}\,\frac{A}{\mathbb{I} \triangleleft A}\circ \qquad \mathsf{u}^{\triangleright}_{\downarrow}\,\frac{A}{A \triangleleft \mathbb{I}}\circ \qquad \mathsf{ref}^{\circ}\,\frac{A \otimes B}{A \triangleleft B}\circ \qquad \mathsf{ref}^{\bullet}\,\frac{A \triangleleft B}{A \otimes B}\bullet$$

$$\mathsf{s}^{\circ}_{\mathsf{L}}\,\frac{A \otimes (B \multimap C)}{(A \multimap B) \multimap C}\circ \qquad \mathsf{s}^{\circ}_{\mathsf{R}}\,\frac{(A \multimap B) \otimes C}{A \multimap (B \otimes C)}\circ \qquad \mathsf{s}^{\bullet}_{\mathsf{L}}\,\frac{(A \multimap B) \multimap C}{A \otimes (B \multimap C)}\bullet \qquad \mathsf{s}^{\bullet}_{\mathsf{R}}\,\frac{A \multimap (B \otimes C)}{(A \multimap B) \otimes C}\bullet$$

$$\mathsf{sq}^{\circ}_{\mathsf{L}}\,\frac{(A \multimap B) \triangleleft C}{A \multimap (B \triangleleft C)}\circ \qquad \mathsf{sq}^{\circ}_{\mathsf{R}}\,\frac{B \triangleleft (A \multimap C)}{A \multimap (B \triangleleft C)}\circ \qquad \mathsf{sq}^{\bullet}_{\mathsf{L}}\,\frac{(A \otimes B) \triangleleft C}{A \otimes (B \triangleleft C)}\bullet \qquad \mathsf{sq}^{\bullet}_{\mathsf{R}}\,\frac{B \triangleleft (A \otimes C)}{A \otimes (B \triangleleft C)}\bullet$$

$$\mathsf{q}^{\circ}_{\downarrow}\,\frac{(A \multimap B) \triangleleft (C \multimap D)}{(A \triangleleft C) \multimap (B \triangleleft D)}\circ \qquad\qquad \mathsf{q}^{\bullet}_{\downarrow}\,\frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}\bullet$$

$$\mathsf{com}^{\otimes}\,\frac{A \otimes B}{B \otimes A} \qquad \mathsf{asso}^{\otimes}\,\frac{(A \otimes B) \otimes C}{A \otimes (B \otimes C)} \qquad \mathsf{asso}^{\triangleleft}_{\mathsf{L}}\,\frac{(A \triangleleft B) \triangleleft C}{A \triangleleft (B \triangleleft C)} \qquad \mathsf{asso}^{\triangleleft}_{\mathsf{R}}\,\frac{A \triangleleft (B \triangleleft C)}{(A \triangleleft B) \triangleleft C}$$

$$\mathsf{u}^{\otimes}_{\downarrow}\,\frac{A}{\mathbb{I} \otimes A} \qquad \mathsf{u}^{\multimap}_{\downarrow}\,\frac{A}{\mathbb{I} \multimap A} \qquad \mathsf{cur}\,\frac{(A \otimes B) \multimap C}{A \multimap (B \multimap C)} \qquad \mathsf{ruc}\,\frac{A \multimap (B \multimap C)}{(A \otimes B) \multimap C}$$

Derivations (where A is a formula)

$$\text{Positive:} \quad \mathcal{P}, \mathcal{Q} \coloneqq \; A \;\Big|\; \mathcal{P} \otimes \mathcal{Q} \;\Big|\; \mathcal{P} \triangleleft \mathcal{Q} \;\Big|\; \mathcal{N} \multimap \mathcal{P} \;\Big|\; \mathsf{r}\,\frac{\mathcal{P}}{\mathcal{Q}}\circ \;\Big|\; \mathsf{r}\,\frac{\mathcal{P}}{\mathcal{Q}}$$

$$\text{Negative:} \quad \mathcal{N}, \mathcal{M} \coloneqq \; A \;\Big|\; \mathcal{N} \otimes \mathcal{M} \;\Big|\; \mathcal{N} \triangleleft \mathcal{M} \;\Big|\; \mathcal{P} \multimap \mathcal{N} \;\Big|\; \mathsf{r}\,\frac{\mathcal{N}}{\mathcal{M}}\bullet \;\Big|\; \mathsf{r}\,\frac{\mathcal{N}}{\mathcal{M}}$$

Figure 1: Formulas, inference rules for system IBV, and the inductive definition of derivations.

Non-associative IBV. We also discuss a weaker logic we call obtained by dropping associativity for the connective $\triangleleft$. For this logic, we provide a cut-elimination sequent calculus we call INML recalled in Figure 2, which is the two-sided single-succedent version of the calculus NML from [1]. As for IBV, we prove the conservativity results of INML with respect to IMLL, as well as NML with respect to INML.

Finally, we prove that by extending the sequent calculus for INML with the rules in the bottom of Figure 2 we provide a sequent calculus for IBV.

Theorem 2. *Let A be a formula. Then $\vdash_{\mathsf{IBV}} A$ iff $\vdash_{\mathsf{INML} \cup \{\mathsf{a\text{-}cut_L}, \mathsf{a\text{-}cut_R}\}} A$.*

$$\mathsf{ax}\,\frac{}{a \vdash a} \qquad \multimap_{\mathsf{R}}\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \multimap_{\mathsf{L}}\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \qquad \otimes_{\mathsf{L}}\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \otimes_{\mathsf{R}}\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\mathbb{I}_R\,\frac{}{\vdash \mathbb{I}} \qquad \mathbb{I}_L\,\frac{\Gamma \vdash A}{\Gamma, \mathbb{I} \vdash A} \qquad \triangleleft\,\frac{\Gamma, A_1, \ldots, A_n \vdash A \quad \Delta, B_1, \ldots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \ldots, A_n \triangleleft B_n \vdash A \triangleleft B}\, n \geq 0$$

$$\mathsf{a\text{-}cut_L}\,\frac{\Gamma \vdash (A \triangleleft B) \triangleleft C \quad A \triangleleft (B \triangleleft C), \Delta \vdash D}{\Gamma, \Delta \vdash D} \qquad \mathsf{a\text{-}cut_R}\,\frac{\Gamma \vdash A \triangleleft (B \triangleleft C) \quad (A \triangleleft B) \triangleleft C, \Delta \vdash D}{\Gamma, \Delta \vdash D}$$

Figure 2: Sequent calculus for INML, and the additional *associative-cut* rules.

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