EXPLORING SEMIGROUP-THEORETIC TECHNIQUES IN LAMBDA CALCULUS

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Group theory is inconsistent with λ-calculus, in the sense that there exists no interpretation of the variety of groups in the variety of combinatory algebras. This follows from a result by Plotkin, Selinger, and Simpson, which shows that the existence of a Mal'cev operation (i.e., congruence permutability) is inconsistent with λ-calculus (see [8]). Semigroup theory might suggest that the situation is not as pessimistic as it may seem. Let $\mathbf{B} := \lambda fgx.f(gx)$ denote the composition combinator. The set of λ-terms modulo β-equivalence, equipped with the product $X \circ Y := \mathbf{B}XY$, forms a semigroup (see [1, Chapter 21]). If one further assumes η-equivalence, this structure becomes a monoid with identity element $\mathbf{I} := \lambda x.x$. The present authors have initiated a project aimed at applying the rich toolkit of semigroup theory to the study of this semigroup. The purpose of this note is dual: to outline the scope of the project and to report on the progress made thus far.

The potential contributions of this project are twofold. Firstly, it continues the tradition of uncovering classical algebraic structures within λ-calculus [6]. Given that the set of all λ-terms does not admit a group structure, it is worthwile to investigate whether certain *subsets* of λ-terms can form groups. Semigroup theory comes to our rescue in this context, since it is not difficult to endow suitable sets of λ-terms with the structure of a semigroup, and appropriate subsets of a semigroup constitute a group (see Sections 1 and 2). In Section 3, we instead demonstrate how symmetric groups can be identified within the λ-calculus, and we characterise the maximal monoids of λ-terms whose invertible elements constitute a symmetric group (see also [7]). Secondly, this project aims at describing the semigroup of λ-terms of a given λ-theory determined by composition (see Section 4). In this respect, Statman [9] has proven that the semigroup of $\lambda\beta$ is SQ-universal, meaning that every countable semigroup can be embedded in the semigroup of the term model of an opportune λ-theory. Given any λ-theory $\mathcal{T}$, we can associate to it the semigroup $(\Lambda_{\mathcal{T}}, \circ)$, where $\Lambda_{\mathcal{T}}$ denotes the set of λ-terms modulo the equivalence relation defined by $\mathcal{T}$, and $\circ$ is the operation defined by composition. Although we do not know whether this assignment from λ-theories to (countable) semigroups is surjective, the remarkable result by Statman shows that for every countable semigroup S, there exists a λ-theory $\mathcal{S}$ such that S embeds into $(\Lambda_{\mathcal{S}}, \circ)$. This suggests that the class of semigroups arising from λ-theories is at any rate very broad.

1. A primer on Green relations

Let S be a semigroup. Certain equivalence relations on S, first studied by Green in the 1950s, have played a central role in the development of semigroup theory. These relations center on the notion of *divisibility* and, notably, all become trivial (i.e., degenerate to the total relation) when S is a group. For a detailed account of semigroup theory, we refer the reader to the classic monograph [4].

From this point onward, we assume that S is a monoid. Let $a, b \in S$. The left and right Green relations, denoted by $\mathcal{L}$ and $\mathcal{R}$, are defined as follows:

(1) $(a, b) \in \mathcal{L}$ if there exist $x, y \in S$ such that $xa = b$ and $yb = a$;
(2) $(a, b) \in \mathcal{R}$ if there exist $x, y \in S$ such that $ax = b$ and $by = a$.

The letter $\mathcal{H}$ denotes their intersection $\mathcal{L} \cap \mathcal{R}$. Given $a \in S$, we write L_a, R_a, and H_a for its equivalence classes under $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$, respectively. A celebrated theorem of Green states that for any $\mathcal{H}$-class $H \subseteq S$, either $H^2 \cap H = \varnothing$ or $H^2 = H$, in which case H forms a subgroup of S. Note that H^2 is a notation for the set $\{h \cdot k \mid h, k \in H\}$. An important corollary of Green's theorem is that if $e \in S$ is an idempotent, then the $\mathcal{H}$-class H_e is a subgroup of S with identity element e. This is the largest subgroup of S containing e. Moreover, no $\mathcal{H}$-class in S can contain more than one idempotent, so the maximal subgroups of S are precisely the $\mathcal{H}$-classes that contain an idempotent. These are necessarily pairwise disjoint. If 1 is the unit of the monoid S, then H_1 is the group of invertible elements, traditionally called the group of units. For example, when S is the monoid of λ-terms modulo η, it follows from a theorem due independently to Dezani and to Bergstra and Klop [1, Theorem 21.2.21] that $H_{\mathbf{I}}$ coincides with the set of finite hereditary permutations (see below).

2. Some special classes of terms

In light of the foregoing section, we aim to characterise nontrivial $\mathcal{H}$-classes of idempotent λ-terms. Given an idempotent term E, we look for elements A and B such that $(A, E) \in \mathcal{L}$ and $(B, E) \in \mathcal{R}$, with the idea of finding $A = B \in \mathcal{L} \cap \mathcal{R} = \mathcal{H}$. A technique that exploits the results concerning the left- and right-invertibility of λ-terms [2, Section 9.3], consists in constructing G, D such that $G \circ D =_\beta \mathbf{I}$ and then in defining $A := D \circ E$ and $B := E \circ G$.

We focus our investigation on certain particularly well-behaved classes of λ-terms (if not stated otherwise, we concentrate on *closed* terms). Note that all semigroups we are going to define are subsemigroups of $(\Lambda_{\lambda\beta}, \circ)$.

Definition 2.1. A λ-term is said to be *regular* if it is of the form

$$M =_\beta \lambda x\, x_1 \dots x_n.\, x\, M_1 \cdots M_k,$$

for some $M_1, \dots, M_k$ and $n \geq 0$. When $n = 0$, we say that M is *strongly regular*.

It is easy to verify that both the set of regular and the set of strongly regular λ-terms are closed under composition, and therefore each forms a semigroup, and indeed a monoid since $\mathbf{I}$ is strongly regular.

Proposition 2.2. *In the monoid of strongly regular terms:*

(1) *M is (right-) left-invertible iff $M =_\beta \mathbf{I}$.*
(2) *$L_{\mathbf{I}} = R_{\mathbf{I}} = \{\mathbf{I}\}$ and therefore $H_{\mathbf{I}} = \{\mathbf{I}\}$.*
(3) *$\mathbf{I}$ is the unique idempotent.*

Turning to regular terms, we characterise regular idempotents.

Lemma 2.3. *Let $E := \lambda x x_1 \dots x_k.x P_1 \cdots P_h$ be a regular term. Then E is idempotent iff $k = h$ and for each $i = 1, \dots, k$, $P_i[x_1 := P_1, \dots, x_k := P_k] =_\beta P_i$.*

Moreover,

Proposition 2.4. *In the monoid of regular terms we have:*

(1) *M is right-invertible iff M is strongly regular;*
(2) *M is left-invertible iff $M = \lambda xy_1 \dots y_n.x$ for some n;*
(3) *$L_{\mathbf{I}} = \{M : M =_\beta \lambda xy_1 \dots y_n.x$ for some $n\}$, $R_{\mathbf{I}}$ is the set of strongly regular terms, and therefore $H_{\mathbf{I}} = \{\mathbf{I}\}$.*

Example 2.5. Let $\mathbf{Y} := \lambda f.\,(\lambda x.\,f(xx))\,(\lambda x.\,f(xx))$ denote the well-known fixed-point combinator, and let $\mathbf{2} := \lambda fx.\,f(fx)$ denote Church's numeral for 2. Then the term $\lambda xy.\,x(\mathbf{Y2}y)$ is a regular idempotent. More complex examples of regular idempotents can be constructed using the [1, 6.5.2. Multiple Fixed Point Theorem].

As in the strongly regular case, the $\mathcal{H}$-class of $\mathbf{I}$ in the monoid of regular terms remains trivial. However, it will be shown in Theorem 3.4 below that there exist regular idempotents E such that H_E is a nontrivial group.

Another important and promising class of λ-terms to consider is that of *linear* λ-terms. Linear λ-terms are of special interest because they enjoy desirable computational properties–most notably, strong normalisation–and this often leads to well-behaved semantic behaviour. We recall that the typing system of linear λ-terms corresponds to the implicative fragment of multiplicative linear logic [3].

Example 2.6. Examples of linear idempotents are $\mathbf{1} := \lambda xx_1.xx_1$, and its generalisations $\mathbf{1}_n := \lambda xx_1 \cdots x_n.xx_1 \cdots x_n$ for all $n \geq 0$.

A particularly rich subclass of linear λ-terms is formed by the η-expansions of the identity that are the terms $\beta\eta$-equivalent to $\mathbf{I}$. This subclass has been investigated by Intrigila and Nesi in [5]. They focused on the subset $\Lambda_\eta^{\mathbf{I}}$ of finite η-expansions of $\mathbf{I}$. They established that composition in this subset corresponds, at the level of Böhm trees, to the union operation. As a consequence, they get the following result, where $\Lambda_\eta^{\mathbf{I}}(n,h)$ is the set of all those η-expansions of $\mathbf{I}$ such that the maximum outdegree of their nodes is n and the maximum depth of their leaves is h.

Proposition 2.7.

(1) [5, Corollary 2] *$\Lambda_\eta^{\mathbf{I}}$ is a commutative idempotent monoid with unit $\mathbf{I}$.*
(2) [5, Proposition 3] *$\Lambda_\eta^{\mathbf{I}}(n,h)$ is a finite submonoid of $\Lambda_\eta^{\mathbf{I}}$.*

In a commutative monoid, Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ all coincide–greatly simplifying the analysis of the semigroup's internal structure.

Another important subclass of linear terms is the set FHP of *finite hereditary permutations*. FHP consists of those linear λ-terms, whose Böhm tree results from a (finite) η-expansion of $\mathbf{I}$ by permuting around the branches (see [2, Definition 9.61]). By [1, Theorem 21.2.21] a λ-term M is $\beta\eta$-invertible (i.e., $M \circ X =_{\beta\eta} X \circ M =_{\beta\eta} \mathbf{I}$ for some X) iff $M \in$ FHP.

Proposition 2.8. FHP *is a monoid with unit* $\mathbf{I}$.

3. Groups in lambda calculus

We now give a characterisation of the largest monoid with unit $\mathbf{1}_n$, whose group $H_{\mathbf{1}_n}$ of invertible elements is isomorphic to the group of all permutations of n elements. In this section λ-terms are considered up to β-equivalence.

We say that a λ-term M is n-invertible if there exists X such that $M \circ X = X \circ M = \mathbf{1}_n$. Let σ be a permutation of $\{1, \dots, n\}$. Then the term $M = \lambda x y_1 \dots y_n . x y_{\sigma_1} \dots y_{\sigma_n}$ is called a permutation of $\mathbf{1}_n$. We denote by Perm_n the set of permutations of $\mathbf{1}_n$.

Lemma 3.1. *The permutations of $\mathbf{1}_n$ are n-invertible and constitute a group under composition with unit $\mathbf{1}_n$.*

We now characterise the λ-terms M such that $\mathbf{1}_n \circ M = M = M \circ \mathbf{1}_n$.

Definition 3.2. Let $N \in \Lambda(x)$. We say that $\lambda x.N$ is *n-good* if, for every leaf node γ in the tree representation of N such that $N(\gamma) = x$, there exists a prefix δ of γ such that the subterm t of N of root δ has the form $t = xY_1 \dots Y_r$ with $r \geq n$.

Lemma 3.3. *Let $n \geq 0$ and M be a closed λ-term. Then we have:*

(1) *$\mathbf{1}_n \circ M = M$ iff M has order $> n$ (i.e., $M =_\beta \lambda x x_1 \dots x_n . N$).*
(2) *$M \circ \mathbf{1}_n = M$ iff M is n-good.*

Let $n \geq 0$. We define: $\mathcal{G}_n$ to be the set of closed λ-terms that are n-good of order $> n$. We say that a λ-term $\lambda x.M$ is *head-linear* if x occurs free exactly once in M. We define $\mathcal{LR}_n$ to be the set of closed terms M such that M is a head-linear regular term of order $> n$ and branching $\geq n$ (i.e., $M =_\beta \lambda x y_1 \dots y_m . x M_1 \dots M_k$ with $m, k \geq n$). We can then state the most important result of this section.

Theorem 3.4. *Let $n \geq 0$.*

(1) *$\mathcal{G}_n$ is a monoid under composition with unit $\mathbf{1}_n$;*
(2) *$\mathcal{LR}_n$ is a submonoid of $\mathcal{G}_n$.*

In both $\mathcal{G}_n$ and $\mathcal{LR}_n$, the group Perm_n is the $\mathcal{H}$-class of the idempotent $\mathbf{1}_n$.

As every finite group can be embedded in a group of permutations,

Corollary 3.5. *For every finite group G, there is n for which $(\mathcal{G}_n, \circ, \mathbf{1}_n)$ is the largest monoid with unit $\mathbf{1}_n$ and G embeds into the group $H_{\mathbf{1}_n}$ of invertible elements of $\mathcal{G}_n$.*

4. Statman's Theorem

We conclude this abstract by presenting some consequences of the aforementioned theorem by Statman [9]. We say that a set X of λI-terms of order 0 is *independent* if, for every $M \in X$ no β-reduct of M contains a β-reduct of any member of X as a proper subterm. For example, the set of terms $\{\Omega \mathbf{n} : n \in \mathbb{N}\}$, where $\Omega := (\lambda x.xx)(\lambda x.xx)$ and $\mathbf{n}$ is the Church numeral for $n \geq 1$, is independent.

Let X be a fixed countable independent set of λI-terms. We denote by X^* the set of words over the alphabet X. The interpretation of concatenation of words in the λ-calculus is given by composition. In $\lambda\beta$ distinct words cannot be equated:

Lemma 4.1 ([9]). *If $P, Q \in X^*$, then $P =_\beta Q$ iff P and Q are the same word.*

Proposition 4.2. *The free semigroup with a countable set of generators can be embedded into the semigroup $(\Lambda_\beta, \circ)$. Therefore, the semigroup $(\Lambda_\beta, \circ)$ generates the variety of semigroups.*

Lemma 4.3. *In the semigroup $(\Lambda_\beta, \circ)$ we have $H_P = \{P\}$ for every word $P \in X^*$.*

Definition 4.4. We say that a λ-theory $\mathcal{T}$ is an *sgλ-theory* if $\mathcal{T}$ is axiomatised by identities between words in X^*.

If $\mathcal{T}$ is an sgλ-theory, then we denote by $sg(\mathcal{T})$ the equational theory in the language of semigroups axiomatised by the identities between words axiomatising $\mathcal{T}$.

Lemma 4.5. [9, Lemma 12] *Let $\mathcal{T}$ be an sgλ-theory and $P, Q \in X^*$. Then $\mathcal{T} \vdash P = Q$ iff $sg(\mathcal{T}) \vdash P = Q$.*

Theorem 4.6. [9, Theorem 13] *The semigroup $(\Lambda_\beta, \circ)$ is SQ-universal, i.e., every countable semigroup can be embedded into the semigroup of the term model of a suitable sgλ-theory.*

Corollary 4.7. *The algebraic lattice of equational theories of semigroups is isomorphic to the algebraic lattice of sgλ-theories.*

The join of two sgλ-theories in the lattice of sgλ-theories coincides with the join of these λ-theories in the lattice of all λ-theories, while the meet does not in general.

5. Conclusions and vistas

At the beginning, we outlined two main goals for this project. Regarding the first–the search for groups within λ-calculus–we are currently working on a generalisation of some results of Section 3 to the case of an arbitrary η-expansion of $\mathbf{I}$ in place of $\mathbf{1}_n$. As for the second goal–developing invariants of λ-theories via semigroup theory–further work is still needed. To each λ-theory $\mathcal{T}$ we can associate the variety of semigroups generated by $(\Lambda_\mathcal{T}, \circ)$. Further, we can define a preorder on λ-theories setting $\mathcal{T} < \mathcal{T}'$ if the variety of $\mathcal{T}'$ is contained in that of $\mathcal{T}$. A characterisation of the order on the equivalence classes associated to the preorder would be illuminating. Note that if $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T} < \mathcal{T}'$ and the lattice of λ-theories gets flattened into this preorder. Finally, we mention that there are other operations than composition that make the set of λ-terms into a semigroup [9, Section 2].

Acknowledgements. We would like to express our gratitude to the referees.

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