An extensional perspective on higher categorical models of linear logic

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The goal of this presentation will be to give detailed examples of "∞-categorical models of linear logic" as defined in our FSCD paper [HM25], motivating them through analogies with more well-known 1- and 2-categorical models. These models can be seen as a generalization of Girard's original model of normal functors [Gir88] and more recent models of species [Fio+08; FGH24] and polynomials [GK13; HM24].

Categorical semantics of linear logic. There are multiple ways to axiomatize what it means for a category to be a model of linear logic. As far as Intuitionistic Linear Logic is concerned, the notion of linear non-linear adjunction encompasses all others, as advocated in [Mel09]. A linear non-linear adjunction is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

between a category with finite products $(\mathcal{M}, \times)$ and a symmetric monoidal closed category $(\mathcal{L}, \otimes)$ such that the left adjoint $L : \mathcal{M} \to \mathcal{L}$ is strongly symmetric monoidal from the cartesian structure on $\mathcal{M}$ to the monoidal structure on $\mathcal{L}$.

Any such adjunction induces a lax-monoidal comonad $LM : \mathcal{L} \to \mathcal{L}$ which models the exponential modality of (intuitionistic) linear logic. The tensor product and monoidal closure on $\mathcal{L}$ give interpretations to the tensor and linear implication connectives of linear logic, and it can be shown that the structure of the linear non-linear adjunction is enough for this to constitute a denotational model of ILL.

Relational models. The simplest and most well-known categorical model of linear logic is the relational model. In the relational model, the formulae of linear logic are interpreted as sets, the proofs of $A \vdash B$ are interpreted as relations $R \subseteq [\![A]\!] \times [\![B]\!]$, and the exponential $!A$ is interpreted as the set $\mathrm{Mul}([\![A]\!])$ of (finite) multisets on $[\![A]\!]$. In this case, the corresponding linear non-linear adjunction is between the monoidal category $\mathcal{L} := \mathrm{Rel}$ (with tensor product given by the cartesian product of underlying sets), and $\mathcal{M} := \mathrm{Rel}_{\mathrm{Mul}}$ the coKleisli category for the comonad Mul on Rel.

An even simpler linear non-linear adjunction involving the category Rel is given by

$$\mathrm{Set} \underset{\mathbf{P}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}} \mathrm{Rel}$$

i.e. the adjunction induced by identifying Rel as the Kleisli category for the powerset monad on Set The left adjoint is strongly monoidal because the monoidal structure on Rel is given by the cartesian product of underlying sets.

This LNL adjunction gives a way to interpret the powerset comonad $\mathbf{P}$ on Rel as an exponential modality of linear logic.

Extensional point of view on relations. Every relation $R \subseteq X \times Y$ induces a union-preserving map between their powersets

$$\begin{aligned} \mathbf{P}(X) &\to \mathbf{P}(Y) \\ U \subseteq X &\mapsto \{y \in Y \mid \exists x \in U, x\ R\ y\} \end{aligned}$$

and every union-preserving map between these powersets is uniquely determined by a relation $R \subseteq X \times Y$ in that way. A poset with arbitrary joins is called a suplattice, and a join-preserving map is called a *suplattice morphism* or *linear map*. In the same way a matrix represents a linear map between vector spaces, a relation represents a linear map betwen suplattices.

From the previous discussion, we see that the fullsubcategory of SupLat on the suplattices of the form $(\mathbf{P}(X), \subseteq)$ is equivalent to the category Rel. The tensor product on Rel extends to a tensor product on SupLat where $E \otimes F$ has the universal property that linear maps $E \otimes F \to G$ correspond to maps $E \times F \to G$ that preserve joins *independently in both variables*.

The multiset comonad on Rel extends to the *cofree commutative comonoid comonad* on SupLat, and the powerset comonad extends to the powerset comonad on SupLat, induced by

$$\mathrm{Set} \underset{\text{forget}}{\overset{F}{\rightleftarrows}} \mathrm{SupLat} \quad (\bot)$$

But there are other interesting exponential comonads on SupLat.

From sets to posets. Let $\mathbb{P}$ be a class of posets. The category $\mathrm{Poset}_{\mathbb{P}}$ of posets E that admit join of families indexed by posets in $\mathbb{P}$ admits a symmetric monoidal structure where the tensor product $E \otimes F$ classifies maps that are "$\mathbb{P}$-linear" independently in both variables. In particular, $\mathrm{Poset}_{\mathrm{all}} = \mathrm{SupLat}$. Moreover, when $\mathbb{P} \subset \mathbb{P}'$, the forgetful functor $\mathrm{Poset}_{\mathbb{P}'} \to \mathrm{Poset}_{\mathbb{P}}$ admits a strongly monoidal left adjoint, a kind of "relative cocompletion".

Writing dir for the class of directed posets, given $\mathbb{P} \subseteq \mathrm{dir}$, it turns out the monoidal structure on $\mathrm{Poset}_{\mathbb{P}}$ is cartesian. Summing everything up, we have the following chain of strongly monoidal left adjoints.

$$(\mathrm{Set}, \times) \underset{\bot}{\rightleftarrows} (\mathrm{Poset}, \times) \underset{\bot}{\rightleftarrows} (\mathrm{Poset}_{\mathrm{dir}}, \times) \underset{\bot}{\rightleftarrows} (\mathrm{SupLat}, \otimes)$$

In particular, this gives three exponential comonads on SupLat. The adjunction with Set induces the powerset comonad $\mathbf{P}$ as before, and it restricts to Rel. The adjunction with Poset gives a variant of the powerset comonad that retains more information about the ordering. The adjunction with $\mathrm{Poset}_{\mathrm{dir}}$ gives the domain-theoretic exponential on Rel.

From sets to posets. Write Porel for the category whose objects are posets and morphisms are ordered relations $E \times F^{\mathrm{op}} \to \mathrm{Bool}$. The functors

$$\begin{aligned} \mathrm{Set} &\to \mathrm{SupLat} \\ X &\mapsto \mathbf{P}(X) := \mathrm{Hom}_{\mathrm{Set}}(X, \mathrm{Bool}) \end{aligned}$$

whose essential image is equivalent to Rel extends to a functor

$$\begin{aligned} \mathrm{Poset} &\to \mathrm{SupLat} \\ E &\mapsto \mathcal{P}(E) := \mathrm{Hom}_{\mathrm{Porel}}(E^{\mathrm{op}}, \mathrm{Bool}) \end{aligned}$$

whose essential image is equivalent to Porel.

Theorem 1. *Under this equivalence, the three previous comonad act on the underlying posets respectively as*

- $E \mapsto \mathbf{P}(E)$ *the free cocompletion of the underlying set of* E*,*
- $E \mapsto \mathcal{P}(E)$ *the free cocompletion of* E*,*
- $E \mapsto \mathcal{F}(E)$ *the free cocompletion of* E *under finite joins.*

While the Mul *comonad acted as the free commutative monoid on underlying sets.*

The relationship between $\mathcal{F}$ and Mul has already been studied for instance in [Ehr12].

From posets to categories. This whole story generalizes to a categorical setting: sets are replaced by (∞-) groupoids, posets by (∞-) categories, Bool by the category Set of sets (or $\mathcal{S}$ of ∞-groupoids). With an additional subtlety: how to generalize the notion of directed poset.

Given a class of (∞−) categories $\mathbb{C}$, write $\mathrm{Cat}_{\mathbb{C}}$ for the (∞−) category of (∞−) categories with $\mathbb{C}$-indexed colimits and functors that preserve such colimits. Then we have a symmetric monoidal structure on $\mathrm{Cat}_{\mathbb{C}}$ as before, and symmetric monoidal left adjoints to the forgetful functors $\mathrm{Cat}_{\mathbb{C}'} \to \mathrm{Cat}_{\mathbb{C}}$ [Lur17].

In particular, writing sift for the class of sifted (∞-) categories and filtr for the class of filtered (∞−) categories, we have the following chain of symmetric monoidal left adjoints.

$$(\mathrm{Grpd}, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Cat}, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Cat}_{\mathrm{filtr}}, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Cat}_{\mathrm{sift}}, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Cat}_{\mathrm{all}}, \otimes)$$

Theorem 2. *The full subcategory of* Cat_{all} *on presheaf categories is equivalent to the category of categories and profunctors, and from this point of view the induced comonads on* Prof *correspond to*

- *the free cocompletion of the underlying groupoid*
- *the free cocompletion*
- *the free cocompletion under finite colimits*
- *the free cocompletion under finite coproducts*

And as before, we can also construct the free exponential by taking cofree commutative comonoids in $\mathrm{Cat}_{\mathrm{all}}$, and the action on the underlying category in Prof will be to take the free symmetric monoidal category, yielding back the exponential from the theory of generalized species of structures [Fio+08; FGH24], a generalization of Girard's original model of normal functors [Gir88].

The general analogy is summed up in table 1.

0-categories (posets)	1-categories	∞-categories
set $X \in \mathsf{Set}$	groupoid $X \in \mathsf{Grpd}$	∞-groupoid $X \in \mathcal{S}$
poset $P \in \mathsf{Poset}$	category $\mathcal{C} \in \mathsf{Cat}$	∞-category $\mathcal{C} \in \mathsf{Cat}_\infty$
relation $r : X \times Y \to \mathsf{Bool}$	functor $F : X \times Y \to \mathsf{Set}$	∞-functor $F : X \times Y \to \mathcal{S}$
$(r;r')(x,z) = \bigvee_y r(x,y) \wedge r'(y,z)$	$(F;F')(x,z) = \operatorname{colim}_y F(x,y) \times F'(y,z)$	
relation $R \subseteq X \times Y$	discrete fibration $Z \to X \times Y$	fibration $Z \to X \times Y$
ordered relation $r : P \times Q^{\mathrm{op}} \to \mathsf{Bool}$	profunctor $F : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathsf{Set}$	∞-profunctor $F : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$
$(r;r')(x,z) = \bigvee_y r(x,y) \wedge r'(y,z)$	$(F;F')(x,z) = \int^y F(x,y) \times F'(y,z)$ (coend formula)	
suplattice	cocomplete category	cocomplete ∞-category
$\mathbf{P}(P) = \mathsf{Bool}^{P^{\mathrm{op}}}$	$\mathbf{P}(\mathcal{C}) = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$	$\mathbf{P}(\mathcal{C}) = \mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$
domain	category with filtered colimits	∞-category with filtered colimits

Table 1: Analogies between 0-, 1- and ∞-categories

References

[Ehr12] Thomas Ehrhard. "The Scott model of linear logic is the extensional collapse of its relational model". In: *Theoretical Computer Science* 424 (2012), pp. 20–45. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2011.11.027. URL: https://www.sciencedirect.com/science/article/pii/S0304397511009467 (cit. on p. 3).

[FGH24] M. Fiore, N. Gambino, and M. Hyland. *Monoidal bicategories, differential linear logic, and analytic functors*. version: 2. May 23, 2024. DOI: 10.48550/arXiv.2405.05774. (Visited on 06/22/2024) (cit. on pp. 1, 3).

[Fio+08] M. Fiore et al. "The cartesian closed bicategory of generalised species of structures". In: *Journal of the London Mathematical Society* 77.1 (Feb. 2008), pp. 203–220. ISSN: 00246107. DOI: 10.1112/jlms/jdm096. (Visited on 06/29/2023) (cit. on pp. 1, 3).

[Gir88] Jean-Yves Girard. "Normal functors, power series and λ-calculus". In: *Annals of Pure and Applied Logic* 37.2 (Feb. 1, 1988), pp. 129–177. ISSN: 0168-0072. DOI: 10.1016/0168-0072(88)90025-5. URL: https://www.sciencedirect.com/science/article/pii/0168007288900255 (visited on 11/14/2022) (cit. on pp. 1, 3).

[GK13] Nicola Gambino and Joachim Kock. "Polynomial functors and polynomial monads". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 154.1 (Jan. 2013), pp. 153–192. ISSN: 0305-0041, 1469-8064. DOI: 10.1017/S0305004112000394. (Visited on 09/29/2022) (cit. on p. 1).

[HM24] Elies Harington and Samuel Mimram. "Polynomials in homotopy type theory as a Kleisli category". In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 4 - Proceedings of MFPS XL, 11 (Dec. 2024). ISSN: 2969-2431. DOI: 10.46298/entics.14786 (cit. on p. 1).

[HM25] Elies Harington and Samuel Mimram. *∞-categorical models of linear logic*. 2025. URL: https://www.lix.polytechnique.fr/Labo/Elies.HARINGTON/papers/ll_infinity.pdf (cit. on p. 1).

[Lur17] Jacob Lurie. *Higher Algebra*. Sept. 18, 2017. URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf (visited on 09/07/2023) (cit. on p. 3).

[Mel09] Paul-André Mellies. "Categorical semantics of linear logic". In: *Panoramas et syntheses* 27 (2009), pp. 15–215 (cit. on p. 1).