# A Tangent on Categorical Models of Differential Linear Logic

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**Abstract for TLLA 2025**

## 1 Introduction

T. Ehrhard and L. Regnier [1], [2] discovered that by symmetrizing some rules of Linear Logic (LL) one obtains a proof calculus that can express the differential of proofs. This sequent calculus is called Differential Linear Logic (DiLL). Looking at its categorical semantics, given by differential categories [3], [4], we find intuitions coming from differential geometry. For example, the formula $!A \otimes A$ plays the role of the tangent space of $A$, where $!A$ is the base space and $A$ the (non-dependant) vector space associated to each point of $!A$. Since DiLL is simply typed, geometric models of DiLL [5], [6] are based on vector spaces as their tangent bundles is a trivial vector bundle (hence, a non dependent construction).
This observation has lead us to wonder whether differential linear logic could be a particular case of a dependent type theory that expresses differentiation. In order to start this investigation we first reformulate models of linear logic (linear-non-linear adjunction) in terms of Grothendieck fibrations, which is a well known semantic of typed theories [7]. Then, we show that the categorical semantic of differential linear logic (differential categories) is expressed in this setting as a tangent functor, that we call a linear tangent functor, from the base category to the linear fibration.

This tangential presentation of models of DiLL is a linear-non-linear reformulation of the usual semantic, which is based on linear categories. Moreover, we think that this fibered perspective is well suited to further generalization as one should be able to adapt already existing formalism of type theories to our context. We have in mind comprehension categories as they provide a semantic for dependent type theory. Our point of view also fits well with already present intuitions that linear logic should be interpreted geometrically as vector bundles [8].

The starting point of this work is the work of M. Kerjean, M. Rogers and V. Maestracci's [9] on a functorial axiomatisation of categorical models of DiLL. They found a more concise and functorial axiomatisation of categorical models of DiLL. They achieved it by placing linear-non-linear adjunction as a basis and asking for the existence of a functor acting as the differential at a point. Our fibered presentation can be understood as a tangential version of their work.
Some of the core ideas presented are also in work of G. Cruttwell, J. Gallagher, JS. P. Lemay and D. Pronk [10, section 2]. They show that one can associate to a coalgebra modality two fibrations above its coKleisli. The categories $\mathcal{L}[X_!]$ and $\mathcal{L}_![X]$ are respectively particular instances of the categories $s(\mathcal{C})$ and $s(\mathcal{C}, \mathcal{L})$ which are defined below. Our work can be seen as a generalization of this part of their work to linear-non-linear adjunctions.

**Notations :**

- For convenience, we assume that the monoidal categories we consider are always strict meaning that the unitors and associators are assumed to be the identity morphism.
- We denote ";" the diagrammatic composition of morphisms : $f; g \coloneqq g \circ f$.

## 2 Quick Overview

We now give a quick overview of the core ideas and constructions of our work.

Categorical models of linear logic are defined as linear-non-linear adjunctions [11]. We first show that every linear-non-linear adjunction can be expressed as a Grothendieck fibration above the cartesian category.

*Definition 2.1 (linear-non-linear adjunction)*: A linear-non-linear adjunction is a symmetric monoidal adjunction between a symmetric monoidal category $(\mathcal{L}, \otimes, 1)$ and a cartesian category $(\mathcal{C}, \times, I)$

$$(\mathcal{C}, \times, I) \underset{(\mathcal{U}, n)}{\overset{(\mathcal{F}, m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{L}, \otimes, 1)$$

$\mathcal{C}$ is called the cartesian category of the LNL adjunction and $\mathcal{L}$ the monoidal category. We always use $X, Y, Z$ for objects of $\mathcal{C}$ and $A, B, C$ for objects of $\mathcal{L}$. A linear-non-linear adjunction is said to be additive when $\mathcal{L}$ is an additive category [3], this means that $\mathcal{L}$ is enriched over commutative monoids and this enrichment is compatible with the monoidal structure of $\mathcal{L}$ (non-standard terminology).

This definition differ from the usual definition of linear-non-linear adjunction in that we do not require $(\mathcal{L}, \otimes, 1)$ to be monoidal closed. This choice has been made in order to capture greater generality and is not of big importance as closure does not play a role in the interpretation of the other connectives of LL and DiLL.
This structure allows the interpretation of LL in the monoidal category $\mathcal{L}$ with ! given by $\mathcal{F} \circ \mathcal{U}$. The contraction and weakening rule are interpreted as natural transformations (in $\mathcal{L}$) with components $\mathbf{c}_A : !A \to !A \otimes !A$ and $\mathbf{w}_A : !A \to 1$. To construct them, we use the colax structure on $\mathcal{F}$ to transport the cocommutative comonoids $(X, \Delta_X, I_X)$ from $\mathcal{C}$ to $\mathcal{L}$ where $\Delta_X : X \to X \times X$ are the diagonal morphisms and $I_X : X \to I$ are the terminal morphisms.

$$\mathbf{c}_X \coloneqq \mathcal{F}(X) \xrightarrow{\mathcal{F}(\Delta_X)} \mathcal{F}(X \times X) \xrightarrow{m^{-1}_{X,X}} \mathcal{F}(X) \otimes \mathcal{F}(X)$$
$$\mathbf{w}_X \coloneqq \mathcal{F}(X) \xrightarrow{\mathcal{F}(I_X)} \mathcal{F}(I) \xrightarrow{m^{-1}_I} 1$$

We now explain how to construct a Grothendieck fibration from any linear-non-linear adjunction. In order to achieve this we take inspiration from a classic construction in the semantic of simple type theory : the simple category [7, chap. 1.2].

*Definition 2.2 (Simple Category)*: Let $(\mathcal{L}, \times, I)$ be a cartesian category. Define $s(\mathcal{C})$ as the category :
- Objects : pairs $(X, J)$ with $X, J \in \mathcal{C}_0$.
- Morphisms : a morphism from $(X, J)$ to $(Y, K)$ is a pair $(f, u)$ with $f : X \to Y$ and $u : X \times J \to K$.

Given $(f, u) : (X, J) \to (Y, K)$ and $(g, v) : (Y, K) \to (Z, L)$, composition is given by :

$$(f, u); (g, v) \coloneqq (f; g, \Delta_X \times \mathrm{id}_J; f \times u; v)$$

The projection functor $P : s(\mathcal{C}) \to \mathcal{C}$ which sends $(X, J)$ to $X$ and $(f, u)$ to $f$ is a split fibration. The product of $\mathcal{C}$ makes $s(\mathcal{C})$ into a fibered cartesian category.

This category interprets simply typed lambda calculus [7, chap. 2]. Similarly, given a linear-non-linear adjunction, we define a category called the linear-non-linear simple category.

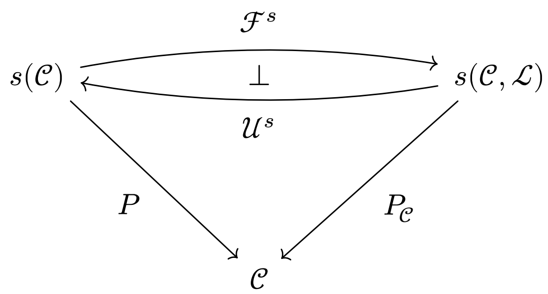
*Definition 2.3 (Linear-non-linear Simple Category)*: The linear-non-linear simple category $s(\mathcal{C}, \mathcal{L})$ of a linear-non-linear adjunction is the category $s(\mathcal{C}, \mathcal{L})$ :
- Objects : pairs $(X, A)$ with $X \in \mathcal{C}_0$ and $A \in \mathcal{L}_0$.
- Morphisms : a morphism from $(X, A)$ to $(Y, B)$ is a pair $(f, u)$ with $f : X \to Y$ and $u : \mathcal{F}(X) \otimes A \to B$.
- The identity of $(X, A)$ is the morphism $(\mathrm{id}_X, \mathbf{w}_X \otimes \mathrm{id}_A)$.

Given $(f, u) : (X, A) \to (Y, B)$ and $(g, v) : (Y, B) \to (Z, C)$, composition is given by :

$$(f, u); (g, v) \coloneqq (f; g \ , \ \mathbf{c}_X \otimes \mathrm{id}_A; \mathcal{F}(f) \otimes u; v)$$

The projection functor $P_{\mathcal{C}} : s(\mathcal{C}, \mathcal{L}) \to \mathcal{C}$ which sends $(X, A)$ to $X$ and $(f, u)$ to $f$ is a split fibration. The product of $\mathcal{L}$ makes $s(\mathcal{C}, \mathcal{L})$ into a fibered monoidal category.

The two categories $s(\mathcal{C})$ and $s(\mathcal{C}, \mathcal{L})$ are respectively the fibered version of the cartesian category $\mathcal{C}$ and the monoidal category $\mathcal{L}$. This is shown by constructing two fibered functors $\mathcal{F}^s : s(\mathcal{C}) \to s(\mathcal{C}, \mathcal{L})$ and $\mathcal{U}^s : s(\mathcal{C}, \mathcal{L}) \to s(\mathcal{C})$ which are involved in a fiberwise linear-non-linear adjunction.

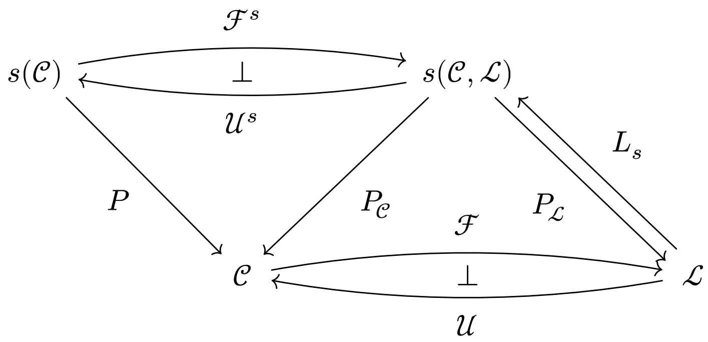



The starting linear-non-linear adjunction is recovered from this construction by taking the fibers above the terminal object of $\mathcal{C}$ : $\mathcal{C} \cong s(\mathcal{C})_I$ and $\mathcal{L} \cong s(\mathcal{C}, \mathcal{L})$. The two functors $\mathcal{U}^s$ and $\mathcal{F}^s$, when restricted to this fiber and passed through those two isomorphisms, identify with $\mathcal{U}$ and $\mathcal{F}$.

Now that we have seen how to construct a fibration from a linear-non-linear adjunction, we explain how asking for the existence of a tangent functor is equivalent to asking that $\mathcal{L}$ is a model of DiLL.

First we need to define two more functors, $P_{\mathcal{L}} : s(\mathcal{C}, \mathcal{L}) \to \mathcal{L}$ and $L_s : \mathcal{L} \to s(\mathcal{C}, \mathcal{L})$.

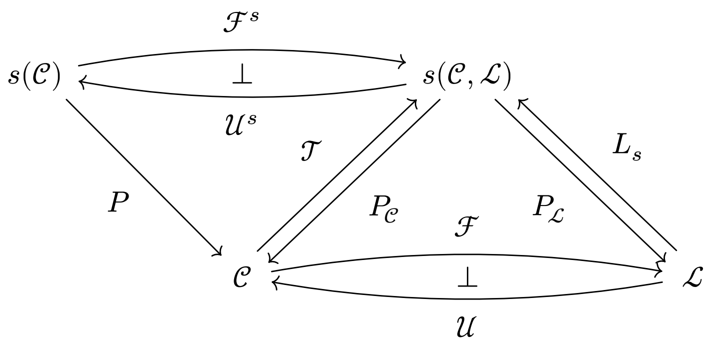
$$P_{\mathcal{L}}(X, A) = \mathcal{F}(X) \otimes A \qquad P_{\mathcal{L}}(f, u) = \mathbf{c}_X \otimes \mathrm{id}_A; \mathcal{F}(f) \otimes u$$
$$L_s(A) = (\mathcal{U}(A), A) \qquad L_s(l) = (\mathcal{U}(l), \mathbf{w}_A \otimes l)$$




*Definition 2.4 (Linear Tangent Functor)*: A linear tangent functor $\mathcal{T}$ for a additive linear-non-linear adjunction is a functor $\mathcal{T} : \mathcal{C} \to s(\mathcal{C},\mathcal{L})$ such that :
1. $\mathcal{T}$ is a section of of $P_{\mathcal{C}}$
2. $\mathcal{T}$ satisfies the linearity rule : $\mathcal{U};\mathcal{T} = L_s$
3. $\mathcal{T}$ satisfies the monoidal rule :

$$\mathcal{T}\left(\eta_{\mathcal{U}(A)}\right) \otimes \mathcal{T}\left(\mathrm{id}_{\mathcal{U}(!B)}\right); n^*_{!A,!B}\left(L_s\left(m_{\otimes_{A,B}}\right)\right) = n^*_{A,!B}(L_s(\mathrm{id}_A \otimes \mathbf{d}_B)); \mathcal{T}\left(\eta_{\mathcal{U}(A\otimes B)}\right)$$

Where $\eta_X : X \to \mathcal{U}\mathcal{F}(X)$ is the unit of the adjunction, $\mathbf{d}_A : !A \to A$ the counit of the adjunction $\mathcal{F} \dashv \mathcal{U}$ and $n^*$ is the reindexing functor relative to $P_{\mathcal{L}}$ of $n_{A,B}$ the lax structure on $\mathcal{U}$.




An additive linear-non-linear adjunction equipped with linear tangent functor is called a tangent linear-non-linear adjunction.

If we think of $(s(\mathcal{C},\mathcal{L}), P_{\mathcal{C}})$ as equipping $\mathcal{C}$ with a notion of (trivial) vector bundle (the fiber above $X$ being the (trivial) vector bundles above $X$ with "Whitney tensor product") then asking for such a tangent functor is quite natural. Notice that the axioms of $\mathcal{T}$ prescribes how it acts on objects of $\mathcal{C}$ of the shape $\mathcal{U}(A)$. This is expected as objects $\mathcal{U}(A)$ play the role of vector spaces and the tangent bundle of a vector space $V$ is a trivial vector bundle $V \times V$ hence, it can be expressed in $s(\mathcal{C},\mathcal{L})$. For more complex spaces that may inhabit $\mathcal{C}$, $\mathcal{T}$ associates it to trivial vector bundle on which we have no information.

We wonder whether a "dependent linear-non-linear fibration"[1] would allows to express in an analogous manner the tangent bundles of more complex spaces. Perhaps such a construction could be to (cartesian) tangent categories what differential categories are to cartesian differential categories.

We now state the main theorem :

[1]Notion yet to be defined but we can take inspiration from the fibered semantic of type theories [7, chap. 10].

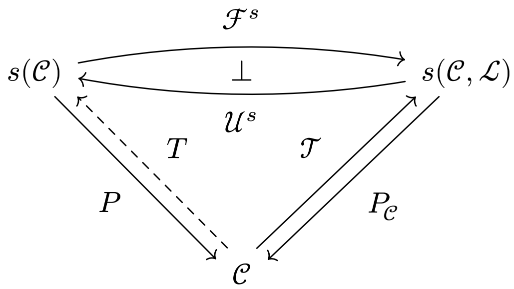
*Theorem 2.1*: Every tangent linear-non-linear adjunction make $\mathcal{L}$ a model of DiLL and reciprocally every model of DiLL (monoidal differential category) is a tangent linear-non-linear adjunction with its Eilenberg-Moore category.

*Proof (Sketch)*: Given a tangent linear-non-linear category define $\partial_A : !A \otimes A \to A$ as the second component of the pair $\mathcal{T}\left(\eta_{\mathcal{U}(A)}\right)$. We show that its a natural transformation. The structure of $s(\mathcal{C}, \mathcal{L})$ allows us to deduce the chain rule. The linearity rule of $\mathcal{T}$ allows us to deduce the linear rule. The monoidal rule of $\mathcal{T}$ allows us to deduce the monoidal rule. This is enough to show that $\mathcal{L}$ is a monoidal differential category.
Reciprocally, given $\mathcal{L}$ a monoidal differential category, it is involved in a linear-non-linear adjunction with its Eilenberg-Moore category $\mathcal{L}^!$. We must now define a linear tangent functor from $\mathcal{L}^!$ to $s(\mathcal{L}^!, \mathcal{L})$. Let $\mathbf{p}_A$ be the comultiplication of !. On objects, every free coalgebra $\mathbf{p}_A$ is sent to $\mathcal{T}(\mathbf{p}_A) := (\mathbf{p}_A, A)$, every non free coalgebra $h : A \to !A$ is sent to $\mathcal{T}(h) := (h, A)$. On morphisms :

- Let $f : \mathbf{p}_A \to \mathbf{p}_B$ be a morphisms between two free coalgebras then, $\mathcal{T}(f) := (f, \partial_A; f; \mathbf{d}_B)$.
- Let $f : \mathbf{p}_A \to h$ be a morphism from a free coalgebra to a non-free coalgebra then, $\mathcal{T}(f) := (f, \partial_A; f)$.
- Let $f : h \to \mathbf{p}_B$ be a morphisms from a non-free coalgebra to a free coalgebra then, $\mathcal{T}(f) := (f, \mathbf{w}_A \otimes (f, \mathbf{d}_B))$.
- Let $f : h \to h'$ be a morphisms between two non-free coalgebras then, $\mathcal{T}(f) := (f, \mathbf{w}_A \otimes f)$

This defines a linear tangent functor.

We finish with a few words on cartesian differential categories.




Given a tangent linear-non-linear adjunction, composing $\mathcal{T}; \mathcal{U}^s$ give rise to a functor $T : \mathcal{C} \to s(\mathcal{C})$. In the case where $\mathcal{L}$ is a Seely category and $\mathcal{C}$ is the coKleisli of !, $T$ sends a morphisms $f : X \to Y$ to $(f, D[f])$ where $D$ is the differential operator of $\mathcal{L}_!$. We recover the usual constructions in the Seely case.
Define $\mathcal{C}_\times$ as the smallest full subcategory of $\mathcal{C}$ containing objects of shape $\mathcal{U}(A)$ and closed by product. Then $\mathcal{C}_\times$ is a cartesian differential with $T$ its associated cartesian tangent structure [12] (as in the Seely case).
This results seams similar Theorem 3.9 of [12], with object of shape $\mathcal{U}(A)$ playing the role of differential objects. As per the discussion page 4, the link between our fibered presentation and tangent category should be investigated.

# Bibliography


[1] T. Ehrhard and L. Regnier, “Differential interaction nets,” *Theoretical Computer Science*, vol. 364, no. 2, pp. 166–195, Nov. 2006, doi: 10.1016/j.tcs.2006.08.003.

[2] T. Ehrhard, “An introduction to differential linear logic: proof-nets, models and antiderivatives,” *Mathematical Structures in Computer Science*, vol. 28, no. 7, pp. 995–1060, Aug. 2018, doi: 10.1017/S0960129516000372.

[3] R. F. Blute, J. R. B. Cockett, J.-S. P. Lemay, and R. A. G. Seely, “Differential Categories Revisited,” *Applied Categorical Structures*, vol. 28, no. 2, pp. 171–235, Apr. 2020, doi: 10.1007/s10485-019-09572-y.

[4] M. P. Fiore, “Differential Structure in Models of Multiplicative Biadditive Intuitionistic Linear Logic: (Extended Abstract),” *Typed Lambda Calculi and Applications*, vol. 4583. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 163–177, 2007. doi: 10.1007/978-3-540-73228-0_13.

[5] J. Clift and D. Murfet, “Cofree coalgebras and differential linear logic,” *Mathematical Structures in Computer Science*, vol. 30, no. 4, pp. 416–457, Apr. 2020, doi: 10.1017/S0960129520000134.

[6] R. Blute, T. Ehrhard, and C. Tasson, “A convenient differential category.”

[7] B. Jacobs, *Categorical logic and type theory*, 1st ed., no. v.141. in Studies in logic and the foundations of mathematics. Amsterdam ; New York: Elsevier Science, 1999.

[8] P.-A. Melliès, “A Functorial Excursion Between Algebraic Geometry and Linear Logic,” in *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, Haifa Israel: ACM, Aug. 2022, pp. 1–13. doi: 10.1145/3531130.3532488.

[9] M. Kerjean, V. Maestracci, and M. Rogers, “Functorial Models of Differential Linear Logic,” Nov. 2023. Accessed: Jan. 13, 2025. [Online]. Available: https://hal.science/hal-04292862

[10] G. Cruttwell, J. Gallagher, J.-S. P. Lemay, and D. Pronk, “Monoidal reverse differential categories,” *Mathematical Structures in Computer Science*, vol. 32, no. 10, pp. 1313–1363, Nov. 2022, doi: 10.1017/S096012952200038X.

[11] P.-A. Melliès, “Categorical Semantics of Linear Logic,” [Online]. Available: https://www.irif.fr/~mellies/papers/panorama.pdf

[12] J. R. B. Cockett and G. S. H. Cruttwell, “Differential Structure, Tangent Structure, and SDG,” *Applied Categorical Structures*, vol. 22, no. 2, pp. 331–417, Apr. 2014, doi: 10.1007/s10485-013-9312-0.