Craig Interpolation for Semi-Substructural Logics

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1 Introduction

Substructural logics are logic systems that lack at least one of the structural rules, weakening, contraction, and exchange. Joachim Lambek's syntactic calculus [10] is a well-known example that disallows weakening, contraction, and exchange. Another example, linear logic, proposed by Jean-Yves Girard [7], is a substructural logic in which weakening and contraction are in general disallowed but can be recovered for some formulae via modalities. Substructural logics have been found in numerous applications from computational analysis of natural languages to the development of resource-sensitive programming languages.

Left skew monoidal categories [13] are a weaker variant of MacLane's monoidal categories where the structural morphisms of associativity and unitality are not required to be bidirectional, they are natural transformations with a particular orientation. Therefore, they can be seen as *semi-associative* and *semi-unital* variants of monoidal categories. Left skew monoidal categories arise naturally in the semantics of programming languages [2], while the concept of semi-associativity is connected with combinatorial structures like the Tamari lattice and Stasheff associahedra [21].

In recent years, in collaboration with Tarmo Uustalu and Noam Zeilberger, we started a research project on *semi-substructural* logics, which is inspired by a series of developments on left skew monoidal categories and related variants by Szlachányi, Street, Bourke, Lack and many others [13, 12, 9, 4, 5].

We call the internal languages of left skew monoidal categories and their variants *semi-substructural* logics, because they are intermediate logics in between (certain fragments of) non-associative and associative intuitionistic linear logic (or Lambek calculus). Semi-associativity and semi-unitality are encoded as follows. Sequents are in the form $S \mid \Gamma \vdash A$, where the antecedent consists of an optional formula S, called stoup, adapted from Girard [8], and an ordered list of formulae Γ. The succedent is a single formula A. We restrict the application of introduction rules in an appropriate way to allow only one of the directions of associativity and unitality.

This approach has successfully captured internal languages for a variety of categories, including (i) left skew semigroup [21], (ii) left skew monoidal [17], (iii) left skew (prounital) closed [15], (iv) left skew monoidal closed categories [14, 19], and (v) left distributive skew monoidal categories with finite products and coproducts [20] through skew variants of the fragments of non-commutative intuitionistic linear logic consisting of combinations of connectives $(\mathsf{I}, \otimes, \multimap, \wedge, \vee)$. Additionally, discussions have covered partial normality conditions, in which one or more structural morphisms are allowed to have an inverse [16], as well as extensions with skew exchange à la Bourke and Lack [18, 20].

All of the aforementioned calculi with sequents of the form $S \mid \Gamma \vdash A$ are cut-free and therefore, by their rule design, they are decidable. Moreover, they all admit sound and complete subcalculi inspired by Andreoli's focusing [3] in which rules are restricted to be applied in a specific order. A focused calculus provides an algorithm to solve both the proof identity problems for its non-focused calculus and coherence problems for its corresponding variant of left skew monoidal category.

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We say a logic $\mathscr{L}$ has Craig interpolation property if for any formula $A \to C$ provable in $\mathscr{L}$, there exists a formula B such that $A \to B$ and $B \to C$ are provable in $\mathscr{L}$, satisfying the variable condition: $\mathsf{var}(B) \subseteq \mathsf{var}(A) \cap \mathsf{var}(C)$ ($\mathsf{var}(A)$ is the set of atomic formulae appearing in A) [6] ($\to$ is the implication connective in $\mathscr{L}$).

In this work, we show that sequent calculi for left skew monoidal (closed) categories enjoy Craig interpolation.

2 Sequent Calculus and Interpolation

Recall the sequent calculus (`LSkG`) for left skew monoidal closed categories from [14], which is a skew variant of non-commutative multiplicative intuitionistic linear logic.

Formulae (Fma) are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes B \mid A \multimap B$, where X comes from a countably infinite set At of atoms, I is a multiplicative unit, $\otimes$ is multiplicative conjunction and $\multimap$ is a linear implication.

A sequent is a triple of the form $S \mid \Gamma \vdash A$, where the antecedent splits into: an optional formula S, called *stoup* [8], and an ordered list of formulae Γ and succedent A is a single formula. The symbol S consistently denotes a stoup, meaning S can either be a single formula or empty, indicated as $S = -$; furthermore, X, Y, and Z always represent atomic formulae.

Derivations in are generated recursively by the following rules:

$$
\begin{array}{cccc}
\dfrac{}{A \mid \quad \vdash A}\ \mathsf{ax} & \dfrac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\ {\multimap}\mathsf{L} & \dfrac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\ \mathsf{IL} & \dfrac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\ {\otimes}\mathsf{L} \\[2ex]
\dfrac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\ \mathsf{pass} & \dfrac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} & \dfrac{}{- \mid \quad \vdash \mathsf{I}}\ \mathsf{IR} & \dfrac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\ {\otimes}\mathsf{R}
\end{array}
\tag{1}
$$

The inference rules in (1) are similar to the ones in the sequent calculus for non-commutative multiplicative intuitionistic linear logic (`NMILL`) [1], but with some crucial differences:

1. The left logical rules IL, ${\otimes}\mathsf{L}$ and ${\multimap}\mathsf{L}$, read bottom-up, are only allowed to be applied on the formula in the stoup position.
2. The right tensor rule ${\otimes}\mathsf{R}$, read bottom-up, splits the antecedent of a sequent $S \mid \Gamma, \Delta \vdash A \otimes B$ and in the case where S is a formula, S is always moved to the stoup of the left premise, even if Γ is empty.
3. The presence of the stoup distinguishes two types of antecedents, $A \mid \Gamma$ and $- \mid A, \Gamma$. The structural rule pass (for 'passivation'), read bottom-up, allows the moving of the leftmost formula in the context to the stoup position whenever the stoup is empty.
4. The logical connectives of `NMILL` typically include two ordered implications $\multimap$ and $\multimapinv$, which are two variants of linear implication arising from the removal of the exchange rule from intuitionistic linear logic. In here, only the left implication $\multimap$ is present.

For a more detailed explanation and an interpretation of the system as a logic of resources, see [14, Section 2]. This calculus is sound and complete wrt. left skew monoidal closed categories and cut-free, i.e., following two rules are admissible:

$$
\dfrac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \dfrac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}
$$

We introduce an equivalence relation on derivations $\stackrel{\circ}{=}$ [14], corresponding to the equational theory of skew monoidal closed categories.

We are interested in if `LSkG` enjoys Craig interpolation. For substructural logic (especially non-commutative logic), we have a general version of interpolation [11]:

> Given $f : \Gamma \vdash C$ and any partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist a formula D and two derivations $g : \Gamma_1 \vdash D$ and $h : \Gamma_0, D, \Gamma_1 \vdash C$, and $\mathsf{var}(D) \subseteq \mathsf{var}(\Gamma_0) \cap \mathsf{var}(\Gamma_0, \Gamma_1, C)$ ($\mathsf{var}(A)$ is the set of atomic formulae appearing in A and $\mathsf{var}(\Gamma)$ means $\bigcup \mathsf{var}(A_i)$ for $A_i \in \Gamma$).

Due to two cut rules in `LSkG`, we should consider two versions of general interpolation:

> (scut-interpolation) Given $f : S \mid \Gamma \vdash C$ and any partition $\langle \Gamma_0, \Gamma_1 \rangle$ of Γ, there exist a formula D and two derivations $g : S \mid \Gamma_0 \vdash D$ and $h : D \mid \Gamma_1 \vdash C$, and $\mathsf{var}(D) \subseteq \mathsf{var}(s(S), \Gamma_0) \cap \mathsf{var}(\Gamma_1, C)$, where $s(S) = I$ if $S = -$ or $s(S) = B$ if $S = B$.
>
> (ccut-interpolation) Given $f : S \mid \Gamma \vdash C$ and any partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist a partition of $\langle \Delta_1, \ldots, \Delta_n \rangle$ of Γ_1, a list of formulae $D_1, \ldots, D_n$ and derivations $g : S \mid \Gamma_0, D_1, \ldots, D_n, \Gamma_2 \vdash C$ and $h_i : - \mid \Delta_i \vdash D_i$ for $i \in [1 \ldots n]$ such that $\mathsf{var}(D_1, \ldots, D_n) \subseteq \mathsf{var}(\Delta_1, \ldots, \Delta_n) \cap \mathsf{var}(s(S), \Gamma_0, \Gamma_2, C)$.

These two statements are proved by mutual induction on derivations.
For scut-interpolation, the critical case is $f = \multimap\mathsf{L}(f', f'')$, with the partition $\langle \Gamma_0, (\Gamma_1, \Gamma_2) \rangle$ for Γ, and two derivations $f' : - \mid \Gamma_0, \Gamma_1 \vdash A$ and $f'' : B \mid \Gamma_2 \vdash C$. Our goals is to find a formula D and derivations $g : A \multimap B \mid \Gamma_0 \vdash D$ and $h : D \mid \Gamma_1, \Gamma_2 \vdash C$. We first apply inductive hypothesis of ccut-interpolation on f' and then get $g' : - \mid \Gamma_0, D_1, \ldots, D_n \vdash A$, $h'_i : - \mid \Delta_i \vdash D_i$, $i \in [1 \ldots n]$. By applying inductive hypothesis of scut-interpolation on f'', we obtain derivations $g'' : B \mid \quad \vdash E$ and $h'' : E \mid \Gamma_2 \vdash C$. Then we construct desired derivations as follows:

$$
\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\overset{g'}{- \mid \Gamma_0, D_1, \ldots, D_n \vdash A} \quad \overset{g''}{B \mid \quad \vdash E}}{A \multimap B \mid \Gamma_0, D_1, \ldots, D_n \vdash E}\multimap\mathsf{L}}{A \multimap B \mid \Gamma_0, D_1, \ldots, D_{n-1} \vdash D_n \multimap E}\multimap\mathsf{R}}}{A \multimap B \mid \Gamma_0 \vdash D_1 \multimap (\ldots (D_n \multimap E) \ldots)}\multimap\mathsf{R}
\qquad
\dfrac{\overset{h'_1}{- \mid \Delta_1 \vdash D_1} \quad \overset{\dfrac{\overset{h'_n}{- \mid \Delta_n, \Gamma_2 \vdash D_n} \quad \overset{h''}{E \mid \Gamma_2 \vdash C}}{D_n \multimap E \mid \Delta_n, \Gamma_2 \vdash C}\multimap\mathsf{L}}{\vdots}}{\vphantom{x}}
$$

$$
\dfrac{\overset{h'_1}{- \mid \Delta_1 \vdash D_1} \quad D_2 \multimap (\ldots (D_n \multimap E) \ldots) \mid \Delta_2, \ldots, \Delta_n, \Gamma_2 \vdash C}{D_1 \multimap (\ldots (D_n \multimap E) \ldots) \mid \Delta_1, \ldots, \Delta_n, \Gamma_2 \vdash C}\multimap\mathsf{L}
$$

Notice that $\Gamma_1 = \Delta_1, \ldots, \Delta_n$, and the variable condition is easy to check.

For ccut-interpolation, the critical case is $f = \mathsf{pass}\ f'$, with the partition $\langle [\,], (A, \Gamma_1), \Gamma_2 \rangle$ and derivation $f' : A \mid \Gamma_1, \Gamma_2 \vdash C$. In this case, we apply inductive hypothesis of scut-interpolation on f' and obtain derivations $g' : A \mid \Gamma_1 \vdash D$ and $h' : D \mid \Gamma_2 \vdash C$, the desired derivations are $\mathsf{pass}\ g' : - \mid A, \Gamma_1 \vdash D$ and $\mathsf{pass}\ h' : - \mid D, \Gamma_2 \vdash C$, i.e. the partition of A, Γ_1 is itself and the list of formulae is the singleton list $[D]$. The variable condition is automatically satisfied.

Given $f : S \mid \Gamma \vdash C$, if we apply scut-interpolation with the partition $\langle \Gamma_0, \Gamma_1 \rangle$, and then apply the admissible scut rule on the resulting g and h, we obtain an equivalence $f \stackrel{\circ}{=} \mathsf{scut}(g, h)$. A similar result also holds for ccut-interpolation.

Theorem. *For any formulae A and C, if $A \multimap C$ is provable, then there exists a formula B such that both $A \multimap B$ and $B \multimap C$ are provable, and* $\mathsf{var}(B) \subseteq \mathsf{var}(A) \cap \mathsf{var}(C)$.

Proof. $A \multimap C$ being provable means that there is a derivation $f : - \mid \quad \vdash A \multimap C$, then by the invertibility of $\multimap\mathsf{R}$ [14], there exists a derivation $f' : - \mid \quad A \vdash C$. By applying scut-interpolation on f' with the partition $\langle [A], [\,] \rangle$, we get a formula B and two derivations $g' : - \mid A \vdash B$ and $h' : B \mid \quad \vdash C$, where $\mathsf{var}(B) \subseteq \mathsf{var}(s(A), [\,]) = \mathsf{var}(A)$ and $\mathsf{var}(B) \subseteq \mathsf{var}([\,], C) = \mathsf{var}(C)$. The formulae $A \multimap B$ and $B \multimap C$ are proved by the derivations $\multimap\mathsf{R}\ g' : - \mid \quad \vdash A \multimap B$ and $\multimap\mathsf{R}(\mathsf{pass}\ h') : - \mid \quad \vdash B \multimap C$, respectively. $\square$

The arguments above also apply to the sequent calculus for left skew monoidal categories in [17].

The statement of ccut-interpolation does not align with the general structure of interpolation in non-commutative substructural logic, because the general form of the statement is unprovable in `LSkG`. To illustrate this, consider the following statement:

> (ccut′-interpolation) Given $f : S \mid \Gamma \vdash C$ and any partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist a formula D and two derivations $g : - \mid \Gamma_1 \vdash D$ and $h : S \mid \Gamma_0, D, \Gamma_2 \vdash C$, and $\mathsf{var}(D) \subseteq \mathsf{var}(\Gamma_1) \cap \mathsf{var}(s(S), \Gamma_0, \Gamma_2, C)$.

The critical case is $f = \otimes\mathsf{R}(f', f'')$ with the partition $\langle \Gamma_0, (\Gamma'_1, \Gamma''_1), \Gamma_2 \rangle$ and two derivations $f' : S \mid \Gamma_0, \Gamma'_1 \vdash A$ and $f'' : - \mid \Gamma''_1, \Gamma_2 \vdash B$. By induction on f' and the partition $\langle \Gamma_0, \Gamma'_1, [\,] \rangle$, and on f'' and the partition $\langle [\,], \Gamma''_1, \Gamma_2 \rangle$ respectively, we have a formula D, $g' : - \mid \Gamma'_1 \vdash D$, and $h' : S \mid \Gamma_0, D \vdash A$ and a formula E, $g'' : - \mid \Gamma''_1 \vdash E$ and $h'' : - \mid E, \Gamma_2 \vdash B$. We obtain $\otimes\mathsf{R}(g', g'') : - \mid \Gamma'_1, \Gamma''_1 \vdash D \otimes E$ and $\otimes\mathsf{R}(h', h'') : S \mid \Gamma_0, D, E, \Gamma_1 \vdash A \otimes B$, so the last step is to produce $D \otimes E$ in the latter. But we get stuck because $\otimes\mathsf{L}$ cannot be applied on formulae in context.

For example, suppose $f = \otimes\mathsf{R}(f', f'') : X \mid Y, Z \vdash (X \otimes Y) \otimes Z$ where $f' : X \mid Y \vdash X \otimes Y$ and $f'' : - \mid Z \vdash Z$, then given the partition $\langle [\,], (Y, Z), [\,] \rangle$, our goal is to find a formula D and two derivations $g : - \mid Y, Z \vdash D$ and $X \mid D \vdash (X \otimes Y) \otimes Z$. Because Y and Z are atomic, the only possibility is that $D = Y \otimes Z$, however, the sequent $X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z$ does not have a proof in `LSkG`.

$$\dfrac{\dfrac{??}{X \mid Y \otimes Z \vdash X \otimes Y} \quad \dfrac{??}{- \mid \quad \vdash Z}}{X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z}\ \otimes\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{}{X \mid \quad \vdash X}\ \mathsf{ax} \quad \dfrac{??}{- \mid \quad \vdash Y}}{X \mid \quad \vdash X \otimes Y}\ \otimes\mathsf{R} \quad \dfrac{\dfrac{\dfrac{??}{Y \mid Z \vdash Z}}{Y \otimes Z \mid \quad \vdash Z}\ \otimes\mathsf{L}}{- \mid Y \otimes Z \vdash Z}\ \mathsf{pass}}{X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z}\ \otimes\mathsf{R}$$

In general, the rule

$$\dfrac{S \mid \Gamma, A, B, \Delta \vdash C}{S \mid \Gamma, A \otimes B, \Delta \vdash C}\ \otimes\mathsf{C}$$

is not admissible in `LSkG`.

3 Formalization

In this ongoing work, we show that sequent calculi for left skew monoidal (closed) categories enjoy Craig interpolation. The proofs of two statements of generalized interpolation are formalized in the proof assistant Agda. The code is available at

`https://github.com/niccoloveltri/code-skewmonclosed/tree/interpolation`.

For the future, we would like to extend the result to other semi-substructural logics in [18, 20].

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