1 Introduction

The geometry of interaction program was proposed by Girard [1] shortly after the inception of linear logic. In opposition to traditional denotational semantics – e.g. domains –, the GOI program aims at giving an account of the proofs and programs which also interpret their dynamical features, i.e. cut-elimination/execution. In its first approximation, the program provides an interpretation of proofs which has been thoroughly studied and led to numerous applications. A second aspect of the program, much less understood, consists in defining models of linear logic based on a reconstruction of types based on the interaction between proofs (or the generalisation of proofs).

The technique, also used in Ludics [2] and Interaction Graphs models [5, 6, 7, 8], is based on bi-orthogonality techniques, very similar to Hyland and Schalk’s double gluing [3], as well as Krivine’s classical realizability [4]. While several constructions exist, it has been difficult to extract the fundamental properties which are used to construct the model and separate them from non-essential properties related to the framework considered (graphs, operators, flows, etc.).

The present work reports on an axiomatic presentation of linear realizability. In a way, this work is inspired by the PCA-based definition of realizability for intuitionistic logic. We therefore define an equivalent to the notion of PCA from which it is possible to define, by a generic construction, models of multiplicative-additive linear logic (and possibly exponentials).

We here only sketch the approach, which is fully developed in [9].

2 Non-localised realizability

The first definition is very close to that of PCA. Instead of having a set of programs equipped with a notion of application and some axioms stating the existence of specific combinators, we here require the existence of an associative operation together with a compatible notion of measurement.

**Definition 1.** A multiplicative linear realizability situation is a tuple \((P, \text{Ex}, \lbrack \cdot, \cdot \rbrack_m)\) where \(P\) is a set, \(\text{Ex} : P \times P \to P\) is an abstract notion of execution, \(\lbrack \cdot, \cdot \rbrack_m : P \times P \to \Theta\) is a measurement in a commutative group \(\Theta\), such that:
• Associativity of execution:
  \[ \text{Ex}(\text{Ex}(p_1, p_2), p_3) = \text{Ex}(p_1, \text{Ex}(p_2, p_3)), \]

• Trefoil property / 2-cocycle:
  \[ [\text{Ex}(p_1, p_2), p_3]_m + [p_1, p_2]_m = [p_1, \text{Ex}(p_2, p_3)]_m + [p_2, p_3]_m. \]

This axiomatic approach does not allow to reconstruct all connectives, but
is enough to recover two essential operations: the linear implication and the
additive conjunction.

One defines the notion of project which is a pair \((a, p)\) where \(a \in \Theta\) and \(p \in P\), as well as a notion of orthogonality on those\(^1\), based on the measurement \([\cdot, \cdot]_m\). The notion of type is then defined as sets of programs equal to their
bi-orthogonal. Given two types \(A, B\), one can naturally define two constructions:

\[ A \rightarrow B = \{ p \mid \forall a \in A, \text{Ex}(p, a) \in B \}, \]
\[ A :: B = \{ \text{Ex}(a, b) \mid \forall a \in A, \forall b \in B \}^\perp \perp. \]

Those constructions can be shown to be dual, i.e. \(A \rightarrow B = (A \otimes B^\perp)^\perp\).

A more involved argument shows that one can consider the set \(\Theta[P]\) (formal linear combinations of elements of \(P\)) and that this set has a naturally
induced structure of linear realisability situation\(^2\). By considering the same
bi-orthogonality construction on those, one can still define the connectives :: and
\(\rightarrow\), but it is also possible to define a cartesian product \&. Putting all of this
together, we have the following result.

**Theorem 2.** From an multiplicative linear realisability situation, one can define
bi-orthogonality based models of linear implication (hom object) and additive
conjunction (cartesian product).

## 3 Localised realizability

The models defined above do not possess low-level constructions corresponding
to non-reversible connectives, i.e. \(\otimes\) and \(\oplus\). This is corrected by considering
localised linear realizability models. Note that all known linear realizability models
are instances of this more involved definition.

In the following, we write \(A \sqcup B\) the symmetric difference of \(A\) and \(B\).

**Definition 3** (A localised linear realizability situation). A localised (multiplicative)
linear realizability situation is given by a boolean algebra \(B\), and a tuple
\((P, \phi, \text{Ex}, [\cdot, \cdot]_m)\) where \(P\) is a set, \(\phi\) is a map \(P \rightarrow B\), the execution satisfies\(^3\)
\(\text{Ex} : P_A \times P_B \rightarrow P_{A \sqcup B}\) is an abstract notion of execution, and \([\cdot, \cdot]_m : P \times P \rightarrow \Theta\)
is a measurement in a commutative group \(\Theta\), such that:

---

\(^1\)The notion of orthogonality is not unique, but we will not go in further details here.

\(^2\)Here again, we are skipping some details: we get an additive situation, which is essentially
the extension of the above definition with an additional map \(P \rightarrow \Theta\) satisfying some conditions.

\(^3\)We write here \(P_A\) the fiber above \(A\), i.e. the subset \(P_A\) of \(P\) such that \(p \in P_A\) implies
\(\phi(p) \in A\).
• (Associativity of execution) When defined,

$$\text{Ex}(\text{Ex}(p_1, p_2), p_3) = \text{Ex}(p_1, \text{Ex}(p_2, p_3)).$$

• (Trefoil – or 2-cocycle – Property) When defined:

$$\|\text{Ex}(p_1, p_2)\|_m + \|p_1, p_2\|_m = \|p_1, \text{Ex}(p_2, p_3)\|_m + \|p_2, p_3\|_m.$$

In this refined framework, one can consider the set $\Theta[P]$, and define projects and types as explained above. The essential difference with the previous case is that the execution of disjoint programs corresponds to a direct sum, allowing for the interpretation of the tensor product. In a similar way, locativity allows for the interpretation of the additive disjunction $\oplus$. It is then possible to show that those constructions are dual to the $\forall$ (or equivalently $\neg\neg$) and $\&$ connectives (which are defined as in the previous section).

**Theorem 4.** From a localised multiplicative linear realizability situation, one can define bi-orthogonality based models of multiplicative-additive linear logic.

**References**


