An axiomatic presentation of linear realisability

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1 Introduction

The geometry of interaction program was proposed by Girard [1] shortly after the inception of linear logic. In opposition to traditional denotational semantics – e.g. domains –, the GOI program aims at giving an account of the proofs and programs which also interprets their dynamical features, i.e. cut-elimination/execution. In its first approximation, the program provides an interpretation of proofs which has been thoroughly studied and led to numerous applications. A second aspect of the program, much less understood, consists in defining models of linear logic based on a reconstruction of types based on the interaction between proofs (or the generalisation of proofs).

The technique, also used in Ludics [2] and Interaction Graphs models [5, 6, 7, 8], is based on bi-orthogonality techniques, very similar to Hyland and Schalk's *double gluing* [3], as well as Krivine's classical realisability [4]. While several constructions exist, it has been difficult to extract the fundamental properties which are used to construct the model and separate them from non-essential properties related to the framework considered (graphs, operators, flows, etc.).

The present work reports on an axiomatic presentation of linear realisability. In a way, this work is inspired by the PCA-based definition of realisability for intuitionnistic logic. We therefore define an equivalent to the notion of PCA from which it is possible to define, by a generic construction, models of multiplicative-additive linear logic (and possibly exponentials).

We here only sketch the approach, which is fully developed in [9].

2 Non-localised realisability

The first definition is very close to that of PCA. Instead of having a set of programs equipped with a notion of application and some axioms stating the existence of specific combinators, we here require the existence of an associative operation together with a compatible notion of measurement.

Definition 1. A multiplicative linear realisability situation is a tuple $(P, \mathrm{Ex}, [\![\cdot,\cdot]\!]_m)$ where P is a set, $\mathrm{Ex} : P \times P \to P$ is an abstract notion of execution, $[\![\cdot,\cdot]\!]_m :$ $P \times P \to \Theta$ is a measurement in a commutative group Θ, such that:

- Associativity of execution:

$$\mathrm{Ex}(\mathrm{Ex}(p_1, p_2), p_3) = \mathrm{Ex}(p_1, \mathrm{Ex}(p_2, p_3)),$$

- Trefoil property / 2-cocycle:

$$[\![\mathrm{Ex}(p_1, p_2), p_3]\!]_m + [\![p_1, p_2]\!]_m = [\![p_1, \mathrm{Ex}(p_2, p_3)]\!]_m + [\![p_2, p_3]\!]_m.$$

This axiomatic approach does not allow to reconstruct all connectives, but is enough to recover two essential operations: the linear implication and the additive conjonction.

One defines the notion of *project* which is a pair (a, p) where $a \in \Theta$ and $p \in P$, as well as a notion of orthogonality on those[1], based on the measurement $[\![\cdot,\cdot]\!]_m$. The notion of *type* is then defined as sets of programs equal to their bi-orthogonal. Given two types $\mathbf{A}$, $\mathbf{B}$, one can naturally define two constructions:

$$\mathbf{A} \multimap \mathbf{B} = \{\mathfrak{p} \mid \forall \mathfrak{a} \in \mathbf{A}, \mathrm{Ex}(\mathfrak{p}, \mathfrak{a}) \in \mathbf{B}\},$$
$$\mathbf{A} :: \mathbf{B} = \{\mathrm{Ex}(\mathfrak{a}, \mathfrak{b}) \mid \forall \mathfrak{a} \in \mathbf{A}, \forall \mathfrak{b} \in \mathbf{B}\}^{\perp\perp}.$$

Those constructions can be shown to be dual, i.e. $\mathbf{A} \multimap \mathbf{B} = (\mathbf{A} \otimes \mathbf{B}^{\perp})^{\perp}$.

A more involved argument shows that one can consider the set $\Theta[P]$ (formal linear combinations of elements of P) and that this set has a naturally induced structure of linear realisability situation[2]. By considering the same bi-orthogonality construction on those, one can still define the connectives :: and $\multimap$, but it is also possible to define a cartesian product &. Putting all of this together, we have the following result.

Theorem 2. *From an multiplicative linear realisability situation, one can define bi-orthogonality based models of linear implication (hom object) and additive conjunction (cartesian product).*

3 Localised realisability

The models defined above do not possess low-level constructions corresponding to non-reversible connectives, i.e. $\otimes$ and $\oplus$. This is corrected by considering *localised linear realisability models.* Note that all known linear realisability models are instances of this more involved definition.

In the following, we write $A \veebar B$ the *symmetric difference* of A and B.

Definition 3 (A localised linear realisability situation)**.** A localised (multiplicative) linear realisability situation is given by a boolean algebra $\mathcal{B}$, and a tuple $(P, \phi, \mathrm{Ex}, [\![\cdot,\cdot]\!]_m)$ where P is a set, ϕ is a map $P \to \mathcal{B}$, the execution satisfies[3] $\mathrm{Ex} : P_A \times P_B \to P_{A \veebar B}$ is an abstract notion of execution, and $[\![\cdot,\cdot]\!]_m : P \times P \to \Theta$ is a measurement in a commutative group Θ, such that:

[1] The notion of orthogonality is not unique, but we will not go in further details here.

[2] Here again, we are skipping some details: we get an *additive* situation, which is essentially the extension of the above definition with an additional map $P \to \Theta$ satisfying some conditions.

[3] We write here P_A the fiber above A, i.e. the subset P_A of P such that $p \in P_A$ implies $\phi(p) \in A$.

- (Associativity of execution) When defined,

$$\mathrm{Ex}(\mathrm{Ex}(p_1, p_2), p_3) = \mathrm{Ex}(p_1, \mathrm{Ex}(p_2, p_3)).$$

- (Trefoil – or 2-cocycle – Property) When defined:

$$[\![\mathrm{Ex}(p_1, p_2), p_3]\!]_m + [\![p_1, p_2]\!]_m = [\![p_1, \mathrm{Ex}(p_2, p_3)]\!]_m + [\![p_2, p_3]\!]_m.$$

In this refined framework, one can consider the set $\Theta[P]$, and define projects and types as explained above. The essential difference with the previous case is that the execution of disjoint programs corresponds to a direct sum, allowing for the interpretation of the tensor product. In a similar way, locativity allows for the interpretation of the additive disjunction $\oplus$. It is then possible to show that those constructions are dual to the $⅋$ (or equivalently $\multimap$) and & connectives (which are defined as in the previous section).

Theorem 4. *From a localised multiplicative linear realisability situation, one can define bi-orthogonality based models of multiplicative-additive linear logic.*

References

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