A supply of functorial models of linear logic

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This is work in progress and may be missing essential references; I want to present it at TLLA to find out what I might be missing.

Consider the relational model Rel of linear logic. Formulas are interpreted as sets, logical connectives are interpreted using categorical structure inherited from the category Set of sets and the exponential modality ! is interpreted by the finite multiset comonad (equivalently, the free commutative monoid comonad). Linear logic proofs relating formulas are interpreted as relations between sets interpreting those formulas.

The present work begins from the observation that there are two types of coefficients implicitly involved in this model.

First, we may view a relation $A \to B$ as a matrix indexed by $A \times B$ with entries in the two-element lattice $\{\top, \bot\}$. Generalizing to a suitable semiring produces *weighted relational models*, [5]. Often this semiring is required to be *complete* or *continuous*. We can also 'categorify', replacing the partially ordered set with a category; a common choice is the category of sets. Up to adjusting some details of the presentation, the latter results in a model closely related to Girard's *normal functors* [3].[1]

Second, the natural numbers $\mathbb{N}$ appear as coefficients in the finite multisets of the exponential modality. More precisely, the set of finite multisets over a set A can be identified with the free $\mathbb{N}$-module with generators in A. Here too we can perform a substitution of semirings: using the two-element semiring instead yields the finite powerset comonad and a 'qualitative' model of linear logic, for instance. More interestingly, $\mathbb{N}$ categorifies to Bij, the groupoid of finite sets and bijections, with the free Bij module over a category A being the free symmetric monoidal category on A, which yields Joyal's *analytic functors* model [4].

[1] The starting point for this work was a close analysis of Girard's work on Normal Functors with Thomas Seiller and William Troiani.

Recently, a bicategorical standard for models of differential linear logic has been established by Fiore, Gambino and Hyland [2], with the analytic functors model as a motivating example. While it has been established that a pseudodistributive law between (pseudo)monads can produce a model of linear logic, there are still a limited number of examples around.

In this talk, we will establish the existence of distributivity laws between two types of pseudomonad induced by *rig categories*, a common generalization of the semirings and categories appearing in the above. In so doing, we unify a broad class of categorical models of linear logic under a common umbrella which may be exploited in the future to compare and transfer results between these models.

Symmetric monoidal categories

Definition 1. Let $\mathbb{C}$ be a locally small category. We define the **free symmetric monoidal category on** $\mathbb{C}$, denoted $\mathcal{M}(\mathbb{C})$ (with monoidal operation $\oplus$ and unit 0), as follows:

- The objects are formal sums $\bigoplus_{i=0}^{n-1} c_i$ of objects of $\mathbb{C}$. Note that $n = 0$ is allowed, and yields the unit object 0.
- Morphisms of $\mathcal{M}(\mathbb{C})$ of type $\bigoplus_{i=0}^{n-1} c_i \to \bigoplus_{i'=0}^{n'-1} c'_{i'}$ exist only when $n = n'$ and consist of a pair $(\sigma, \vec{f})$ where $\sigma : [n] \to [n]$ is a permutation of the n-element set $[n] = \{0, \ldots, n-1\}$ and $\vec{f} = \{f_i : c_i \to c'_{\sigma(i)}\}_{i=1}^{n}$ is a set of morphisms of $\mathbb{C}$.
- The monoidal sum is defined by 'concatenation' as one would expect.

Free monoidal categories form the foundation of our framework. Observe that there is a full and faithful functor $\mathbb{C} \to \mathcal{M}(\mathbb{C})$ sending c to the singleton sum at that object.[2]

We can define a 'tensor product' of monoidal categories enabling us to build new monoidal categories.

Definition 2. Let $\mathcal{E}, \mathcal{E}'$ be symmetric monoidal categories. We define the **tensor product of $\mathcal{E}$ and $\mathcal{E}'$** (over $\mathrm{FinSet}_{\mathrm{bij}}$), denoted $\mathcal{E} \boxtimes \mathcal{E}'$, to be the symmetric monoidal category whose objects are formal monoidal sums of

[2] Beware that for us, 'monoidal functors' will refer to *strong* monoidal functors unless otherwise stated, in contrast with weaker *lax* or *oplax* monoidal functors.

objects of the form $X \boxtimes X'$ (with $X \in \mathcal{E}$ and $X' \in \mathcal{E}'$), with $0 \boxtimes 0'$ as the unit. We add isomorphisms,

$$X \boxtimes (X'_0 \oplus X'_1) \cong (X \boxtimes X'_0) \oplus (X \boxtimes X'_1) \tag{1}$$

$$(X_0 \oplus X_1) \boxtimes X' \cong (X_0 \boxtimes X') \oplus (X_1 \boxtimes X') \tag{2}$$

$$0 \boxtimes X' \cong 0 \boxtimes 0' \cong X \boxtimes 0', \tag{3}$$

each of which is required to be natural and compatible with symmetries in each argument. Morphisms are composites of these with permutations of summands and morphisms of the form $\bigoplus_{i=1}^{n} x_i \boxtimes x'_i$.

Rig categories

The free symmetric monoidal category category $\mathcal{M}(1)$ on the trivial one-object category 1 can be identified with the groupoid Bij of finite sets and bijections. Just as the free commutative monoid on one generator $\mathbb{N}$ is a (commutative) *rig*, Bij carries the structure of a *rig category*.

Definition 3. A **rig category** (also known as a *bimonoidal category*) is a pseudomonoid in the monoidal 2-category $(\mathrm{MON}, \boxtimes, 1)$. More concretely, a rig category is a symmetric monoidal category $(\mathcal{R}, \oplus, 0)$ carrying a monoidal functor $\otimes : \mathcal{R} \boxtimes \mathcal{R} \to \mathcal{R}$ called the **monoidal product**, symmetry isomorphisms[3] and a **unit object** 1 for the monoidal product. Being monoidal means that $\otimes$ distributes over $\oplus$ on both sides and 0 is an absorbing element for $\otimes$, in the sense that $0 \otimes B \cong 0 \cong B \otimes 0$.

A **rig functor** is a functor $F : \mathcal{R} \to \mathcal{R}'$ between rig categories equipped with natural isomorphisms $F(X \oplus X') \cong F(X) \oplus F(X')$, $F(X \otimes X') \cong F(X) \otimes F(X')$, $F(0) \cong 0$ and $F(1) \cong 1$ commuting with the structural isomorphisms as one would expect.

Remark 4. The coherence results for the structural isomorphisms of rig categories were established by Laplaza [6]. Loregian and Trimble focus on the special case where $\oplus$ is a coproduct in [7] and supplement this structure with a 'derivation'; we may explore the need for this extra structure later on. Baez, Moeller and Trimble study the further specialised case of rig categories enriched in vector spaces in [1] in the context of representation theory. This

[3]Beware that symmetry isomorphisms are not a general requirement; rather, we are restricting attention to **symmetric** rig categories.

illustrates the potential wealth of examples which the present work might give access to if successful.

Objects of rig categories will serve as coefficients in our constructions, with the categorical structure reflecting relationships between these. It should not be surprising, given the analogy with commutative ri(n)gs, that we can define modules of rig categories.

Definition 5. Let $\mathcal{R}$ be a rig category. A (left) **$\mathcal{R}$-module** consists of a monoidal category $\mathcal{E}$ equipped with a monoidal functor $\bullet : \mathcal{R} \boxtimes \mathcal{E} \to \mathcal{E}$ called an **$\mathcal{R}$-action** (the tensor product in the domain comes from the additive structure on $\mathcal{R}$), as well as **structural isomorphisms** $S \bullet (R \bullet X) \cong (S \otimes R) \bullet X$ and $1 \bullet X \cong X$ natural in objects $R, S \in \mathcal{R}$ and $X \in \mathcal{E}$.

An **$\mathcal{R}$-linear functor** $(\mathcal{E}, \bullet) \to (\mathcal{E}', \bullet')$ is a monoidal functor $\mathcal{E} \to \mathcal{E}'$ respecting the $\mathcal{R}$-action up to isomorphisms required to be coherent with the structural isomorphisms on the domain and codomain.

For $\mathcal{R}$ a (locally small) rig category and $\mathbb{C}$ a small category, we can now define

$$\mathcal{M}_{\mathcal{R}}(\mathbb{C}) \coloneqq \mathcal{R} \boxtimes \mathcal{M}(\mathbb{C}), \tag{4}$$

the *free $\mathcal{R}$-module over* $\mathbb{C}$. As the name would suggest, this has the structure of an $\mathcal{R}$-module given by $R \bullet (S \boxtimes C) \coloneqq (R \otimes S) \boxtimes C$.

Another way to construct an $\mathcal{R}$-module is with functors with codomain $\mathcal{R}$. We switch notation here, since the rig category involved will serve as a supply of coefficients in the other sense mentioned in the introduction.

Lemma 6. *Let $\mathbb{C}$ be a small category and $\mathcal{S}$ any rig category. Then $[\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$ is an $\mathcal{S}$-module via the 'point-wise action':*

$$(S \bullet F)(C) \coloneqq S \otimes (F(C)).$$

As stated, the construction of Lemma 6 ignores any $\mathcal{R}$-linear structure on $\mathbb{C}$, and for an arbitrary $\mathcal{S}$ there need not be a functor $\mathbb{C} \to [\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$ to act as the unit of a prospective monad structure for the functor $\mathbb{C} \mapsto [\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$. To complete the picture, we therefore constrain both the second rig category $\mathcal{S}$ and the class of categories $\mathbb{C}$ which we consider, as follows.

Lemma 7. *Let $\mathcal{S}$ be a rig category and $\mathbb{C}$ a small category enriched over (the multiplicative structure of) $\mathcal{S}$. Then there is a Yoneda embedding $\mathbb{C} \to [\mathbb{C}^{\mathrm{op}}, \mathcal{S}]$, and such functors assemble to make $[(-)^{\mathrm{op}}, \mathcal{S}]$ a relative pseudomonad on the class of small $\mathcal{S}$-categories.*

Note that a category cannot be enriched in Bij in a non-trivial way, so this is a constraint on $\mathcal{S}$ as well as $\mathbb{C}$. For simplicity, we treat only subcategories of Set here, since going further would require discussion of enriched monoidal categories. Recall that for a small monoidal category $\mathbb{C}$, Day convolution provides a monoidal structure on $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$ making the Yoneda embedding a monoidal functor.

Proposition 8. *Let $\mathbb{C}$ be an $\mathcal{R}$-module. Then $[\mathbb{C}^{\mathrm{op}}, \mathrm{Set}]$ admits an $\mathcal{R}$-module structure making the Yoneda embedding an $\mathcal{R}$-module homomorphism.*

This construction yields the distributive law witnessing that $\mathcal{M}_{\mathcal{R}}(-)$ defines a comonad on the Kleisli category of $[(-)^{\mathrm{op}}, \mathrm{Set}]$, so we end up with a model of linear logic. It remains to reconstruct differentiation!

If there is time, we will discuss conditions on $\mathcal{S}$ (and/or constraints on $\mathbb{C}$) which enable these constructions to be reproduced for more general rig categories of weights. If successful, this work will show how varying the rig categories involved yields a systematic understanding of how 'decategorifying' affects the content of models.

References

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