Laplace Transformation and Symmetry in Differential Linear Logic

Marie Kerjean
CNRS, Université Sorbonne Paris Nord, France
kerjean@lipn.fr

Jean-Simon Pacaud Lemay
Macquarie University, Australia
js.lemay@mq.edu.au

Abstract

This talk is an extension of our accepted paper at FSCD2024. Compared to the FSCD2024 talk, in the TLLA talk we will focus on semantics, examples and future work. We will also take a deeper look into the relation between our FSCD2024 paper about Laplace transformations and our LICS2023 paper about co-promotion (where early work of the latter was presented at TLLA2022).

Differential Linear Logic (DiLL) [6], introduced by Ehrhard and Regnier [7], introduces the concept of differentiation in Linear Logic (LL), by symmetrizing three out of the four rules for the aptly called exponential connective !. Recall that in LL, the four exponential structural rules which dictate the use of $!A$ are: the weakening rule w, the contraction rule c, the dereliction rule d, and the promotion rule P[1].

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \mathsf{P}$$

As is well known, the promotion rule can be equivalently replaced by two rules: the functorial promotion rule $!_{\mathsf{f}}$ and the digging rule p.

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\ !_{\mathsf{f}} \qquad \frac{\Gamma, !!A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \mathsf{p}$$

DiLL then adds the co-structural rules which are the co-weakening rule $\overline{\mathsf{w}}$, the co-contraction rule $\overline{\mathsf{c}}$, and the co-dereliction rule $\overline{\mathsf{d}}$.

$$\frac{\vdash}{\vdash !A}\ \overline{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \overline{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \overline{\mathsf{d}}$$

The co-dereliction rule $\overline{\mathsf{d}}$ expresses differentiation, while the co-contraction rule $\overline{\mathsf{c}}$ and the co-weakening rule $\overline{\mathsf{w}}$ are necessary for cut-elimination purposes. This beautifully results in a symmetry between the structural rules and co-structural rules, that has however never been properly explained. In this talk, we explain this symmetry using the *Laplace transform* [10]. Applied to the digging, the Laplace transformation results in the co-digging rule [9]:

$$\frac{\Gamma, !!A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ \overline{\mathsf{p}}$$

this is equal to p

Differentiation on proofs

The key concept in LL is that a proof of a sequent $A \vdash B$ will be a *linear* proof, making use of A exactly once and not allowing contraction nor weakening on A. This is opposed to a proof of $!A \vdash B$, which can make a non-linear usage of A by using contraction or weakening. The basic rule of LL is that you can forget about linearity. Hence, the dereliction rule d transforms a linear proof into a non-linear proof, which intuitively is done so by just forgetting about the linearity property. DiLL takes the reverse path by introducing a co-dereliction rule $\overline{\mathsf{d}}$, which, after a cut, allows the transformation of a non-linear proof $!A \vdash B$ into a linear proof $A \vdash B$. From a semantical point of view, linearizing a non-linear function (which interprets a proof) is done so via differentiation. This analogy is made precise by introducing new

[1] These rules are presented with bilateral sequent for simplicity, but they could also be made monolateral by using the exponential disjunction ?, which is the dual of !.

cut-elimination rules between $\bar{\mathsf{d}}$ and structural rules. The cut-elimination between d and $\bar{\mathsf{d}}$ results in a cut between their premises, and this represents the fact that differentiating at 0 a linear function returns the same linear function. The cut-elimination between promotion p and $\bar{\mathsf{d}}$ is more intricate and uses $\bar{\mathsf{c}}$ and $\bar{\mathsf{w}}$: it represents the chain rule, which is the formula expressing how to differentiate a composition of functions.

Rules of DiLL can also be understood through the notions of functions and *distributions*. Naively, distributions are linear scalar maps which are computed on smooth functions. Let us for now suggestively denote $\mathcal{C}^\infty(A,B) := \mathcal{L}(!A,B)$ the set of smooth maps from A to B, and $A \multimap B := \mathcal{L}(A,B)$ the set of linear maps from A to B. Now, in Classical DiLL, elements of $!A$ can be interpreted as distributions, so we may suggestively write $!A \subseteq \mathcal{C}^\infty(A,I) \multimap I$. In most models, I is often interpreted as the field of real or complex numbers. Now, for each element x of A, the dereliction rule gives us the Dirac distribution at x, which is the distribution $\delta_x \in\, !A$ which evaluates a smooth function at x, so $\delta_x(f) = f(x)$. For finite-dimensional vector spaces, or in the model of convenient vector spaces [1], it is sufficient to define what a non-linear map does on Dirac distributions. So, on Dirac distributions, the structural maps, which correspond to the structural rules of LL and the co-structural rules of DiLL, are given as follows:

$$\begin{gathered}\mathsf{p}_A(\delta_x) = \delta_{\delta_x} \qquad \mathsf{d}_A(\delta_x) = x \qquad \mathsf{c}_A(\delta_x) = \delta_x \otimes \delta_x \qquad \mathsf{w}_A(\delta_x) = 1 \\ \bar{\mathsf{d}}_A(x) = \mathsf{D}_0(\_)(x) \qquad \bar{\mathsf{c}}_A(\delta_x \otimes \delta_y) = \delta_{x+y} \qquad \bar{\mathsf{w}}_A(1) = \delta_0\end{gathered} \tag{1}$$

where for the co-dereliction $\bar{\mathsf{d}}$, the D is the differential operator, that is, for a smooth function f, $\mathsf{D}_x(f)(y)$ is the derivative of f at point x along the vector y. We highlight that on the whole space $!A$, the co-contraction $\bar{\mathsf{c}} : !A \otimes !A \to !A$ is interpreted as the *convolution of distributions*:

$$\bar{\mathsf{c}}_A(\phi \otimes \psi) = \phi * \psi := f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y)).$$

Moreover, the structural rules of LL can also be naturally expressed on functions. Indeed, in Classical DiLL, we have an involutive duality $*$ where A^* is the linear dual of A, that is, $A^* = A \multimap I$. Using the linear dual, one also introduces the connector $?A = (!A^*)^*$, which is interpreted as a space of smooth functions, $?A \subseteq \mathcal{C}^\infty(A^*, I)$. We also get the multiplicative disjunction $A \parr B = (A^* \otimes B^*)^*$, which we may think of as a completed tensor product. Then the contraction $\mathsf{c}^?_A : ?A \parr ?A \to ?A$ is interpreted by the pointwise multiplication of scalar functions, the weakening $\mathsf{w}^?_A : \mathbb{K} \to ?A$ maps scalars r to constant functions $cst_r : x \mapsto r$, and the dereliction $\mathsf{d}^?_A : A \to ?A$ maps elements of A to their evalutation at a point x:

$$\mathsf{c}^?_A(f \otimes g) = f \cdot g \qquad \mathsf{w}^?_A(r) = \mathrm{cst}_r \qquad \mathsf{d}^?_A(x) = (\ell \in A^* \mapsto \ell(x)) \tag{2}$$

A higher-order Laplace transform

The categorification of functional analysis and differential geometry entertains close links with the semantics of the sequent calculus for LL and DiLL. Differential categories were introduced by Blute, Cocket, and Seely [2], and originated from the semantics of DiLL. Since their introduction, differential categories now have a rich mathematical literature and have been quite successful in categorifying various important concepts from differential calculus and differential geometry, as well as various other aspects of differentiation throughout mathematics and computer science. We follow this line of research. Following the categorification of the exponential functions in a differential category by the second named author in [12], and the completion of DiLL by the addition of a co-digging rule by the authors in [9], here we give a categorical interpretation of the Laplace transform and study its properties. We explain why it is the reason behind the symmetry in DiLL rules, which we exploit categorically.

The Laplace transform is a central component of calculus and engineering, as it changes differential equations into polynomial equations. As such, the Laplace transform is a very useful tool for solving differential equations. In its first-order version, the Laplace transform takes a function $f : \mathbb{R} \to \mathbb{R}$ to a function $\mathscr{L}(f) : \mathbb{C} \to \mathbb{R}$, defined as:

$$\mathscr{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt$$

However, this first-order version does not necessarily fit well with the categorical semantics of DiLL. Instead of using integration to make functions act on functions, one can use distributions by following the general idea of interpreting distributions as generalized functions. Consider a distribution ϕ with compact support, that is, $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})'$ is a linear form on the space of smooth functions (where $F' := \mathcal{L}(F,\mathbb{R})$ is the space of linear scalar functions on a vector space F). Then we may write:

$$\mathscr{L}(\phi)(s) = \phi(t \mapsto e^{-st})$$

So for a higher-order distribution $\phi \in \mathcal{C}^\infty(E,\mathbb{R})'$, where E stands for a possibly infinite-dimensional vector space, we get:

$$\mathscr{L}\colon \begin{cases} \mathcal{C}^\infty(E,\mathbb{R})' \longrightarrow \mathcal{C}^\infty(E',\mathbb{R}) \\ \qquad\qquad \phi \longmapsto \left(x^* \mapsto \left(\phi\left(t \mapsto e^{x^*(t)}\right)\right)\right) \end{cases} \tag{3}$$

Following the intuitions developed above, this gives us a new understanding of the Laplace transformation in terms of connectives of LL, resulting in a natural transformation of type $\mathscr{L}_A : {!A} \to {?A}$. This idea was only recently noticed in the context of DiLL, thanks to the higher-order presentation of the Laplace transform in a specific polarized model of DiLL discussed in [9, Prop V.8].

Since we have a categorical understanding of higher-order distribution theory, we, therefore, have all the ingredients in hand to axiomatize $\mathscr{L}$ categorically in a suitable differential category. For now, we simply state that we categorify the Laplace transform as a natural transformation of type $\mathscr{L} : {!A} \to {?A}$, which we call a **Laplace transformation**. Semantically, the axioms say that $\mathscr{L}$ transforms the interpretation of co-structural rules into the interpretation of structural rules:

$$\mathscr{L} : {!A} \mapsto {?A}; \overline{\mathsf{w}} \mapsto \mathsf{w}; \overline{\mathsf{c}} \mapsto \mathsf{c}; \overline{\mathsf{d}} \mapsto \mathsf{d}; \overline{\mathsf{p}} \mapsto \mathsf{p}.$$

These are all analogues of very well-known facts in calculus. For example, the Laplace transform converts convolution into multiplication, which is recaptured by the fact that our Laplace transformation $\mathcal{L}$ turns $\overline{\mathsf{c}}$ into $\mathsf{c}^?$.

All these equations are intrinsically linked with the basic properties of the exponential function e^x. Indeed, generalizations of the exponential function in a differential category were defined by the second named author in [12], and are axiomatized by analogues of three fundamental properties of the exponential function: that $e^{x+y} = e^x e^y$ and $e^0 = 1$, and also that e^x is its own derivative. We will explain how the notion of a Laplace transformation is fundamentally linked to that of a generalized exponential function on the monoidal unit I.

Moreover, since we are in the monoidal closed, we may uncurry the Laplace transformation to get an extranatural transformation $\partial_A : {!A^*} \otimes {!A} \to I$, which we call a **Laplace evaluator**, or take the dual to get a natural transformation $\ell_A : {!A^*} \to ({!A})^*$, which we call a **Laplace distributor**. In a closed setting with a co-digging, this is equivalent to the introduction of a dinatural transformation that we call the **relative Laplace evaluator** $\ell_{A,B} : !\,(A \multimap {!B}) \otimes {!A} \to {!B}$.

Many well-known and important examples of differential categories have a Laplace transformation including: (weighted) relations, finiteness spaces, Köthe spaces, and convenient vector spaces.

Future work

A natural path to consider is generalizing this story from isomix star-autonomous categories to *linearly distributive categories* [3]. Indeed, the diagrams the Laplace transformation $\mathcal{L} : {!A} \to {?A}$ satisfies can easily be written down in a linearly distributive category with the proper notion of exponentials [4, 5]. So one could study Laplace transformations in a linearly distributive setting. However, the linearly distributive generalization of differential categories has not yet been properly defined or studied. So hopefully the new notion of Laplace transformation will motivate the development of such a theory.

Work is also needed on concrete models of Laplace transforms. The original intuition for the categorification of the Laplace transform came from higher-order work in functional analysis [8, 9], in which two kinds of functions with different exponential growth model the two types of exponential connectives, applying to formulas with different polarities [11]. The Laplace transformation then changes distributions on one type of function into distributions on the other type of function. Understanding the categorical interplay between the Laplace transformation and polarity might lead to a better axiomatization of star-autonomous linear differential categories.

References

[1] R. Blute, T. Ehrhard, and C. Tasson. A convenient differential category. *Cahiers de topologie et géométrie différentielle catégoriques*, 2012.

[2] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Differential categories. *Math. Struct. Comput. Sci.*, 2006.

[3] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114(2), 1997.

[4] J. R. B. Cockett and R.A.G. Seely. Linearly distributive functors. *Journal of Pure and Applied Algebra*, 143(1-3), 1999.

[5] J.R.B. Cockett and P. V. Srinivasan. Exponential modalities and complementarity (extended abstract). In Kohei Kishida, editor, Proceedings of the Fourth International Conference on *Applied Category Theory,* Cambridge, United Kingdom, 12-16th July 2021, volume 372 of *Electronic Proceedings in Theoretical Computer Science*. Open Publishing Association, 2022.

[6] T. Ehrhard. An introduction to differential linear logic: proof-nets, models and antiderivatives. *Mathematical Structures in Computer Science*, 2017.

[7] T. Ehrhard and L. Regnier. Differential interaction nets. *Theoretical Computer Science*, 364(2), 2006.

[8] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui. Un théorème de dualité entre espaces de fonctions holomorphes à croissance exponentielle. *Journal of Functional Analysis*, 171(1), 2000.

[9] M. Kerjean and J.-S. P. Lemay. Taylor Expansion as a Monad in Models of DiLL. In *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, 2023.

[10] M. Kerjean and J.-S. P. Lemay. Laplace distributors and laplace transformations for differential categories. 2024.

[11] O. Laurent. *Etude de la polarisation en logique.* Thèse de Doctorat, Université Aix-Marseille II, March 2002.

[12] J.-S. P. Lemay. Exponential functions in cartesian differential categories. *Applied Categorical Structures*, 29, 2021.