Interpolation in Extensions of Linear Logic

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Interpolation is one of the most fundamental and thoroughly studied concepts in logic. It was originally investigated for classical logic by W. Craig in [4], and since then has found numerous links to other topics in logic as well as influential applications (such as to verification [11]). There are many variants of interpolation that have been formulated for particular, specialized purposes but, for the purposes of this work, two types of interpolation are especially noteworthy. The variant of interpolation first isolated by Craig may be stated, for a propositional logical system $\mathcal{L}$, as follows: We say that $\mathcal{L}$ has the *Craig interpolation property* if for any formulas φ, ψ,

> if $\varphi \to \psi$ is provable in L, then there exists a formula α whose variables are among those contained in both φ and ψ such that each of $\varphi \to \alpha$ and $\alpha \to \psi$ are provable in L.

The Craig interpolation property has been studied for a wide range of different logical systems, and has proven to be especially amenable to analysis via proof-theoretic methods. On the other hand, the demand that a given logical system has the Craig interpolation property is rather strenuous. For example, among substructural logics satisfying the exchange rule (of which linear logic is especially prominent example) it is known (see [9]) that the Craig interpolation property is strictly stronger than the following *deductive interpolation property*, where here $\vdash_{\mathsf{L}}$ denotes the consequence relation associated to the propositional logic L:

> If $\varphi \vdash_{\mathsf{L}} \psi$, then there exists a formula α whose variables are among those contained in both φ and ψ such that each of $\varphi \vdash_{\mathsf{L}} \alpha$ and $\alpha \vdash_{\mathsf{L}} \psi$.

This study focuses on interpolation in axiomatic extensions of classical linear logic LL, multiplicative-additive linear logic MALL, and several related systems. Both LL and MALL can be shown to have the Craig interpolation property using analytic proof systems (see, e.g. [13]), and consequently these systems also have the deductive interpolation property. However, interpolation in extensions of LL and MALL have received relatively little attention. The main result of this work illustrates that the deductive interpolation property, at least in some course sense, very common among such extensions.

Theorem 1. *Each of $\vdash_{\mathsf{LL}}$ and $\vdash_{\mathsf{MALL}}$ has continuum-many axiomatic extensions with the deductive interpolation property.*

The previous result stands in contrast to well known results in adjacent logical contexts, where, for example, Maksimova has shown [10] that, of the continuum-many consistent superintuitionistic logics, only 7 have the deductive interpolation property (equivalent in this context to the Craig interpolation property). While the extensions with the deductive interpolation property that we identify in Theorem 1 do not have the Craig interpolation property, the identified extensions of LL have a weak form of Craig interpolation that makes reference to the exponential $!$. This may be stated as follows.

Theorem 2. *Suppose that* L *is an axiomatic extension of* LL. *Then* L *has the deductive interpolation property if and only if* L *has the following guarded form of the Craig interpolation property:*

> *If* $!\varphi \to !\psi$ *is provable in* L, *then there exists a formula* α *whose variables are among those contained in both* φ *and* ψ *such that each of* $!\varphi \to !\alpha$ *and* $!\alpha \to !\psi$ *are provable in* L.

The results announced in Theorems 1 and 2 can be easily adapted to related contexts. For example, we show also that the full Lambek calculus with exchange (see, e.g., [8]) has continuum-many axiomatic extensions with the deductive interpolation property.

The methodology we use to prove the previously announced theorems is inherently algebraic: It relies on the well-known connection between the deductive interpolation property for a logic and the amalgamation property for its associated class of algebraic models (see, e.g., [5]). This work can thus also be understood as an illustration of (or tutorial on) algebraic methods in linear logic, which has not historically been studied from an algebraic point of view. Initial steps toward an algebraic treatment of linear logic were first taken by Avron in [2], where LL was first presented in terms of consequence relations. Later, in unpublished work that became folklore among algebraic logicians, Aglianò showed in [1] that the equivalent algebraic semantics of LL in the sense of [3]. The equivalent algebraic semantics for classical linear logic LL is given by the variety (AKA equational class) of *girales*, i.e., algebraic structures of the form $\langle A, \wedge, \vee, \cdot, 0, 1, !\rangle$, where $\langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ is a commutative residuated lattice with $(x \to 0) \to 0 = x$ (see [8]) and $!\colon A \to A$ is a unary operation, reminiscent of an S4-like modal box operator, satisfying the identities:

1. $!(x \wedge y) = !x \cdot !y$.
2. $!!x = !x \leq x$.
3. $!1 = 1$.

Here the connective ? may be understood as derivative of ! in the sense that $?x = \neg!\neg x$.

In order to find continuum-many axiomatic extensions of LL with the deductive interpolation property, we find continuum-many varieties of girales with the amalgamation property (see [6] for relevant definitions). These varieties are constructed by first considering suitably chosen quasivarieties of abelian groups, each with the amalgamation property. The abelian groups contained in these quasivarieties are then transformed into girales using an algebraic construction (introduced in this work) that preserves the amalgamation property, and the latter are used as generating algebras for the varieties we are interested in. The examples we construct are sufficiently transparent to lift the amalgamation property from the generating algebras using existing tools (see, e.g., [6, 12]), but also sufficiently flexible that they may also be used for algebraic models of MALL as well as the full Lambek calculus with exchange and several other related logics. Thus, along the way to Theorem 1, we also obtain the following result.

Theorem 3. *There are continuum-many varieties of girales with the amalgamation property.*

Further details on this work may be found in our preprint [7].

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