Abstract

This work presents the correspondence between an ultrafilter of the ground model which naturally arises from realizability algebras and the ultrafilter which generates the realizability model.

1 Context

Griffin [4] enables to extend Curry-Howard correspondence to classical logic. Indeed, in [4] it has been shown that Pierce’s Law \( \neg A \rightarrow A \rightarrow A \) is an admissible type for \( \text{cc} \). This has been a major achievement in the constructivization of mathematics, since proof-programs correspondence was hitherto limited to intuitionistic logic. About this latter, in 1945 Kleene [5] exposed a method, namely realizability, which employed recursive functions as witnesses for the satisfiability of a formula for a fixed language, which eventually was a generalisation of intuitionistic semantics. Combining Griffin’s result with Kleene’s tradition, Krivine [6] has developed realizability for classical set theory, introducing a structure called \textit{realizability algebra}.

Definition 1. A realizability algebra is a tuple \( A = (\Lambda_c, \Pi, \succ, \bot) \), where:

- \( \Lambda_c \) is the set of terms generated by
  \[ t := x | \text{cc} | \lambda x.t | (t)t \]  
  with \( k \pi \) \textit{continuation} of \( \pi \);

- \( \Pi \) is the set of \textit{stacks} generated by
  \[ \pi := \pi_0 | t \pi \]  
  for a fixed set \( \Pi_0 \) of \textit{stack constants};

- \( \succ \) is a reduction relation \( (t, \pi) \succ (u, \rho) \) between \textit{processes} \( (t, \pi), (u, \rho) \), for \( t, u \in \Lambda_c, \pi, \rho \in \Pi \). For sake of readability (and history), we note \( t \pi \) for a process \( (t, \pi) \). \( \succ \) is defined as the transitive closure of following rules:
  \[ \lambda:\beta t \ast u \pi \succ t[u/x] \pi, \]  
  (grab)
  \[ (t) u \ast \pi \succ t \ast u \pi, \]  
  (push)
  \[ \text{cc} \ast t \pi \succ t \ast k \pi \pi, \]  
  (save)
  \[ k \pi \ast t \rho \succ t \pi; \]  
  (restore)

- \( \bot \), called \textit{pole}, is a fixed subset of processes closed for anti-reduction, i.e., for \( t, u \in \Lambda_c, \pi, \rho \in \Pi \),
  \[ t \pi \succ u \rho, u \rho \in \bot \implies t \pi \in \bot. \]

\( \bot \) induces a notion of orthogonality between sets of \( \Lambda_c \)-terms and sets of stacks. Then, for any set of stacks \( X \subseteq \Pi \) we can define
  \[ X \bot := \{ t \in Q | \forall \pi \in X(t \pi \in \bot) \}. \]
In order to have a consistent set of realized formulæ, fix a set $Q$ of $\Lambda$, such that $\mathbb{C} \subset Q$ and $\forall \mathbb{C} \in \Pi(t \prec \pi \not\prec \perp)$ $Q$ is the set of quasi-formulæ. For any set $X \subseteq \Pi$, $X$ is realized $X^{\perp} \cap Q \not= \emptyset$. For any formulæ $F$ of a language $\mathcal{L}$, a set $\mathcal{F}$ is associated to it, and $F$ is said to be realized if $(Q \cap \mathcal{F}^{\perp}) \not= \emptyset$, where $\mathcal{F}^{\perp} \subseteq \Lambda$ is the orthogonal of $\mathcal{F}$ with the respect to $\perp$. A term $t \in \Lambda$ is a realizer for $F$, in symbols $t \vdash F$, if $t \in (\{\mathcal{F}\}^{\perp} \cap Q)$. We denote $\vdash F$ if there exists a realizer for $F$.

This new application has therefore produced set-theoretical models that supply programs associated to ZF-theorems.

During last twenty years, Krivine has realized relevant mathematical principles, like the Axiom of Dependant Choice (DC$_N$), which can be viewed as a further extension of proof-programs correspondence to (at least) real analysis. Recent developments have extended even further this correspondence, realizing choice principles on arbitrary cardinals and large cardinals axioms (in [2, 3]). Nowadays, realizability stands as a well-grounded technique, which enables to built ZF-models encompassing a “constructive behaviour”. For instance, while the formulæ $\top \wedge \perp \rightarrow \perp$ can be considered the same one in a classical set-up, from a computational point of view these have a slightly different meaning, the former behaving has a right projection, the latter as a left one, hence they are not realized by the same program in general. In fact, assuming the existence of such a program $\diamondsuit$, verifying $\diamondsuit \vdash \top \wedge \perp \rightarrow \perp$, $\diamondsuit \vdash \perp \wedge \top \rightarrow \perp$, introduces non-deterministic processes in the underlying calculus. Furthermore, it turns out that $\diamondsuit$ transforms the realizability model in a forcing one. Forcing technique, a wide-spread tool of modern set-theory developed by Cohen [1] in 1963, can be viewed as a special case of realizability, where every formulæ is realized by the same program, thus it is considered as a trivialization of realizability.

2 Renovating realizability

We present an improved formalism for Krivine’s realizability, developed by Fontanella, Geoffroy and Matthews (in [2, 3, 7]) which strengthens this framework with a forcing-like definition of the realizability model.

For a fixed model $M$ of ZF, the realizability model $\mathcal{N}$ generated by a realizability algebra $\mathcal{A} \in \mathcal{M}$ is a first-order model satisfying formulæ of $\mathcal{L} = \{\not\in, \subseteq, \mathbb{C}\}$. $\mathcal{L}$ is a slight modification of set theory signature, due to technical reasons, which defines a conservative extension of ZF, namely $\mathcal{ZF}^\mathcal{C}$

$$\mathcal{ZF}^\mathcal{C} := \left\{ \begin{array}{l}
\subseteq\text{-Extensionality} \equiv \forall x \forall y (x \subseteq y \leftrightarrow \exists z \in y (x \sim z)); \\
\subseteq\text{-Extensionality} \equiv \forall x \forall y (x \subseteq y \leftrightarrow \forall z \in x (z \in y)); \\
\text{Foundation} \equiv \forall x \forall y (x \in y \leftrightarrow \forall z \in y (x \in z)); \\
\text{Comprehension} \equiv \forall x \forall y \exists z \forall \forall y (y \in x \rightarrow F(y, x) \rightarrow F(a, x)); \\
\text{Paring} \equiv \forall x \forall y \exists z \forall \forall y (y \in x \rightarrow (x \in a \wedge F(x, a))); \\
\text{Union} \equiv \forall x \forall y \exists z \forall \forall y (y \in x \rightarrow (x \in a \wedge b \in c)); \\
\text{Power Set} \equiv \forall x \forall y \exists z \forall \forall y (y \in x \leftrightarrow (x \in a \wedge z \in x)); \\
\text{Collection} \equiv \forall x \forall y \exists z \forall \forall y (y \in x \rightarrow F(x, y, \bar{x}) \rightarrow \exists y \in b F(x, y, \bar{x})); \\
\text{Infinity} \equiv \forall x \forall y \exists z \forall \forall y (a \in b \rightarrow (x \in y) \rightarrow (y \in b F(x, y, \bar{x})))
\end{array} \right\} \mathcal{F} \in \mathcal{F}_{\mathcal{C}}.$$

$\not\in$ is the negation of membership, $\subseteq$ is the subset relation. $\not\in$ is the negation of a non-extensional membership relation. Extensional equality, defined as usual by use of $\subseteq$, is denoted as $\equiv$. Non-extensional equality is denoted as $\equiv$.

Since the realizability relation is defined inductively on the structure of formulæ, it suffices to well-define it for atomic formulæ of the language. For this purpose, we introduce names.

Definition 2. The class of $\mathcal{A}$-names is defined inductively as

- $M^A_0 := \emptyset;$

We denote $\Lambda^c$ the set of closed $\lambda$-terms. The last condition is necessary to obtain a model. Observe that for $\perp \not= \emptyset$, $Q \not= \Lambda_c$. Indeed, if $t \prec \pi \not\prec \perp$, for any $t \leq \Pi$, $(k_\pi)t \not\prec t \wedge t \rightarrow (k_\pi)t \not\prec \Pi$, thus $(k_\pi)t \not\in Q$.

$\equiv x \equiv y := x \subseteq y \wedge y \subseteq x$

$\equiv x \equiv y := \forall z (x \equiv z \leftrightarrow y \equiv z)$
Following Definition 1, it is possible to associate a realizer elements of the ground model is defined, denoted as improvement in the comprehension of pre-existing results. A class of canonical representatives for As expected, by induction on \(\langle F, \alpha, \pi \rangle \) in \(\mathcal{M}^A\) is defined by induction of the structure of \(\mathcal{L}\)-formulae into \(\varphi(\Pi)\). Indeed, we will define by induction on the structure of \(F \in \mathcal{F}_\mathcal{L}\) the set of falsity values \(\|F\| \subseteq \varphi(\Pi)\). Names play a fundamental role in the atomic-formulæ cases and the universal-quantifier case. In order to define \(\|F\|\), a definition of rank for names is needed.

**Definition 3.** For every \(a \in M^A\), we define the rank of \(a\) in \(M^A\) as

\[
\text{rank}^A(a) = \min\{\alpha \in \text{Ord} | a \in M^A_{\alpha+1}\}.
\]

**Definition 4.** We define \(\|a \not\in b\| := \{\pi \in \Pi | (a, \pi) \in b\}\) for every \(a, b \in M^A\). Moreover, by induction on \((\text{rank}^A(a), \text{rank}^A(b))\), we set:

- \(\|a \not\in b\| := \bigcup_{\eta \in M^A} \{t \cdot t', \pi \in \Pi | (\langle c, \pi \rangle \in b, t \vdash c \subseteq a, t' \vdash a \subseteq c)\}\)
- \(\|a \subseteq b\| := \bigcup_{\eta \in M^A} \{t \cdot \pi \in \Pi | (\langle c, \pi \rangle \in a, t \vdash c \not\in b)\}\)

The set of falsity values \(\|F\| \subseteq \varphi(\Pi)\) for a formula \(F \in \mathcal{F}_\mathcal{L}\) is defined by induction of the structure of \(F\):

- \(\|\top\| := \emptyset, \|\bot\| := \Pi\);
- atomic cases as above;
- \(\|G_1 \rightarrow F_2\| := \{t \cdot \pi \in \Pi | t \in \|F_1\|\not\subseteq, \pi \in \|F_2\|\}\);
- \(\|\forall x F(x)\| := \bigcup_{\eta \in M^A} \|F(\eta)\|\).

Following Definition 4, it is possible to associate a realizer \(t \in \|F\|\not\subseteq\) for any formula \(F\) - if it exists. As expected, ZF-axioms are realized (see \(\mathcal{M}^A\)).

\(M^A\) allows to define basic objects of \(\mathcal{N}\) in a more explicit way, consequently it represents an improvement in the comprehension of pre-existing results. A class of canonical representatives for elements of the ground model is defined, denoted as \(\mathcal{N}^M\)

**Definition 5.** By induction on \(\rho(a)\) we define \(\eta(b, \pi) = \{\langle \eta(b), \pi \rangle | b \in a, \pi \in \Pi\}\). We denote \(\eta(M) = \{\eta(a) | a \in M\}\).

Among the elements of \(\mathcal{N}^M\), \(\mathcal{N}^2 = \{\eta(b, \pi) | b = 0, 1; \pi \in \Pi\}\) turns out to have a relevant role, as the canonical representative of 2 in \(\mathcal{N}\) may contain arbitrary copies of 0 and 1, distinguished by the relation \(\varepsilon\) introduced with the language \(\mathcal{L}\) above. The cardinality of \(\mathcal{N}^2\) is strictly related with realizability model as \(\vdash \forall x \in \mathcal{N}^2(x = \top \lor x = \bot)\) if, and only if, \(\mathcal{N}\) is a forcing model. The left-to-right implication holds if one assumes that an instruction quote is in \(\Lambda_e\), this term acting like an enumerator for closed \(\lambda_e\)-terms. Thus, \(\mathcal{N}^2\) allows to establish whether \(\mathcal{M}\) produces a forcing model or not.

In order to get more information about the nature of \(\mathcal{N}\), it is possible to add a boolean-algebra structure on \(\mathcal{N}^2\), induced by the minimal boolean algebra \(\langle 2, \leq, 0, 1 \rangle\) in \(\mathcal{M}\), to fix a complete theory containing the one of realized formulæ. Let \(\langle \mathcal{N}^2, \leq, 0, 1 \rangle\) be the induced algebra. With respect to the extensional equality \(\approx\), it is the minimal boolean-algebra of \(\mathcal{N}\). \(\langle \mathcal{N}^2, \leq, 0, 1 \rangle\) is a powerful tool to investigate the structure of the realizability model. Indeed, \(\bot\)-orthogonality induces on the powerset of \(\Pi\) a preorder \(\leq\) defined as:

**Definition 6.** For any \(X, Y \in \varphi(\Pi), X \leq Y\) if, and only if, \(Q \cap (X \rightarrow Y) \not\subseteq 0\) (or \(\vdash X \rightarrow Y\)), where

\[
X \rightarrow Y := \{t \cdot \pi \in \Pi | t \in X\not\subseteq, \pi \in Y\}.
\]

\(^4\text{It is a subclass of } M^A.\)
The induced poset has a boolean-algebra structure \((\mathcal{P}(\Pi), \subseteq, \Pi, \emptyset)\), which can be related via representatives \(u_X \in M^A\), for \(X \in \mathcal{P}(\Pi)\), to the boolean algebra on \(\mathcal{P}(\Pi)\).

**Definition 7.** For \(X \in \mathcal{P}(\Pi)\), \(u_X := \{\lnot 0, \pi| \pi \in X\} \).

**Theorem 1.** The following results hold:

1. For any \(X \in \mathcal{P}(\Pi)\), \(N \models u_X \simeq \lnot 0 \lor u_X \simeq \lnot 1\);
2. For any formula \(F\) of \(\mathcal{L}_\mathcal{L}\), \(\models F\) if, and only if, \(N \models u_{||F||} \simeq \lnot 0\);
3. \(\mathfrak{S} := \{X \in \mathcal{P}(\Pi)| N \models u_X \simeq \lnot 0\}\) is an ultrafilter of \((\mathcal{P}(\Pi), \subseteq)\).

The theorem states that (1.) any \(u_X\) is extensionally equal to \(\lnot 0\) or \(\lnot 1\), i.e. it is contained in \(\mathcal{P}(\Pi)\) (2.) formule whose falsity value is sent to \(\lnot 0\) in \(N\) are precisely those that are realized, (3.) these formulæ generate a filter in \(\mathcal{P}(\Pi)\) which is contained in an ultrafilter \(\mathfrak{S}\) of \(\mathcal{P}(\Pi)\), thus \(\mathfrak{S}\) determines a complete theory containing any realized formula.

**Proof.** 1. Fix \(X \in \mathcal{P}(\Pi)\).

First, we show that \(\lambda x.x \models u_X \subseteqc \lnot 1\).

\[
||u_X \subseteqc \lnot 1|| = \bigcup_{c \in M^A} \{t \cdot \pi | t \in ||c \neq \lnot 1||, \pi \in ||c \neq u_X||\}.
\]

For \(c \neq \lnot 0\), \(||c \neq u_X|| = \emptyset\), thus

\[
||u_X \subseteqc \lnot 1|| = \{t \cdot \pi | t \in ||\lnot 0 \neq \lnot 1||, \pi \in ||\lnot 0 \neq u_X||\} = \{t \cdot \pi | t \in ||\bot||, \pi \in X\} = \Pi \rightarrow X
\]

which is realized by the identity \(\lambda x.x\).

Next, we show that \(\models \lnot 0 \in c \rightarrow \lnot 1 \subseteqc c\) for any name \(c \in M^A\). Consider falsity values associated with the formula

\[
||\lnot 0 \in c \rightarrow \lnot 1 \subseteqc c|| = \{t \cdot u \cdot \pi | t \in ||\lnot 0 \in c||, u \in ||\lnot 0 \notin c||, \pi \in \Pi\}.
\]

Let \(t \in ||\lnot 0 \in c||,\) then \(t(uW) \in ||\bot||\) for \(t \in ||\lnot 0 \in c||\). Then, \(\lambda x.y.x(yWW) \models \lnot 0 \in c \rightarrow \lnot 1 \subseteqc c\), in particular for \(c = u_X\).

Observe that for \(\pi \in ||\lnot 0 \notin u_X||, k_x \in ||\lnot 0 \in u_X||\). Consider now

\[
||\forall x(u_X \neq \lnot 1 \rightarrow x \notin u_X)|| = \{t \cdot \pi | t \in ||(u_X \subseteq \lnot 1 \land \lnot 1 \subseteq u_X) \rightarrow \bot|, \pi \in X\};
\]

as just shown, \(I \models u_X \subseteq \lnot 1\); moreover, from the discussion above, \((\lambda x.y.x(yWW))k_\pi \in ||\lnot 1 \subseteq u_X||\) for any \(\pi \in \Pi\), thus \(tI(\lambda x.y.x(yWW))k_\pi \in ||\bot||\). This shows that

\[
\lambda x.xc\lambda k.x((\lambda x.y.x(yWW))k) \models \forall x(u_X \neq \lnot 1 \rightarrow x \notin u_X)
\]

and implies that there exists a realizer for \(u_X \neq \lnot 1 \rightarrow \forall x(x \notin u_X)\).

Lastly, it suffices to show that \(\models \forall x(\forall y \neq x) \leftrightarrow x \simeq \lnot 0\). The formula can be reduce to \(\forall x(\forall y \neq x) \leftrightarrow x \simeq \lnot 0\).

We first show the left-to-right implication. \(||\forall x(\forall y \neq x) \rightarrow x \subseteq \lnot 0|| = \bigcup_{c \in M^A} \{t \cdot \pi | t \in ||\forall y \neq c||, \pi \in ||c \subseteq \lnot 0||\} \). Let \(t \cdot \pi \in ||\forall x(\forall y \neq x) \rightarrow x \subseteq \lnot 0||\) for a fixed \(c \in M^A\). Then,

\[
\pi = u \cdot \rho \in \bigcup_{d \in M^A} \{u \cdot \rho | u \in ||d \notin \lnot 0||, \pi \in ||d \notin c||\}.
\]

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5This is not the case from the non-extensional point of view: for \(X \neq \Pi, N \models (u_X \neq \lnot 1 \land u_X \neq \lnot 1)\).

6\(x \subseteqc y := \forall z (z \neq y \rightarrow z \neq x)\). It is easy to show that \(\text{ZF}_x \models \forall x, y (x \subseteqc y \rightarrow x \subseteqc y)\).

7\(Y := (\lambda x.\lambda y.(y)(x)y)(\lambda x.\lambda y.(y)(x)y), W := (Y)(\lambda x.\lambda y.(y)xx)\). It is easy to see that for any \(c \in M^A\), \(W \models c \subseteqc c\), then \(\lambda x.xW \models \exists x \subseq c\).
Since \( t \cdot \rho \in \perp \), \( \lambda y.x \models \forall x(\forall y(y \neq x) \rightarrow x \subseteq \emptyset) \). We show the left-to-right implication. \( \vdash \forall x(x \subseteq \emptyset \rightarrow \forall y(y \neq x)) \). Fix such a falsity value \( t \cdot \pi \). \( t \in \| \forall y(y \neq \emptyset \rightarrow y \neq c) \| \). It is easy to see that \( \| d \notin \emptyset \| = \emptyset \) for any \( d \in M^A \), hence any \( u \in Q \) realizes \( d \notin \emptyset \), which implies \( tu \in \| d \notin c \| \). Without loss of generality, suppose \( u = I \). Then, \( tI \in \| \forall y(y \neq c) \| \), which proves \( \lambda x.xI \vdash \forall x(x \subseteq \emptyset \rightarrow \forall y(y \neq x)) \).

To conclude, we showed that for any \( X \in \Pi \)
\[
\vdash u_X \not\models \top \rightarrow \forall x(x \neq u_X),
\]
which implies that
\[
\vdash u_X \not\models \top \rightarrow u_X \models \emptyset.
\]
2. For any closed formula \( F \in \mathcal{F}_L \) with parameters in \( M^A \)
\[
\vdash F \iff \exists t \in Q \forall \pi \in \| F \| ((t \cdot \pi \in \perp) \iff \exists t \in Q (t \models \emptyset \not\models u_{||F||}) \iff \exists t \in Q (t \models \forall x(x \neq u_{||F||}) \iff \not\models u_{||F||} \models \emptyset).
\]
3. Consider \( \mathcal{G} = \{ X \in \mathcal{G}(\Pi) | N \models u_X \models \emptyset \} \).
- \( \mathcal{G} \) is upward closed for \( \leq \). Let \( X \in \mathcal{G}, Y \in \mathcal{G}(\Pi), X \leq Y \). By hypothesis, there exists \( u \models \forall x(x \neq u_X) \), \( t \models X \rightarrow Y \), the latter equivalent to \( t \models \forall x(x \neq u_X) \rightarrow \forall x(x \neq u_Y) \). Thus, \( (t)u \models \forall x(x \neq u_Y) \).
- \( \mathcal{G} \) is closed for meets. Fix \( X, Y \in \mathcal{G} \). Then, \( \| \forall x(x \neq u_{X \land Y}) \| = \| \emptyset \not\models u_{X \land Y} \| = X \land Y \). By hypothesis, there exists \( t, u \in Q \) such that \( t \models X, u \models Y \), thus \( \lambda x.xtu \models \forall x(x \neq u_{X \land Y}) \).
- \( \mathcal{G} \) is maximal. Consider \( X \in \mathcal{G}(\Pi) \). We show that \( \vdash (u_X \models \emptyset \models \perp) \rightarrow u_{X \rightarrow \perp} \models \emptyset \), which is equivalent to \( \vdash (\forall x(x \neq u_X) \rightarrow \perp) \rightarrow \forall x(x \neq u_{X \rightarrow \perp}) \). Observe that
\[
\| \forall x(x \neq u_X) \rightarrow \perp \| = \{ t, \pi \mid t \in \| \forall x(x \neq u_X) \|, \pi \in \Pi \}
\]
\[
= \{ t, \pi \mid t \in \| \emptyset \not\models u_X \|, \pi \in \Pi \}
\]
\[
= \{ t, \pi \mid t \in X \models \perp, \pi \in \Pi \}
\]
\[
= \| \emptyset \not\models u_{X \rightarrow \perp} \|
\]
\[
= \| \forall x(x \neq u_{X \rightarrow \perp}) \|
\]
Thus, \( \lambda x.xI \vdash (\forall x(x \neq u_X) \rightarrow \perp) \rightarrow \forall x(x \neq u_{X \rightarrow \perp}) \).

References


