Injectivity of the coherent model for a fragment of connected *MELL* proof-nets

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1 Introduction

A *semantics* or *interpretation* is a function that associates with every term t of a rewriting system a mathematical object $[\![t]\!]$. Semantics must be invariant with respect to the rewriting rules: if t reduces to t', then $[\![t]\!] = [\![t']\!]$. This invariance implies the inclusion of the *syntactic equivalence* relation on terms, which identifies the terms that can be transformed into each other by applying rewriting rules, in the *semantic equivalence* relation that, on the other hand, identifies the terms with the same interpretation.

Another interesting equivalence relation on the terms of a rewriting system is the *observational equivalence*. Roughly speaking, two terms are observationally equivalent if they normalize to the same value in any context of some type which is considered as ground, or "observable". This notion originated in the field of programming languages semantics: in order to compare two pieces of code, it is very natural to plug them into a program that produces an observable value, such as a number and check whether or not they produce the same value. Every reasonable notion of observational equivalence includes the semantic equivalence of every interpretation. We say that a semantics is *injective* if the induced semantic equivalence coincides with syntactic equivalence, *fully abstract* if it coincides with observational equivalence. In general, full abstraction fails when the interpretation is not surjective. A classic example is Scott's continuous model, which is not fully abstract for PCF because the "parallel or" function is not PCF-definable, as shown in [7]. On the contrary, when the semantics is surjective, we say that *full completeness* holds. This last property, which was originally studied in [1], is often exploited as a sufficient condition for full abstraction, for instance in [5].

In the second half of the last century, with the discovery of Curry-Howard's correspondence, the study of these equivalence relations, historically at the heart of theoretical computer science, became relevant in proof theory: a proof can be seen as a program, or a term of a rewriting system, whose execution corresponds to the cut-elimination procedure. In Gentzen's classical logic LK, however, the syntactic equivalence is trivial: it identifies all proofs of the same formula. In particular, every semantic equivalence is trivial.

Linear logic, introduced by Jean-Yves Girard in [4] in 1987, is a refinement of classical and intuitionistic logic in which formulas are treated as resources: the structural rules of contraction and weakening are restricted and the semantic equivalence induced by the models of linear logic is non-trivial.

In this context, we ask ourselves the question of finding a canonical object representing the proofs in the same class of semantic equivalence. Formalization through proof-nets removes the redundant information of sequent calculus concerning the order of application of the rules and allows us to define the cut-elimination procedure through local manipulations of graphs. It is therefore an appropriate formalism to study the dynamics of normalisation and to prove fundamental properties of the system such as strong normalisation (see [4] and [6]).

In linear logic, the question of injectivity was first addressed by Tortora de Falco in [9], where he produced counter-examples to the injectivity of multiset-based coherent semantics for multiplicative exponential linear logic without units (*MELL*). On the other hand, the injectivity of relational semantics for the full multiplicative

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exponential fragment of linear logic was recently proven by de Carvalho in [3] by employing the powerful Taylor expansion technique, that allows us to represent a proof-net as the infinite series of its linear approximations.

We resume the work on the injectivity of multiset-based coherent semantics which was started in [9]. It was conjectured that coherent semantics is injective for connected *MELL* proof-nets and it was provided a sufficient condition to reach this conclusion: the existence of an injective experiment for every connected proof-net which only consists of axiom, tensor, dereliction and contraction nodes. This justifies our definition of proof-structure (Definition 2.1). It was also proven, using the (C)-pairs technique, that this condition is met if one assumes that every contraction node is terminal. In this work, we prove the injectivity of coherent semantics for connected $(?\parr)LL_{pol}$ proof-nets (Section 5). This is done through the introduction of the notion of atomic pre-experiment (Section 3) and a generalization of (C)-pairs (Section 4).

2 Proof-nets and experiments

In this section, we focus on a particular subsystem of cut-free *MELL* proof-nets. Formulas are generated by the following grammar, where $X, X^\perp$ denote dual atomic formulas:

$$A ::= X \mid X^\perp \mid A \otimes A \mid ?A$$

Definition 2.1. A *proof-structure* is a non-empty labelled directed graph R such that its vertices, called *nodes*, have exactly one label among ax, $\otimes$, $?$, $\bullet$, its arcs are labelled by formulas, every arc of R is called a *premise* of its head and a *conclusion* of its tail and every node of R labelled by:

- ax is called an *axiom*, has no premises and exactly two conclusions, labelled by dual atomic formulas;
- $\otimes$ is called a *tensor*, has exactly one conclusion, labelled by a formula $A \otimes B$, and exactly two premises, one of which is called its *left premise* and is labelled by A, whereas the other is called its *right premise* and is labelled by B;
- $?$ is called a *why not* and has exactly one conclusion, labelled by a formula of the shape $?A$. Such a node has all of its premises labelled by A and is called a *weakening* when it has no premises, a *dereliction* when it has exactly one premise, a *contraction* otherwise;
- $\bullet$ is called a *conclusion* and possesses exactly one premise and no conclusions.

The *type* of an arc a of R is the formula labelling a. A *switching* of R is a function φ mapping every dereliction or contraction node of R to one of its premises. The *switching graph* of R induced by φ is the proof-structure R^φ obtained from R by replacing the head of a with a fresh conclusion, for each premise a of a contraction node n of R such that $\varphi(n) \neq a$. We say that R is a *proof-net* if the underlying undirected graph of every switching graph of R is acyclic, a *connected* proof-net if such graphs are also connected. Two nodes m, m' of R are φ-connected if there exists a path from m to m' in the underlying undirected graph of R^φ. If R is a proof-net, then such a path is unique (if it exists) and is written $\theta^\varphi_{m,m'}$. We will assume that a proof-structure comes with a total order of its conclusion nodes, called its *interface*.

Notation 2.1. Let R be a proof-structure and let a be an arc of R. We write n_a for the node of which the arc a is a conclusion. If R is a proof-net, φ is a switching of R and a, a' are arcs of R, we write $\theta^\varphi_{a,a'}$ for the path $\theta^\varphi_{n_a,n_{a'}}$.

We now turn our attention from syntax to semantics, by recalling the notion of coherence space.

Notation 2.2. Let A and B be sets. We write $A \times B$ for the cartesian product of A and B and $\mathcal{M}_{\mathrm{fin}}(A)$ for the set of finite multisets of elements of A. Also, for all $x = (a, b) \in A \times B$, we define $\mathbf{pr}_1(x) := a$ and $\mathbf{pr}_2(x) := b$.

Definition 2.2. A *coherence space* $\mathcal{A}$ is an ordered pair $(|\mathcal{A}|, \coh_\mathcal{A})$, where $|\mathcal{A}|$ is a set, called *web* and $\coh_\mathcal{A}$ is a binary reflexive and symmetric relation on the web called *coherence*. Strict coherence is written $\frown_\mathcal{A}$. A clique of $\mathcal{A}$ is a subset C of $|\mathcal{A}|$ such that $x \coh_\mathcal{A} y$ for every $x, y \in C$. The coherence space $\mathcal{A}^\perp$ is defined by $|\mathcal{A}^\perp| := |\mathcal{A}|$ and $\coh_{\mathcal{A}^\perp} := |\mathcal{A}|^2 \setminus \frown_\mathcal{A}$. We then define *incoherence* in $\mathcal{A}$ as $\asymp_\mathcal{A} := \coh_{\mathcal{A}^\perp}$ and we write $\smile_\mathcal{A}$ for strict incoherence. Subscripts are omitted when they are clear from the context. Lastly, an *anticlique* of $\mathcal{A}$ is just a clique of $\mathcal{A}^\perp$.

	$[\![\cdot]\!]_{\mathbf{Rel}}$	$\lvert[\![\cdot]\!]_{\mathbf{Coh}}\rvert$	$x \frown y$
$A \otimes B$	$[\![A]\!]_{\mathbf{Rel}} \times [\![B]\!]_{\mathbf{Rel}}$	$[\![A]\!]_{\mathbf{Coh}} \times [\![B]\!]_{\mathbf{Coh}}$	$\mathbf{pr}_1(x) \frown \mathbf{pr}_1(y)$ and $\mathbf{pr}_2(x) \frown \mathbf{pr}_2(y)$
$?A$	$\mathcal{M}_{\mathrm{fin}}([\![A]\!]_{\mathbf{Rel}})$	$\mathcal{M}_{\mathrm{clfin}}([\![A]\!]_{\mathbf{Coh}}^{\perp})$	$x = y$ or $x \cup y \notin \mathcal{M}_{\mathrm{clfin}}([\![A]\!]_{\mathbf{Coh}}^{\perp})$

Table 1: The relational and coherent interpretations of non-atomic formulas, where $M \in \mathcal{M}_{\mathrm{clfin}}([\![A]\!]_{\mathbf{Coh}}^{\perp})$ if and only if $M \in \mathcal{M}_{\mathrm{fin}}(\lvert[\![A]\!]_{\mathbf{Coh}}\rvert)$ and, for every $u, v \in M$, we have $u \asymp_{[\![A]\!]_{\mathbf{Coh}}} v$.

We now assume that we have an interpretation of atomic formulas by sets (resp. by coherence spaces), that is a map $[\![\cdot]\!]_{\mathbf{Rel}}$ (resp. a map $[\![\cdot]\!]_{\mathbf{Coh}}$) that associates with each atomic formula X a set $[\![X]\!]_{\mathbf{Rel}}$ (resp. a coherence space $[\![X]\!]_{\mathbf{Coh}}$) in such a way that $[\![X^{\perp}]\!]_{\mathbf{Rel}} = [\![X]\!]_{\mathbf{Rel}}$ (resp. $[\![X^{\perp}]\!]_{\mathbf{Coh}} = [\![X]\!]_{\mathbf{Coh}}^{\perp}$) for every atomic formula X. The interpretation of non-atomic formulas is then defined by induction on their logical complexity, according to Table 1. Now, in order to compute the semantics of a proof-structure, we need the notion of experiment.

Definition 2.3. Let R be a proof-structure. A *relational* (resp. *coherent*) *experiment* of R is a function e which associates with every arc of type A of R an element of its relational interpretation (resp. an element of the web of its coherent interpretation) and such that:

- If α is a conclusion of an axiom node of R, then $e(\alpha) = e(\alpha^{\perp})$;
- If a is the conclusion of a tensor node of R with left premise b and right premise c, then $e(a) = (e(b), e(c))$;
- If a is the conclusion of a why not node of R with premises $a_1, \ldots, a_k$, then $e(a) = \{e(a_1), \ldots, e(a_k)\}$.

We say that e is *injective* if $e(\alpha_1) \neq e(\alpha_2)$ for all distinct arcs α_1, α_2 of R of the same atomic type. If $(c_1, \ldots, c_h)$ is the sequence of the premises of the conclusion nodes in the interface order, then $(e(c_1), \ldots, e(c_h))$ is called the *result* of e. Finally, the *relational* (resp. *coherent*) *semantics* $[\![R]\!]_{\mathbf{Rel}}$ (resp. $[\![R]\!]_{\mathbf{Coh}}$) of R is the set of the results of all relational (resp. coherent) experiments of R.

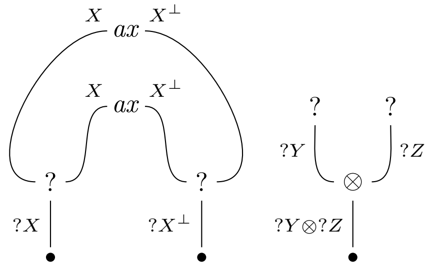
Remark 2.1. Both relational and coherent semantics identify a non-atomic axiom and the η-expansion of that axiom, which is the canonical proof-net with the same conclusions as the non-atomic axiom and which only uses atomic axioms. This justifies our choice of only allowing atomic axioms in Definition 2.1.

Remark 2.2. Let R be a *proof-net*, let $(c_1, \ldots, c_h)$ be the sequence of the premises of the conclusion nodes of R in the interface order, let C_i be the type of c_i for all $i \in \{1, \ldots, h\}$ and let $\parr\Gamma := (C_1 \parr \cdots) \parr C_h$. Then $[\![R]\!]_{\mathbf{Coh}}$ is a clique of $[\![\parr\Gamma]\!]_{\mathbf{Coh}}$ (see [4]). We also have $[\![R]\!]_{\mathbf{Coh}} = [\![R]\!]_{\mathbf{Rel}} \cap \lvert[\![\parr\Gamma]\!]_{\mathbf{Coh}}\rvert$ (see [8]). Hence, the injectivity of coherent semantics for a fragment of proof-nets entails the injectivity of relational semantics for that fragment.

Remark 2.3. Every function mapping distinct axioms of R to distinct points of the relational interpretations of their conclusions trivially induces an injective relational experiment of R. On the other hand, the existence of an injective *coherent* experiment of R is non-trivial: whenever a is the conclusion of a contraction with premises $a_1, \ldots, a_k$ of type A, we have $e(a) \in \lvert[\![?A]\!]_{\mathbf{Coh}}\rvert$, or equivalently $e(a_i) \asymp e(a_j)$ for all $i, j \in \{1, \ldots, k\}$.

Example 2.1. Figure 1 provides an example of proof-net for which there is no injective coherent experiment. Observe that this is *not* a connected proof-net. The connected component of this proof-net with the tensor and weakening nodes is only necessary if a correspondence with a *MELL* sequent calculus proof is desired.

All the coherent experiments of the proof-net R in Figure 1 associate *the same* element with both axioms of R. In fact, if x is associated with the upper axiom and y with the lower one, then we must have $x \asymp_{[\![X]\!]_{\mathbf{Coh}}} y$ and $x \asymp_{[\![X^{\perp}]\!]_{\mathbf{Coh}}} y$, which entails $x = y$. As a result, no coherent experiment can distinguish R from the proof-net R' obtained by crossing the two arcs labelled by X in R. In this case this is harmless, since R and R' are indeed *the same* proof-net. However, by slightly complicating this crucial example, one can obtain two distinct proof-nets having the same coherent interpretation, as shown in [9]. This proves that coherent semantics is not injective for *MELL* proof-nets and hints at the fact that the existence of an injective coherent experiment is intimately related to the injectivity of coherent semantics. More precisely, the following sufficient condition for the "local" injectivity of coherent semantics was provided in [9].



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^{\perp}}\,ax}{\vdash X, X^{\perp}, ?Y}\,?w \qquad \dfrac{\dfrac{}{\vdash X, X^{\perp}}\,ax}{\vdash X, X^{\perp}, ?Z}\,?w}{\vdash X, X, X^{\perp}, X^{\perp}, ?Y \otimes ?Z}\,\otimes}{\vdash ?X, ?X, ?X^{\perp}, ?X^{\perp}, ?Y \otimes ?Z}\,?de \times 4}{\vdash ?X, ?X^{\perp}, ?Y \otimes ?Z}\,?co \times 2$$

Figure 1: A proof-net for which there is no injective coherent experiment and a corresponding sequent calculus proof.

Lemma 2.1. *Let R be a connected proof-net for which there exists an injective coherent experiment. Then, for every connected proof-net R' with the same conclusions as R and such that $[\![R]\!]_{\mathbf{Coh}} = [\![R']\!]_{\mathbf{Coh}}$, we have $R = R'$.*

In [9] it was also conjectured that, if we restrict ourselves to *connected* proof-nets, then coherent semantics is injective. Such a result, which would establish a very interesting relationship between the syntactic property of connectedness and the semantic property of injectivity, could be proven by using the following intermediate result, which is a consequence of Lemma 2.1.

Corollary 2.1. *If, for every connected proof-net R, an injective coherent experiment of R exists, then coherent semantics is injective for the subsystem of connected proof-nets.*

It is well known that, in the absence of weakenings, all sequent calculus proofs can be desequentialized into connected proof-nets. Therefore, by Corollary 2.1 and by the results in [9] that allow us to restrict ourselves to the fragment we have considered at the beginning of this section (namely, those concerning the linearization and $\parr$-mutilation procedures), we obtain the following result, which further motivates our goal of proving the existence of an injective coherent experiment for all connected proof-nets.

Corollary 2.2. *If, for every connected proof-net R, an injective coherent experiment of R exists, then coherent semantics is injective for MELL without weakenings.*

3 Atomic pre-experiment

We have just seen in Section 2 that the injectivity of coherent semantics boils down to the existence of injective coherent experiments. When trying to define such an experiment for a given proof-net, some choices are forced from the very beginning, while others may be postponed. For instance, if two premises of a contraction node are atomic arcs of type X, conclusions of axioms n and n', then every injective coherent experiment must associate with n and n' two incoherent elements of $|[\![X]\!]_{\mathbf{Coh}}|$. On the other hand, by considering for instance a proof-net with no contraction nodes, it is clear that no particular choice is forced. Following this observation, we define an abstraction and generalization of injective experiments which is partial: the notion of *pre-experiment*. The idea is to forget the elements of the webs of the coherence spaces and retain the coherence and incoherence relations, which will always be strict because our goal is to produce an *injective* experiment.

Notation 3.1. If α is a conclusion of an axiom n of a proof-structure R, we denote by $\alpha^{\perp}$ the other conclusion of n. We also write P_R for the set of unordered pairs $\{a, a'\}$ such that a, a' are two distinct arcs of R of the same type and P_R^{at} the set of those elements $\{\alpha, \alpha'\} \in P_R$ such that the type of α, α' is atomic.

Definition 3.1. If R is a proof-structure, a *pre-experiment* of R is a partial function $e\colon P_R^{at} \to \{\frown, \smile\}$, where $\frown$ and $\smile$ are just two formal symbols, called *coherence* and *incoherence*, such that, for every $\{\alpha, \alpha'\} \in P_R^{at}$, if $e(\{\alpha, \alpha'\})$ is defined, then $e(\{\alpha^{\perp}, \alpha'^{\perp}\})$ is also defined and we have $e(\{\alpha, \alpha'\}) \neq e(\{\alpha^{\perp}, \alpha'^{\perp}\})$. In addition, the pre-experiment e uniquely extends to a partial function $\bar{e}\colon P_R \to \{\frown, \smile\}$, which is defined by induction on the type A of the arcs a, a' of a pair $\{a, a'\} \in P_R$:

- If A is an atomic type, then $\bar{e}(\{a,a'\}) = e(\{a,a'\})$;
- If $A = B \otimes C$ for some types B and C, then a, a' are conclusions of tensor nodes of R having left premises b, b' and right premises c, c' respectively. We define:

$$\bar{e}(\{a,a'\}) = \begin{cases} \frown & \text{if } \bar{e}(\{b,b'\}) = \bar{e}(\{c,c'\}) = \frown \\ \smile & \text{if } \bar{e}(\{b,b'\}) = \smile \text{ or } \bar{e}(\{c,c'\}) = \smile \end{cases}$$

  If neither of the conditions on the right holds, then the partial function $\bar{e}$ is undefined on the pair $\{a,a'\}$;
- If $A = ?B$ for some type B, then a, a' are conclusions of why not nodes of R with premises $b_1, \ldots, b_k$ and $b'_1, \ldots, b'_{k'}$ respectively, for some strictly positive integers k and k'. We define:

$$\bar{e}(\{a,a'\}) = \begin{cases} \frown & \text{if } \exists\, i \in \{1,\ldots,k\}, i' \in \{1,\ldots,k'\} \colon \bar{e}(\{b_i, b'_{i'}\}) = \frown \\ \smile & \text{if } \forall\, i \in \{1,\ldots,k\}, i' \in \{1,\ldots,k'\} \colon \bar{e}(\{b_i, b'_{i'}\}) = \smile \end{cases}$$

  Again, if neither of the conditions on the right holds, the partial function $\bar{e}$ is undefined on $\{a,a'\}$.

Notation 3.2. If e is a pre-experiment of a proof-structure R and $\{a,a'\} \in P_R$, we can unambiguously denote by $e(a,a')$ the element $\bar{e}(\{a,a'\})$.

A key property that pre-experiments must satisfy in order to be extendable to a total injective experiment is the requirement that any two premises of any contraction are *not* coherent.

Definition 3.2. A pre-experiment e of a proof-structure R is *admissible* if, for every two distinct premises a, a' of the same contraction node of R, we do *not* have $e(a,a') = \frown$, meaning that $e(a,a')$ is either $\smile$ or undefined. We say that e is *complete* if e is admissible and defined on every pair of premises of a contraction node of R.

Remark 3.1. Every total extension of a complete pre-experiment of R is essentially an injective experiment.

The following definition now presents a particular pre-experiment of connected proof-nets.

Notation 3.3. We denote by $|\theta|$ the support of a path θ, that is its set of arcs and $\mathcal{N}(\theta)$ its set of vertices.

Definition 3.3. The *atomic pre-experiment* of a connected proof-net R is defined by the condition:

$$e_{at}(\alpha,\alpha') = \frown \iff \forall\, \varphi \text{ switching of } R \colon \alpha, \alpha' \in |\theta^{\varphi}_{\alpha,\alpha'}|$$

One easily proves that every pair of premises of an atomic contraction is incoherent.

Lemma 3.1. *Let R be a connected proof-net and let $\{\alpha,\alpha'\} \in P^{at}_R$ such that α, α' are two premises of the same contraction of R. Then $e_{at}(\alpha,\alpha') = \smile$.*

Remark 3.2. If every contraction of R is atomic, then e_{at} is a complete pre-experiment of R.

To characterize the coherence assigned by the atomic pre-experiment to any pair of arcs of the same type, possibly non-atomic, we introduce the notion of tree above an arc.

Definition 3.4. Let R be a proof-structure. If a is an arc of R, the *distance of a from an axiom* is the smallest non-negative integer h for which there is a descent path[1] $a_0 \ldots a_h$ of R such that a_0 is a conclusion of an axiom of R and $a_h = a$. The *tree above a*, written T_a, is then defined by induction on the distance d of a from an axiom:

- If $d = 0$, then the tree above a is the arc a in which the label of the head is replaced by $\bullet$;
- Otherwise, the arc a is the conclusion of a tensor or why not node n of R with premises $b_1, \ldots, b_\ell$. The tree above a is produced by first identifying for all indices $i \in \{1,\ldots,\ell\}$ the head of b_i in T_{b_i} and the tail of a, then replacing the labels of the tail and of the head of a with the label of n and $\bullet$ respectively.

Lemma 3.2. *Let R be a connected proof-net. For every pair $\{a,a'\} \in P_R$ we have:*

$$e_{at}(a,a') = \frown \iff \textit{no contraction of } R \textit{ occurs in } T_a \textit{ nor in } T_{a'} \textit{ and } \forall\, \varphi \textit{ switching of } R \colon a, a' \in |\theta^{\varphi}_{a,a'}|$$

An immediate consequence is the admissibility of the atomic pre-experiment.

Corollary 3.1. *Let R be a connected proof-net. Then e_{at} is admissible.*

[1] A descent path is a sequence of arcs $a_0 \ldots a_h$ such that the head of a_{i-1} is the tail of a_i for all $i \in \{1,\ldots,h\}$.

4 Generalization of (C)-pairs

The notion of (C)-pair is designed for producing an injective experiment such that every two conclusions of the same type of the proof-net are incoherent. It relies crucially on the fact that we deal with *connected* proof-nets: the incoherence of a pair of conclusions (a, a') is guaranteed by atomic arcs above a and a' which are involved in the paths $\theta^{\varphi}_{a,a'}$, where φ is any switching of the proof-net. Given that any two conclusions of the same type can be turned into premises of a terminal contraction and conversely, this technique, originally introduced in [9], can be used to prove the existence of an injective experiment in presence of terminal contraction nodes.

While in [9] the result was established under the hypothesis that every contraction node is terminal, here we generalize the notion of (C)-pair to proof-nets in which atomic contraction nodes may occur. From now on, we write R for a connected proof-net in which only axioms, tensors, derelictions and atomic contractions occur. We assume, for the sake of simplicity, that every conclusion of an axiom of R is a premise of a why not node. This is done without loss of generality, by adding dereliction nodes under atomic arcs where why not nodes are absent. The old notion of (C)-pair is well defined for the switching graphs of R, so we begin by recalling it. We use here a different terminology to avoid any confusion with the new notion of (C)-pair we will introduce later.

Definition 4.1. Let φ be a switching of R and let $\{a, a'\} \in P_{R^{\varphi}}$, $\{\alpha_0, \alpha'_0\} \in P^{at}_{R^{\varphi}}$ such that a, a' are conclusions of R^{φ} and such that α_0, α'_0 are arcs of T_a and of $T_{a'}$ respectively. We say that (α_0, α'_0) is a (φ)-*pair* for (a, a') if:

$$n_{\alpha_0} \in \mathcal{N}(\theta^{\varphi}_{a,a'}) \quad \text{or} \quad n_{\alpha'_0} \in \mathcal{N}(\theta^{\varphi}_{a,a'})$$

Remark 4.1. For every switching φ of R, there always exists a (φ)-pair for (a, a') and there exist at most two.

Definition 4.2. Let $\{a, a'\} \in P_R$, $\{\alpha, \alpha'\} \in P^{at}_R$ such that a, a' are conclusions of R and such that α, α' are the conclusions of two atomic why not nodes, the first in T_a and the second in $T_{a'}$. We say that (α, α') is a (C)-*pair* for (a, a') if there exists a switching φ of R such that $(\varphi(n_\alpha), \varphi(n_{\alpha'}))$ is a (φ)-pair for (a, a').

Remark 4.2. There always exists a (C)-pair for (a, a').

The following result expresses the fact that the incoherence of a (C)-pair guarantees the incoherence of the corresponding conclusions of the proof-net.

Lemma 4.1. *If e is a pre-experiment of R, if (α, α') is a (C)-pair for (a, a') and $e(\alpha, \alpha') = \smile$, then $e(a, a') = \smile$.*

The atomic pre-experiment always assigns incoherence on (C)-pairs that are unique over the corresponding conclusions of the proof-net.

Lemma 4.2. *If (α, α') is the unique (C)-pair for (a, a'), then $e_{at}(\alpha, \alpha') = \smile$ and $e_{at}(a, a') = \smile$.*

5 Injectivity for connected $(?\parr)LL_{pol}$ proof-nets

We now consider the fragment $(?\parr)LL_{pol}$ of linear logic which is defined by the following grammar:

$$N, M ::= X \mid ?X \mid ?P \parr N \mid N \parr ?P \qquad\qquad P, Q ::= X^{\perp} \mid !X^{\perp} \mid !N \otimes P \mid P \otimes !N$$

In this section, we see that coherent semantics is injective for connected $(?\parr)LL_{pol}$ proof-nets. By definition of injective experiment we can assume, without loss of generality, that X and $X^{\perp}$ are the only atomic types. Now we apply the linearization procedure given in [9] and, as a result, we obtain the $L((?\parr)LL_{pol})$ proof-net $L(R)$, in which the of course and auxiliary door nodes of R disappear, as well as the borders of the boxes. The fragment $L((?\parr)LL_{pol}))$ is defined by the following grammar:

$$N, M ::= X \mid ?X \mid ?P \parr N \mid N \parr ?P \qquad\qquad P, Q ::= X^{\perp} \mid N \otimes P \mid P \otimes N$$

We now apply the $\parr$-mutilation procedure, defined in [9], in a very particular way. For each $\parr$ node of $L(R)$, we replace its conclusion with its premise of type N, whereas its premise of type $?P$ becomes the premise of a fresh conclusion node. Then, whenever $a_1, \ldots, a_k$ are the premises of the why not nodes having as conclusions all and only the premises of the fresh conclusions nodes of type $?P$ for some P, we replace these why not nodes with a unique terminal why not node that has $a_1, \ldots, a_k$ as premises. The resulting proof-net is denoted by $L(R)^{\parr}$ and is a $L((?\parr)LL_{pol})^{\parr}$ proof-net, where the fragment $L((?\parr)LL_{pol})^{\parr}$ is expressed by the grammar:

$$N, M ::= X \mid ?X \qquad A, B := ?P \qquad P, Q ::= X^{\perp} \mid N \otimes P \mid P \otimes N$$

Observe that every contraction n of $L(R)^{\parr}$ is either atomic or terminal. We can assume that every atomic arc of $L(R)^{\parr}$ is a premise of a why not node, like in Section 4 and we can remove every terminal contraction of $L(R)^{\parr}$. The resulting proof-net, denoted by $L(R)^{\parr}_*$, is a $L((?\parr)LL_{pol})^{\parr}_*$ proof-net, where $L((?\parr)LL_{pol})^{\parr}_*$ is defined by:

$$N, M ::= ?X \qquad P, Q ::= ?X^{\perp} \mid ?X \otimes P \mid P \otimes ?X$$

Remark 5.1. Every $L((?\parr)LL_{pol})^{\parr}_*$ formula different from $?X$ has precisely one occurrence of subformula $?X^{\perp}$, that is, up to associativity, it has the shape:

$$?X \otimes \cdots \otimes ?X \otimes ?X^{\perp} \otimes ?X \otimes \cdots \otimes ?X$$

Consequently, for every $\{a, a'\} \in P_R$ such that a, a' are conclusions of $L(R)^{\parr}_*$, either there is a unique (C)-pair for (a, a'), or there exists a (C)-pair (α, α') for (a, a') such that α and α' are arcs of type $?X$.

Each step of the procedure we just described preserves the connectedness of the switching graphs, so $L(R)^{\parr}_*$ is a connected proof-net and we can then use the (C)-pairs technique provided in Section 4. The following result is now a consequence of Lemma 2.1 and other sufficient conditions given in [9].

Lemma 5.1. *If, for every connected $(?\parr)LL_{pol}$ proof-net R, there is a complete pre-experiment of $L(R)^{\parr}_*$ such that every two conclusions of the same type are incoherent, then coherent semantics is injective for connected $(?\parr)LL_{pol}$ proof-nets.*

We finally prove the sufficient condition expressed by the previous result.

Lemma 5.2. *If R is a connected $(?\parr)LL_{pol}$ proof-net, then there exists a complete pre-experiment e of $L(R)^{\parr}_*$ such that any two conclusions of the same type are incoherent.*

Proof. Let $R' := L(R)^{\parr}_*$. By Remark 3.2, we know that e_{at} is a complete pre-experiment of R'. Now consider a pair $\{a, a'\} \in P_{R'}$ such that a, a' are conclusions of R' and such that $e_{at}(a, a')$ is undefined. By Lemma 4.2 and Remark 5.1 combined, there exists a (C)-pair (α, α') for (a, a') such that α and α' are arcs of type $?X$. We then define e as any (complete) pre-experiment of R' extending e_{at} in such a way that, whenever $\{a, a'\} \in P_{R'}$ is a pair of conclusions of R' such that $e_{at}(a, a')$ is undefined, we have $e(\alpha, \alpha') = \smile$, where (α, α') is a (C)-pair for (a, a') such that α and α' are arcs of type $?X$. Notice that e is well defined because we are adding incoherence only on pairs of arcs of type X. We can therefore conclude that e is a complete pre-experiment of R' and that $e(a, a') = \smile$ for every $\{a, a'\} \in P_{R'}$ with a, a' conclusions of R' by Lemma 4.1. □

Corollary 5.1. *Coherent semantics is injective for connected $(?\parr)LL_{pol}$ proof-nets.*

Lastly, given that connected $(?\parr)LL_{pol}$ proof-nets embed the simply typed λI-calculus, which is the simply typed λ-calculus without weakenings (see [2]), we also have a proof of the following result.

Corollary 5.2. *Coherent semantics is injective for the simply typed λI-calculus.*

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