Normalization in Multiplicative Exponential Linear Logic via Taylor expansion[*]

Stefano Catozi[†]

1. Introduction

The aim of denotational semantic is to give a mathematical interpretation to syntactical objects, like proofs or programs, in such a way that when an object t evaluates to s, then their interpretations will coincide. One possible denotational model to interpret the λ-calculus is the category **Rel** of sets and relations. A way to present this model is via a non-idempotent intersection type system. Being the set of such types for a λ-term t invariant under evaluation, this set defines a denotation $[\![t]\!]$ that coincides with the interpretation in the relational model. Despite its simplicity, we can extract a lot of computational information from this interpretation, in particular we can measure the execution time of any program by purely semantic means. In fact, as shown by de Carvalho in [Car07], the interpretation of a λ-term in the relational model allows to compute the exact number of steps needed by the Krivine machine [Kri07] to compute its (head) normal form. These results have been revisited recently in [Acc24], in which the number of β-reduction steps are computed, instead of the steps of the Krivine machine. A wave of works adapted these ideas in various settings, for example in [BL13], the authors have been able to measure the length of longest β-reduction sequence in strongly normalizing terms.

One of the main notions introduced by linear logic [Gir87] is the one of proof-nets: a special kind of graphs that allows to represent proofs in a purely geometrical way. This new graphical syntax is also an interesting computational object since the procedure of cut-elimination can be defined as transformations on these graphs. In this work we focus on the Multiplicative Exponential fragment of linear logic (MELL), and in particular on the *untyped* version of its proof-nets, a framework that is expressive enough to encode the full λ-calculus. While it is well known that typed proof-nets are strongly normalizable, this property does not hold in the untyped case. The category **Rel** is also a denotational model for MELL, the

[*]This work can be found here

[†]Université Sorbonne Paris Nord, catozi@lipn.univ-paris13.fr

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interpretation of a proof-net π in this model can be defined using the notion of experiment: an experiment is simply an annotation of the edges of π. The interpretation of a proof-net is then the set of the resulting annotations of its conclusions for all possible experiments. If we look at the well known call-by-name translation of λ-terms into proof-nets of MELL, we can see that the annotations given by experiments are a proper generalization of the non-idempotent intersection type assignment on t.

The results of de Carvalho can be reformulated for proof-nets. In [DPF11] and [CF16], the authors have been able to use the relational interpretation of the untyped version of MELL proof-nets to obtain a characterization of those who are head, weakly or strongly normalizable. Furthermore, by means of a careful investigation of the structure of experiments, they have been able to compute the exact number of cut-elimination steps leading an untyped MELL proof-net to its (head) normal form.

Differential linear logic was introduced by Ehrhard and Regnier in [ER05]. In this framework the notion of derivation represents, in a certain way, the converse of the usual operation of dereliction, since it turns a non linear proof into a linear one. A notion of Taylor expansion can then be defined: at the syntactic level, the Taylor expansion decomposes a proof-net in an infinite formal sum of *resource-nets*, each of which contains resources that can be used only a fixed number of times. Intuitively, given a proof-net π, an element of the Taylor expansion T_π is a resource-net obtained from π by replacing each box b in π with $k_b \in \mathbb{N}$ copies of its content, recursively. The relational interpretation and the Taylor expansion are strongly related. In particular, for a MELL proof-net in normal form, an element of its Taylor expansion can be seen as a canonical representative of an equivalence class of some points of its interpretation. This relation is studied in depth, for example, in [GDP14] and [Gue13].

2. Content

Our goal is to study the connection between the relational interpretation and the Taylor expansion of untyped MELL proof-nets by translating the results in [DPF11, CF16] into the formalism of Taylor expansion. In the following we will call the untyped MELL proof-nets just *nets*. The following definition about nets can be found in [DPF11, CF16]. Given a net π and one of its nodes c, we define, as usual, the notion of depth as the number of boxes in which c is contained, we then call *head cut* a cut node that has depth 0. We define the cut-elimination steps for nets in the usual way and, given two nets π and π', we write $\pi \rightsquigarrow_h \pi'$ if π is reduced to π' by executing a head cut. We will not detail the cut-elimination for resource-nets here, but we follow the definitions in [ER05]. We just remark that we will write $t \rightsquigarrow T$ for a cut-elimination step on the resource-net t, where T is a set of resource-nets. Furthermore, defining the size $s(t)$ as the number of edges in t, we have that, for each $t' \in T$, $s(t') = s(t) - 2$. We start by stating some results of simulation for cut-elimination between nets and their Taylor expansion.

Lemma 2.1. *Let* $\pi \rightsquigarrow_h \pi'$

1. *For each* $t \in T_\pi$ *it exists* $T' \subset T_{\pi'}$ *such that* $t \rightsquigarrow T'$
2. *For each* $t' \in T_{\pi'}$ *there exist* $t \in T_\pi$ *and* $T' \subset T_{\pi'}$ *such that* $t \rightsquigarrow T'$ *and* $t' \in T'$

These results imply that, given a net π, the normal form of its Taylor expansion $NF(T_\pi)$ is invariant under cut-elimination. This is the exact counterpart of what happens in the relational interpretation, where this property holds for the interpretation of a net. Furthermore, these properties can be seen also as a counterpart of the subject reduction and expansion in an non-idempotent intersection type system.

We now want to characterize the head (**HN**), weak (**WN**) and strong (**SN**) normalization of nets under cut-elimination by defining three different subsets of the normal form of their Taylor expansion. Let $\mathcal{RN}^+$ be the set of resource-nets such that all co-contractions have at least one premise. Moreover, let $\neg e$ be the *non erasing* cut-elimination strategy for resource-nets that do not reduces cuts for which one of the two active node is a weakening. The following results hold.

Theorem 2.2. *Let* π *be a net*

1. $\pi \in \mathbf{HN} \iff NF(T_\pi) \neq \emptyset$
2. $\pi \in \mathbf{WN} \iff NF(T_\pi) \cap \mathcal{RN}^+ \neq \emptyset$
3. $\pi \in \mathbf{SN} \iff NF^{\neg e}(T_\pi \cap \mathcal{RN}^+) \neq \emptyset$

These results correspond to what has been proved in [Oli18] for the case of λ-calculus. In order to prove these results we use in a crucial way the linearity of the elements of the Taylor expansion, in particular the fact that each step of cut-elimination strictly decreases their size. This is closely related to combinatorial proofs of normalization for the λ-calculus, as used, for example, in [BKV17, BL13]. These proofs exploit the fact that, in a non-idempotent intersection type system, when a term t reduces to t' then there exists a typing derivation for t' whose size is smaller than the one of t.

We say that a net is in $\neg e$-normal form if the only cut nodes present in the net are erasing cuts. We can prove that, if a net π is in $\neg e$ normal form, then $NF^{\neg e}(T_\pi \cap \mathcal{RN}^+) \neq \emptyset$. This allow us to prove the following as an easy corollary of Theorem 2.2.

Theorem 2.3 (Conservation Theorem). $\mathbf{SN} = \mathbf{WN}^{\neg e}$

The untyped version of proof-nets that we use can straightforwardly be enriched by a typing of the edges, in which a MELL formula is associated to each edge. This results in the standard notion of a typed proof-net of MELL, for which we have the following theorem, proved in [CF16].

Proposition 2.4. *If π is a typed MELL proof-net, then $\pi \in \mathbf{WN}^{\neg e}$*

The fact that typed proof-nets are strongly normalizable is well known, but we can still give a new proof of this result as a straightforward corollary of the previous proposition and of our presentation of the Conservation Theorem (2.3).

Furthermore, we can compute the exact length of the reduction sequences transforming a net into its (head) normal form by carefully choosing a specific element of its Taylor expansion. Again, this has a strong relation with what happens in the relational interpretation, where similar results can be obtained by selecting some particular points of the interpretation of the net. Let $E(T_\pi) = \{t \in T_\pi \mid NF(t) \neq \emptyset\}$, we can prove the following theorem, exploiting the strict decrease in size of the elements of the Taylor expansion under cut-elimination.

Theorem 2.5. *Let π, π' two nets, with π' in head normal form and $\pi \rightsquigarrow_h^n \pi'$ a cut-elimination sequence of length n. Let $t \in E(T_\pi)$ such that $s(t) = min\,\{s(p) \mid p \in E(T_\pi)\}$*

$$n = \frac{s(t) - s(NF(t))}{2}$$

Similar results are obtained for the number of steps required to take a weakly normalizable net to its normal form and for the number of steps of the longest cut-elimination sequence needed to bring a strongly normalizable net to its normal form.

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