Stability Property for the Call-by-Value λ-calculus through Taylor Expansion

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Abstract

We prove the *Stability Property* for the *call-by-value* λ-calculus (CbV in the following) (Theorem 3.1). This result states necessary conditions under which the contexts of the CbV λ-calculus commute with intersections of approximants. This is an important non-trivial result, which implies the sequentiality of the calculus (Corollary 3.2). We prove it via the tool of resource approximation [ER08], whose power has been shown, e.g. in [BM20, Bar22, CA23]. This technique is usually conceived for the ordinary λ-calculus[1], but it can be easily defined for the CbV setting [KMP20].Our proof is the adaptation of the ordinary one given in [BM20], with some minimal technical modification due to the fact that in the CbV setting one linearises terms in a slightly different way than usual (cfr. $!(A \multimap B)$ vs $!A \multimap B$). The content of this contribution is taken from the PhD thesis [Bar21].

1 CbV and its resource approximation in a nutshell

Definition 1.1 (CbV λ-calculus). *The set Λ_{cbv} of the CbV λ-terms is the same as for the ordinary λ-calculus. The set* Val *of* values *contains exactly variables and abstractions. k-Contexts (k different holes occurring any number of times) are defined as for ordinary λ-calculus. The reduction $\rightarrow_{\mathrm{v}} \subseteq \Lambda_{\mathrm{cbv}} \times \Lambda_{\mathrm{cbv}}$ of CbV λ-calculus is defined in [KMP20, Definition 1.5] and it is confluent ([KMP20, Proposition 1.18]).*

Definition 1.2 (Resource CbV λ-calculus). *[KMP20, Def. 3.1] The set $\Lambda^{\mathrm{r}}_{\mathrm{cbv}}$ of the resource-CbV λ-terms is the set $\Lambda^{\mathrm{r}}_{\mathrm{cbv}} := \mathrm{Val}^{\mathrm{r}} \cup \mathrm{Simp}^{\mathrm{r}}$ of* CbV-terms, *where the sets $\mathrm{Val}^{\mathrm{r}}$ of* resource values *and $\mathrm{Simp}^{\mathrm{r}}$ of* resource simple terms *are defined by mutual induction (plus α-equivalence) by:*

$$\mathrm{Val}^{\mathrm{r}}: \quad v ::= \quad x \mid \lambda x.s \qquad\qquad \mathrm{Simp}^{\mathrm{r}}: \quad s ::= \quad s_1 s_2 \mid [v_1, \ldots, v_n]$$

The reduction $\rightarrow_{\mathrm{r}} \subseteq \Lambda^{\mathrm{r}}_{\mathrm{cbv}} \times \mathscr{P}(\Lambda^{\mathrm{r}}_{\mathrm{cbv}})$ of the resource CbV calculus is defined in [KMP20, Definition 3.3, 3.4].

Proposition 1.3. *[KMP20, Proposition 3.6] The reduction $\rightarrow_{\mathrm{r}}$ is confluent and strongly normalising.*

Definition 1.4 (Qualitative CbV-Taylor expansion). *[KMP20, Def. 3.9] The* (qualitative) CbV-Taylor expansion *is the following map $\mathcal{T} : \Lambda_{\mathrm{cbv}} \rightarrow \mathcal{P}(\mathrm{Simp}^{\mathrm{r}})$:*

$$\mathcal{T}(x) := \{\, [x, \overset{(n)}{\ldots}, x] \mid n \in \mathbb{N}\} \qquad\qquad \mathcal{T}(\lambda x.M) := \{\, [\lambda x.s_1, \ldots, \lambda x.s_n] \mid n \in \mathbb{N} \text{ and } s_i \in \mathcal{T}(M)\}$$

$$\mathcal{T}(M_1 M_2) := \{s_1 s_2 \mid s_i \in \mathcal{T}(M_i)\}.$$

One defines as usual the set $\mathrm{NF}\mathcal{T}(M) := \bigcup_{s \in \mathcal{T}(M)} \mathrm{nf}(M) \subseteq \mathrm{nf}(\mathrm{Simp}^{\mathrm{r}})$. The inclusion $\mathrm{NF}\mathcal{T}(M) \subseteq \mathrm{NF}\mathcal{T}(N)$ defines as usual a partial preorder $M \leq N$, as well as its symmetric closure equivalence $=_\tau$.

Remark 1.5. *One has $\mathrm{nf}(MN) = \mathrm{nf}(\mathrm{nf}(M)\mathrm{nf}(N))$ and $\mathrm{nf}(\lambda x.M) = \lambda x.\mathrm{nf}(M)$, whenever the written normal forms exist. Moreover, $\mathrm{NF}\mathcal{T}(\lambda x.M) := \{\, [\lambda x.s_1, \ldots, \lambda x.s_n] \mid n \in \mathbb{N} \text{ and } s_i \in \mathrm{NF}\mathcal{T}(M)\}$ and $\mathrm{NF}\mathcal{T}(MN) := \bigcup_{s_i \in \mathrm{NF}\mathcal{T}(M)} \mathrm{nf}(s_1 s_2)$.*

The following result appears in [KMP20, Lemma 4.6]. [Bar21] gives a more concise inductive proof.

Theorem 1.6 (Monotonicity Property). *Any n-context C is monotone w.r.t. $\leq$.*

[1] Sometimes, in contrapposition to the CbV λ-calculus, this is addressed as “Call by Name”. Strictly speaking however, the ordinary λ-calculus does *not* follow a supposed “CbN evaluation” since there is no restriction on redexes’ firing.

Remark that Theorem 1.6 implies that contexts are well-defined functions on the quotient $\Lambda_{\mathrm{cbv}}/_{=_{\mathcal{T}}}$.

We can simulate $\to_{\mathrm{v}}$ via $\twoheadrightarrow_{\mathrm{r}}$. As for the ordinary λ-calculus, this is one of the fundamental features of a notion of approximation.

Proposition 1.7 (Simulation Property)**.** *[KMP20, Lemma 4.4] If $M \to_{\mathrm{v}} N$ then:*

1. *for all $s \in \mathcal{T}(M)$ there exists $\mathbb{T} \subseteq \mathcal{T}(N)$ s.t. $s \twoheadrightarrow_{\mathrm{r}} \mathbb{T}$*
2. *for all $s' \in \mathcal{T}(N)$ s.t.*[2] *$s' \not\to_0 0$, there exists $s \in \mathcal{T}(M)$ s.t. $s \twoheadrightarrow_{\mathrm{r}} s' + \mathbb{T}$ for some sum $\mathbb{T}$.*

A CbV λ-theory is a congruence containing the reflexive symmetric and transitive closure $=_{\mathrm{v}}$ of $\to_{\mathrm{v}}$.

Corollary 1.8 (Taylor normal form λ-theory)**.** *The equivalence $=_{\tau}$ is a CbV λ-theory.*

The proofs of the following results are either trivial or appear in [KMP20, Lemma 4.9].

Remark 1.9. *If $V \in \mathrm{Val}$ then $\mathcal{T}(V) \subseteq !\,\mathrm{Val}^{\mathrm{r}}$.*

Remark 1.10. *Let $t \in \Lambda^{\mathrm{r}}_{\mathrm{cbv}}$ normal and belonging to $\mathcal{T}(M)$. If $M \to_{\mathrm{v}} N$ then $t \in \mathcal{T}(N)$.*

Proposition 1.11. *If $t \in \mathrm{NF}\mathcal{T}(M)$, there exists $N \in \Lambda_{\mathrm{cbv}}$ s.t. $M \twoheadrightarrow_{\mathrm{v}} N$ and $t \in \mathcal{T}(N)$.*

Proposition 1.12 (Partition Property)**.** *For all $t, s \in \mathcal{T}(M)$ s.t. $t \neq s$, we have $\mathrm{nf}(t) \cap \mathrm{nf}(s) = \emptyset$.*

As for the ordinary λ-calculus, this last property is the key non-trivial ingredient of the proof of the Stability Property. It says that $\mathrm{NF}\mathcal{T}(M)$ is partitioned by the family $\{\mathrm{nf}(t) \mid t \in \mathcal{T}(M) \text{ and } \mathrm{nf}(t) \neq \emptyset\}$.

2 Rigid resource terms

In this section, as for the usual λ-calculus, we consider "rigid" terms/contexts, in which we fix an enumeration of the resources appearing in the bags (hence the permutations in the Definitions 2.2 and 2.3). This allows us to obtain Lemmas 2.5 and 2.6.

Definition 2.1 (CbV resource-contexts)**.** *The set* $\mathrm{Cxt}^{\mathrm{r}}_k$ *of* CbV resource-k-contexts *is defined as:* $\mathrm{Cxt}^{\mathrm{r}}_k := \mathrm{Val}^{\mathrm{r}}_k \cup \mathrm{Simp}^{\mathrm{r}}_k$*, where* $\mathrm{Val}^{\mathrm{r}}_k$ *and* $\mathrm{Simp}^{\mathrm{r}}_k$ *are defined by mutual induction (*without *α-equivalence):*

$$\mathrm{Val}^{\mathrm{r}}_k: \quad c^v ::= \quad \Box_1 \mid \cdots \mid \Box_k \mid x \mid \lambda x.c^s \qquad\qquad \mathrm{Simp}^{\mathrm{r}}_k: \quad c^s ::= \quad c^s_1 c^s_2 \mid [c^v_1, \ldots, c^v_n]$$

We extend the Taylor expansion on each Cxt_k by: $\mathcal{T}(\Box_i) := \{\, [\Box_i, \overset{(n)}{\ldots}, \Box_i] \mid n \in \mathbb{N}\} \subseteq \mathrm{Simp}^{\mathrm{r}}_k$.

Definition 2.2 (Rigid CbV λ-terms)**.**

1. *A* rigid k-context *is built as a resource k-context but taking lists*[3] *of rigid k-contexts instead of bags of resource k-contexts. In particular, a rigid term is a rigid context with no occurrences of the holes, taken modulo α-equivalence. As for CbV-terms, rigid contexts are divided into rigid value-contexts and rigid simple-contexts (and this distincion coincides with that of terms when a context has no holes).*
2. *Let c be a resource k-context. We define a set* $\mathrm{Rigid}(c)$ *of* rigid k-contexts *associated with c, whose elements are called the* rigids of c*, by mutual induction on* $\mathrm{Val}^{\mathrm{r}}_k$ *and* $\mathrm{Simp}^{\mathrm{r}}_k$ *as follows:*

$$\mathrm{Rigid}(\Box_i) = \{\Box_i\} \qquad\qquad \mathrm{Rigid}(x) = \{x\}$$

$$\mathrm{Rigid}(\lambda x.c_0) = \{\lambda x.c^\bullet_0 \mid c^\bullet_0 \in \mathrm{Rigid}(c_0)\} \qquad\qquad \mathrm{Rigid}(c_0 c_1) = \{c^\bullet_0 c^\bullet_1 \mid c^\bullet_i \in \mathrm{Rigid}(c_i)\}$$

$$\mathrm{Rigid}([c_1, \ldots, c_k]) = \{\langle c^\bullet_{\sigma(1)}, \ldots, c^\bullet_{\sigma(k)}\rangle \mid \sigma \text{ permutation and } c^\bullet_i \in \mathrm{Rigid}(c_i)\}.$$

The above definition makes sense since one immediately sees that if c is a resource k-value/simple-context then any of its rigids $c^\bullet$ is a rigid k-value/simple-context.

Definition 2.3. *Let $c^\bullet$ be a rigid of a CbV resource k-context c and, for $i = 1, \ldots, k$, let $\vec{v}^i := \langle v^i_1, \ldots, v^i_{\deg_{\Box_i}(c)}\rangle$ be a list*[4] *of resource* values *(that is, elements of* $\mathrm{Val}^{\mathrm{r}}$*). We define, by mutual induction on $\mathrm{Val}^{\mathrm{r}}_k$ and $\mathrm{Simp}^{\mathrm{r}}_k$, a resource term $c^\bullet\langle\vec{v}^1, \ldots, \vec{v}^k\rangle \in \Lambda^{\mathrm{r}}_{\mathrm{cbv}}$ s.t. if $c \in \mathrm{Val}^{\mathrm{r}}_k$ (resp. $\in \mathrm{Simp}^{\mathrm{r}}_k$) then $c^\bullet\langle\vec{v}^1, \ldots, \vec{v}^k\rangle \in \mathrm{Val}^{\mathrm{r}}$ (resp. $\in \mathrm{Simp}^{\mathrm{r}}$). The definition goes as follows:*

[2] The condition $s' \not\to_0 0$ refers to a particular reduction $\to_0$, which we did not specify, see [KMP20].

[3] We use $\langle\cdot, \ldots, \cdot\rangle$ to denote lists.

[4] If $\deg_{\Box_i}(c) = 0$ we mean the empty list.

1. *If* $c = \Box_i$ *then* $c^\bullet = \Box_i$*; we set* $c^\bullet\langle\langle\rangle, \dots, \langle\rangle, \langle v_1^i\rangle, \langle\rangle, \dots, \langle\rangle\rangle := v_1^i$
2. *If* $c = x$ *then* $c^\bullet = x$*; we set* $c^\bullet\langle\langle\rangle, \dots, \langle\rangle\rangle := x$
3. *If* $c = \lambda x.c_0$ *then* $c^\bullet = \lambda x.c_0^\bullet$ *where* $c_0^\bullet$ *is a rigid of* c_0*; we set* $c^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle = \lambda x.c_0^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle$
4. *If* $c = c_1c_2$*, then* $c^\bullet = c_1^\bullet c_2^\bullet$ *where* $c_i^\bullet$ *is a rigid of* c_i*, and each list* $\vec{v}^{\,i}$ *is a concatenation* $\vec{v}^{\,i} =: \vec{w}^{\,i1}\vec{w}^{\,i2}$ *where the lists* $\vec{w}^{\,ij}$ *have exactly* $\deg_{\Box_i}(c_j)$ *elements; we set:*
$$c^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle := c_1^\bullet\langle\vec{w}^{11}, \dots, \vec{w}^{k1}\rangle c_2^\bullet\langle\vec{w}^{12}, \dots, \vec{w}^{k2}\rangle.$$
5. *If* $c = [c_1, \dots, c_n]$*, then* $c^\bullet = \langle c_{\sigma(1)}^\bullet, \dots, c_{\sigma(n)}^\bullet\rangle$ *where* σ *is a permutation and* $c_i^\bullet$ *is a rigid of* c_i*, and each list* $\vec{v}^{\,i}$ *is a concatenation* $\vec{v}^{\,i} =: \vec{w}^{\,i1} \cdots \vec{w}^{\,in}$ *where the lists* $\vec{w}^{\,ij}$ *have exactly* $\deg_{\Box_i}(c_{\sigma(j)})$ *elements; we set:*
$$c^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle := [c_{\sigma(1)}^\bullet\langle\vec{w}^{\,11}, \dots, \vec{w}^{\,k1}\rangle, \dots, c_{\sigma(n)}^\bullet\langle\vec{w}^{\,1n}, \dots, \vec{w}^{\,kn}\rangle].$$

Remark 2.4. *If* $v \twoheadrightarrow_{\mathrm{r}} \mathbb{V}$ *then* $c^\bullet\langle\cdots, \langle\cdots, v, \cdots\rangle, \cdots\rangle \in \Lambda_{\mathrm{cbv}}^{\mathrm{r}} \twoheadrightarrow_{\mathrm{r}} \{c^\bullet\langle\cdots, \langle\cdots, w, \cdots\rangle, \cdots\rangle \mid w \in \mathbb{V}\}$.

The following lemmas will be used in the proof of Theorem 3.1. If $\vec{v}$ is a list we denote with $[\vec{v}]$ the multiset associated with $\vec{v}$ (same elements but disordered).

Lemma 2.5. *Let* C *be a* k*-context and* $c_1, c_2 \in \mathcal{T}(C)$ *(hence* c_1, c_2 *are resource* k*-contexts). Let* $c_1^\bullet$*,* $c_2^\bullet$ *be rigids respectivly of* c_1*,* c_2*. For* $i = 1, \dots, k$*, let* $\vec{v}^{\,i} = \langle v_1^i, \dots, v_{\deg_{\Box_i}(c_1)}^i\rangle$ *and* $\vec{u}^{\,i} = \langle u_1^i, \dots, u_{\deg_{\Box_i}(c_2)}^i\rangle$ *be lists of resource* values*. If* $c_1^\bullet\langle\vec{v}^{\,1}, \dots, \vec{v}^{\,k}\rangle = c_2^\bullet\langle\vec{u}^{\,1}, \dots, \vec{u}^{\,k}\rangle$ *then* $c_1 = c_2$ *and* $[\vec{v}^{\,i}] = [\vec{u}^{\,i}]$ *for all* i*.*

Proof. Induction on C.

Case $C = \Box_i$. Then $c_1 = [\Box_i, \overset{(n)}{\dots}, \Box_i]$, $c_2 = [\Box_i, \overset{(m)}{\dots}, \Box_i]$, $\vec{v}^{\,i} = \langle v^{i1}, \dots, v^{in}\rangle$, $\vec{u}^{\,i} = \langle u^{i1}, \dots, u^{im}\rangle$ and $\vec{v}^{\,j} = \langle\rangle = \vec{u}^{\,j}$ for $j \neq i$. So $[v^{i1}, \dots, v^{in}] = c_1^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle = c_2^\bullet\langle\vec{u}^1, \dots, \vec{u}^k\rangle = [u^{i1}, \dots, u^{im}]$, thus $n = m$, i.e. $c_1 = c_2$.

Case $C = x$. Then $c_1 = [x, \overset{(n)}{\dots}, x]$, $c_2 = [x, \overset{(m)}{\dots}, x]$ and $\vec{v}^{\,i} = \langle\rangle = \vec{u}^{\,i}$. So $[x, \overset{(n)}{\dots}, x] = c_1^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle = c_2^\bullet\langle\vec{u}^1, \dots, \vec{u}^k\rangle = [x, \overset{(m)}{\dots}, x]$, thus $n = m$, i.e. $c_1 = c_2$.

Case $C = \lambda x.C_0$. Then, for $i = 1, 2$, one has $c_i = [\lambda x.c_{i1}, \dots, \lambda x.c_{in_i}]$ with $c_{ij} \in \mathcal{T}(C_0)$ for all i, j. So $c_i^\bullet = \langle\lambda x.c_{i\sigma_i(1)}^\bullet, \dots, \lambda x.c_{i\sigma_i(n_i)}^\bullet\rangle$ where σ_i is a permutation on n_i elements. By Definition 2.3 we have that $c_1^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle$ and $c_2^\bullet\langle\vec{u}^1, \dots, \vec{u}^k\rangle$ are equal to:
$$[\,\lambda x.c_{i\widetilde{\sigma}_i(1)}^\bullet\langle\vec{w}^{\,i11}, \dots, \vec{w}^{\,ik1}\rangle, \dots, \lambda x.c_{i\widetilde{\sigma}_i(n_i)}^\bullet\langle\vec{w}^{\,i1n_i}, \dots, \vec{w}^{\,ikn_i}\rangle]$$
respectively if $i = 1$ or $i = 2$, where $\widetilde{\sigma}_i$ is some permutation on n_i elements and where the concatenation $\vec{w}^{ij1} \cdots \vec{w}^{ijn_i}$ gives $\vec{v}^j$ if $i = 1$ and gives $\vec{u}^j$ if $i = 2$. From $c_1^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle = c_2^\bullet\langle\vec{u}^1, \dots, \vec{u}^k\rangle$ we get that $n_1 = n_2 =: n$ and that there exists a permutation ρ on n elements which identifies each term of the bag $c_1^\bullet\langle\vec{v}^1, \dots, \vec{v}^k\rangle$ with the respective one of the bag $c_2^\bullet\langle\vec{u}^1, \dots, \vec{u}^k\rangle$, that is, for all $j = 1, \dots, n$, one has:
$$c_{1j}^\bullet\langle\vec{w}^{\,1\,1\,\widetilde{\sigma_1}^{-1}(j)}, \dots, \vec{w}^{\,1\,k\,\widetilde{\sigma_1}^{-1}(j)}\rangle = c_{2\rho(j)}^\bullet\langle\vec{w}^{\,2\,1\,\widetilde{\sigma_2}^{-1}(\rho(j))}, \dots, \vec{w}^{\,2\,k\,\widetilde{\sigma_2}^{-1}(\rho(j))}\rangle.$$
The inductive hypothesis gives $c_{1j} = c_{2\rho(j)}$ for all $j = 1, \dots, n$ and, putting $h := \widetilde{\sigma_1}^{-1}(j)$, we have $\vec{w}^{\,1\,i\,h} = \vec{w}^{\,2\,i\,\widetilde{\sigma_2}^{-1}(\rho(\widetilde{\sigma_1}(h)))}$ for all $i = 1, \dots, k$. Now, the former equalities give $c_1 = c_2$, while the latter give $[\vec{v}^i] = [\vec{w}^{1i1} \cdots \vec{w}^{1in}] = [\vec{w}^{2i1} \cdots \vec{w}^{2in}] = [\vec{u}^i]$.

Case $C = C'C''$. Analogous and easier than the above case. □

Lemma 2.6. *Let* C *be a* k*-context and* $V_1, \dots, V_k \in \mathrm{Val}$*. Then:*
$$\mathcal{T}(C\langle V_1, \dots, V_k\rangle) = \{c^\bullet\langle\vec{v}^{\,1}, \dots, \vec{v}^{\,k}\rangle \mid c \in \mathcal{T}(C),\ c^\bullet \text{ rigid of } c,\ [v_1^i, \dots, v_{\deg_{\Box_i}(c)}^i] \in \mathcal{T}(V_i)\}.$$

Proof. Induction on C. Let us show only one of the two base cases, namely the one for $C = \Box_i$.
$$\begin{aligned} & \{c^\bullet\langle\vec{v}^{\,1}, \dots, \vec{v}^{\,k}\rangle \mid c \in \mathcal{T}(C),\ c^\bullet \text{ rigid of } c,\ [v_1^i, \dots, v_{\deg_{\Box_i}(c)}^i] \in \mathcal{T}(V_i)\} \\ = \ & \{\langle\Box_i, \overset{(n)}{\dots}, \Box_i\rangle\langle\langle\rangle, \overset{(i-1)}{\dots}, \langle\rangle, \langle v_1^i, \dots, v_n^i\rangle, \langle\rangle, \overset{(k-i)}{\dots}, \langle\rangle\rangle \mid n \in \mathbb{N},\ [v_1^i, \dots, v_n^i] \in \mathcal{T}(V_i)\} \\ = \ & \{\,[v_1^i, \dots, v_n^i] \mid [v_1^i, \dots, v_n^i] \in \mathcal{T}(V_i)\} = \mathcal{T}(V_i) = \mathcal{T}(C\langle V_1, \dots, V_k\rangle). \end{aligned}$$
□

3 Stability

Given $\emptyset \neq \mathcal{X} \subseteq \Lambda_{\mathrm{cbv}}$, we can consider $\bigcap_{M\in\mathcal{X}} \mathrm{NF}\mathcal{T}(M) \subseteq \Lambda^{\mathrm{r}}_{\mathrm{cbv}}$, which is the inf (w.r.t. $\leq$) of $\mathcal{X}$ in $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$. Note that even if an M such that $M =_\mathcal{T} \inf \mathcal{X}$ exists, it need not be unique in Λ_{cbv}, but it is unique in $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$. If it exists, we say that $\inf \mathcal{X}$ *is definable in* $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$. The proof presented below is an adaptation of the proof for the ordinary λ-calculus [BM20]. Knowing that the proof technique can be adapted also for the $\lambda\mu$-calculus [Bar22], the fact that one can adapt it for CbV as well, can be seen as yet an extra strength of Taylor resource approximation. We conclude this section with Corollary 3.2, which expresses a notion of sequentiality of the calculus.

Theorem 3.1 (Stability Property). *Let $C : \Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}} \times \overset{(n)}{\cdots} \times \Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}} \to \Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$ be an n-context and fix non-empty $\mathcal{X}_1, \dots, \mathcal{X}_n \subseteq \mathrm{Val}/_{=_\mathcal{T}}$, each upper bounded by some value. If $\inf \mathcal{X}_i$ is definable in $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$ by a value for all i, then the* inf *of the image of C on $\mathcal{X}_1 \times \overset{(n)}{\cdots} \times \mathcal{X}_n$ is definable in $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$ and it is:*

$$C\langle \inf_{N_1\in\mathcal{X}_1} N_1, \dots, \inf_{N_n\in\mathcal{X}_n} N_n\rangle = \inf_{\substack{N_1\in\mathcal{X}_1\\ \cdots \\ N_n\in\mathcal{X}_n}} C\langle N_1, \dots, N_n\rangle.$$

Proof. Wlog we can consider that the $\mathcal{X}_i$'s are in Val. Since they are upper bound by a value, for $i = 1, \dots, n$ there exists $L_i \in \mathrm{Val}$ s.t. $\bigcup_{N\in\mathcal{X}_i} \mathrm{NF}\mathcal{T}(N) \subseteq \mathrm{NF}\mathcal{T}(L_i)$. Since the $\inf \mathcal{X}_i$'s are definable in $\Lambda_{\mathrm{cbv}}/_{=_\mathcal{T}}$ by a value, let $V_1, \dots, V_n \in \mathrm{Val}$ s.t. $\mathrm{NF}\mathcal{T}(V_i) = \bigcap_{N\in\mathcal{X}_i} \mathrm{NF}\mathcal{T}(N)$. It suffices now to show that $\mathrm{NF}\mathcal{T}(C\langle V_1, \dots, V_n\rangle) = \bigcap_{N_1\in\mathcal{X}_1} \cdots \bigcap_{N_n\in\mathcal{X}_n} \mathrm{NF}\mathcal{T}(C\langle N_1, \dots, N_n\rangle)$.

($\subseteq$). Immediate by Theorem 1.6.

($\supseteq$). Let $t \in \bigcap_{\vec{N}\in\vec{\mathcal{X}}} \mathrm{NF}\mathcal{T}(C\langle N_1, \dots, N_n\rangle)$ (where $\vec{N} := (N_1, \dots, N_n)$ and $\vec{\mathcal{X}} := (\mathcal{X}_1, \dots, \mathcal{X}_n)$). For every $\vec{N} \in \vec{\mathcal{X}}$, by Lemma 2.6 there exists a CbV n-resource-context $c_{\vec{N}} \in \mathcal{T}(C)$ and, for every $i = 1, \dots, n$, a list $\vec{v}^{\,i}_{\vec{N}} = \langle v^{i1}_{\vec{N}}, \dots, v^{id_i}_{\vec{N}}\rangle$ (where $d_i := \deg_{\square_i}(c_{\vec{N}})$) with $[\vec{v}^{\,i}_{\vec{N}}] \in \mathcal{T}(N_i)$ and such that $t \in \mathrm{nf}(c^\bullet_{\vec{N}}\langle \vec{v}^{\,1}_{\vec{N}}, \dots, \vec{v}^{\,n}_{\vec{N}}\rangle)$, for $c^\bullet_{\vec{N}}$ a rigid of $c_{\vec{N}}$. Confluence allows to factorize the reduction from $c^\bullet_{\vec{N}}\langle \vec{v}^{\,1}_{\vec{N}}, \dots, \vec{v}^{\,n}_{\vec{N}}\rangle$ to t as follows:

$$c^\bullet_{\vec{N}}\langle \mathrm{nf}(v^{11}_{\vec{N}}), \dots, \mathrm{nf}(v^{1d_1}_{\vec{N}}), \dots, \mathrm{nf}(v^{n1}_{\vec{N}}), \dots, \mathrm{nf}(v^{nd_n}_{\vec{N}})\rangle \twoheadrightarrow_{\mathrm{r}} \mathrm{nf}(c^\bullet_{\vec{N}}\langle \vec{v}^{\,1}_{\vec{N}}, \dots, \vec{v}^{\,n}_{\vec{N}}\rangle) \ni t.$$

Wlog $d_i \geq 1$ for all i and $\mathrm{nf}(v^{ij}_{\vec{N}}) \neq \emptyset$ for all i, j. In fact, the holes s.t. $d_i = 0$ can be ignored, and if both $d_i \geq 1$ and $\mathrm{nf}(v^{ij}_{\vec{N}}) \neq \emptyset$ for some j, then by the previous line we would have $t \in \mathrm{nf}(\emptyset)$, a contraddiction.

So for all $i = 1, \dots, n$ and $j = 1, \dots, d_i$, there exists $w^{ij}_{\vec{N}} \in \mathrm{nf}(v^{ij}_{\vec{N}})$ such that:

$$\mathrm{nf}(c^\bullet_{\vec{N}}\langle \vec{w}^{\,1}_{\vec{N}}, \dots, \vec{w}^{\,n}_{\vec{N}}\rangle) \ni t \tag{1}$$

and being $N_i \in \mathcal{X}_i$ which is bounded by L_i, we have $[\vec{w}^{i}_{\vec{N}}] \in \mathrm{nf}([\vec{v}^{i}_{\vec{N}}]) \subseteq \mathrm{NF}\mathcal{T}(N_i) \subseteq \mathrm{NF}\mathcal{T}(L_i)$. From the inclusion $[\vec{w}^{i}_{\vec{N}}] \in \mathrm{NF}\mathcal{T}(L_i)$ we obtain, thanks to Remark 1.9 because L_i is a value, a simple term $[\vec{u}^{\,i}_{\vec{N}}] \in \mathcal{T}(L_i)$ such that:

$$[\vec{w}^{i}_{\vec{N}}] \in \mathrm{nf}([\vec{u}^{i}_{\vec{N}}]) \tag{2}$$

i.e. they have the same number of elements and $\mathrm{nf}(u^{ij}_{\vec{N}}) \ni w^{ij}_{\vec{N}}$ for all $i, j, \vec{N}$. By composing thus a reduction from $c^\bullet_{\vec{N}}\langle \vec{w}^{\,1}_{\vec{N}}, \dots, \vec{w}^{\,n}_{\vec{N}}\rangle$ to t with a reduction from $u^{ij}_{\vec{N}}$ to $w^{ij}_{\vec{N}}$, we find that $t \in \mathrm{nf}(c^\bullet_{\vec{N}}\langle \vec{u}^{\,1}_{\vec{N}}, \dots, \vec{u}^{n}_{\vec{N}}\rangle)$. This holds for all $\vec{N} \in \vec{\mathcal{X}}$, i.e.:

$$t \in \bigcap_{\vec{N}\in\vec{\mathcal{X}}} \mathrm{nf}(c_{\vec{N}}\langle \vec{u}^{\,1}_{\vec{N}}, \dots, \vec{u}^{n}_{\vec{N}}\rangle). \tag{3}$$

Now, Lemma 2.6 gives $c^\bullet_{\vec{N}}\langle \vec{u}^{\,1}_{\vec{N}}, \dots, \vec{u}^{n}_{\vec{N}}\rangle \in \mathcal{T}(C\langle L_1, \dots, L_n\rangle)$. But since the L_i's are independent from $N_1, \dots, N_n$, and thanks to (3), we can apply Proposition 1.12, and obtain that the set $\{c^\bullet_{\vec{N}}\langle \vec{u}^{\,1}_{\vec{N}}, \dots, \vec{u}^{n}_{\vec{N}}\rangle \mid \vec{N} \in \vec{\mathcal{X}}\}$ is actually a singleton. Therefore, Lemma 2.5 tells us that also the terms $c^\bullet_{\vec{N}}$ and the bags $[\vec{u}^{\,i}_{\vec{N}}]$ are independent from $\vec{N} \in \vec{\mathcal{X}}$. The unique element of the previous sigleton has hence shape $c^\bullet\langle \vec{u}^{\,i}, \dots, \vec{u}^{\,n}\rangle$, with $c^\bullet$ a rigid of a $c \in \mathcal{T}(C)$, and $[\vec{u}^{\,i}] \in \mathcal{T}(L_i)$. Recalling now that $[\vec{w}^{\,i}_{\vec{N}}] \in \mathrm{NF}\mathcal{T}(L_i)$, we can apply Proposition 1.11 in order to obtain, for each $i = 1, \dots, n$, an $L'_{[\vec{w}^{\,i}_{\vec{N}}]} \in \Lambda_{\mathrm{cbv}}$ such that $L_i \twoheadrightarrow_{\mathrm{v}} L'_{[\vec{w}^{\,i}_{\vec{N}}]}$ and $[\vec{w}^{\,i}_{\vec{N}}] \in \mathcal{T}(L'_{[\vec{w}^{\,i}_{\vec{N}}]})$. Remark that these $L_{[\vec{w}^{\,i}_{\vec{N}}]}$'s must in fact be values, since they are reducts of the values

L_i. Consider now the set $\{[\vec{w}^{\,i}_{\vec{N}}] \mid \vec{N} \in \vec{\mathcal{X}}\}$, which can be *a priori* infinite. Since for i fixed, the set $\{[\vec{u}^{\,i}_{\vec{N}}] \mid \vec{N} \in \vec{\mathcal{X}}\}$ is a singleton $\{[\vec{u}^{\,i}]\}$, (2) entails that $[\vec{w}^{i}_{\vec{N}}] \in \mathrm{nf}([\vec{u}^{i}])$, and our set $\{[\vec{w}^{\,i}_{\vec{N}}] \mid \vec{N} \in \vec{\mathcal{X}}\}$ must thus in fact be finite. Therefore we can invoke confluence in order to say that the *finitely many* $L'_{[\vec{w}^{\,i}_{\vec{N}}]}$'s share a commond reduct, call it L'_i, which as the notation shows is now independent from $[\vec{w}^{\,i}_{\vec{N}}]$ (but still depends on i). Of course L'_i is also a reduct of L_i, and it is still a value. Also, since each $[\vec{w}^{\,i}_{\vec{N}}]$ belongs to $\mathcal{T}(L'_{[\vec{w}^{\,i}_{\vec{N}}]})$ and is normal, by Remark 1.10 we have $\{[\vec{w}^{\,i}_{\vec{N}}] \mid \vec{N} \in \vec{\mathcal{X}}\} \subseteq \mathcal{T}(L'_i)$. Thus we can apply Lemma 2.6 and find that, for every $\vec{N} \in \vec{\mathcal{X}}$, we have:

$$c^{\bullet}\langle \vec{w}^{\,1}_{\vec{N}}, \ldots, \vec{w}^{n}_{\vec{N}}\rangle \in \mathcal{T}(C\langle L'_1, \ldots, L'_n\rangle). \tag{4}$$

But now thanks to (1) (which holds for all $\vec{N} \in \vec{\mathcal{X}}$) and (4), we can apply again Proposition 1.12 in order to find that the set $\{c^{\bullet}\langle \vec{w}^{\,1}_{\vec{N}}, \ldots, \vec{w}^{n}_{\vec{N}}\rangle \mid \vec{N} \in \vec{\mathcal{X}}\}$ is a singleton. Again by Lemma 2.5, we have that all the bags $[\vec{w}^{\,1}_{\vec{N}}], \ldots, [\vec{w}^{n}_{\vec{N}}]$ for $\vec{N} \in \vec{\mathcal{X}}$, coincide respectively to some bags $[\vec{w}^{\,1}], \ldots, [\vec{w}^{n}]$ which are independent from $\vec{N} \in \vec{\mathcal{X}}$. So the only element of the previous singleton has shape $c^{\bullet}\langle \vec{w}^{\,1}, \ldots, \vec{w}^{n}\rangle$, and by (1) we get:

$$t \in \mathrm{nf}(c^{\bullet}\langle \vec{w}^{\,1}, \ldots, \vec{w}^{n}\rangle). \tag{5}$$

Now, for all i, remembering what we found already, we have $[\vec{w}^{\,i}] = [\vec{w}^{\,i}_{\vec{N}}] \in \mathrm{NF}\mathcal{T}(N)$ for all $N \in \mathcal{X}_i$. That is,

$$[\vec{w}^{\,i}] \in \bigcap_{N \in \mathcal{X}_i} \mathrm{NF}\mathcal{T}(N) = \mathrm{NF}\mathcal{T}(V_i) \tag{6}$$

where we finally used the hypothesis. From (6) and Lemma 2.6 one can now easily conclude that $t \in \mathrm{nf}(c^{\bullet}\langle \vec{w}^{1}, \ldots, \vec{w}^{n}\rangle) \subseteq \mathrm{NF}\mathcal{T}(C\langle V_1, \ldots, V_n\rangle)$. □

As usual, one obtains as corollary the non-existence of the following *parallel-or*. We use the usual encoding of pairs: $(M, N) := \lambda z.zMN$. Remark that a pair is a value.

Corollary 3.2 (No parallel-or). *There is* no $\texttt{Por} \in \Lambda_{\mathrm{cbv}}$ *s.t. for all* $M, N \in \Lambda_{\mathrm{cbv}}$,

$$\begin{cases} \texttt{Por}(M,N) =_{\mathcal{T}} \texttt{True} & \text{if } M \neq_{\mathcal{T}} \Omega \text{ or } N \neq_{\mathcal{T}} \Omega \\ \texttt{Por}(M,N) =_{\mathcal{T}} \Omega & \text{if } M =_{\mathcal{T}} N =_{\mathcal{T}} \Omega. \end{cases}$$

Proof. Otherwise, for $C := \texttt{Por}\,\square$, $\mathcal{X} = \{(\texttt{True}, \Omega), (\Omega, \texttt{True})\}$ (upper bounded by the value $(\texttt{True}, \texttt{True})$), and the value $V = (\Omega, \Omega) =_{\mathcal{T}} \inf\{(\texttt{True}, \Omega), (\Omega, \texttt{True})\}$, Theorem 3.1 would give the contradiction:

$$\texttt{True} =_{\mathcal{T}} \inf\{C\langle(\texttt{True}, \Omega)\rangle, C\langle(\Omega, \texttt{True})\rangle\} =_{\mathcal{T}} C\langle(\Omega, \Omega)\rangle =_{\mathcal{T}} \Omega.$$ □

Remark 3.3. *Here Ω is taken as representative of "operationally meaningless" term in the calculus. However, one should see what happens with the more appropriate CbV notion studied in [AGK24].*

4 Final comments

The CbV λ-calculus that we have used is of the form given in [KMP20]. However, one could argue that the canonical formulation of CbV should be on the lines of the one given in [AGK24] (explicit substitutions and a distant action reduction). The first work would therefore be to reproduce our proof for that syntax (or at least prove that Stability in either setting is equivalent to Stability in the other).

Moreover, we remark that Section 2, where we have to consider lists instead of multisets, is quite annoying. This detour also appears, in the exact same form, for the ordinary setting. If one would directly define resource approximation with lists (usually called the *rigid/polyadic resource approximation* [OA22, MPV18]), this annoying detour would probably disappear (its content would still be there, but in a different shape). The second work would therefore be to reformulate the whole theory of approximation of CbV (and even the ordinary one!) in a rigid/polyadic way.

Finally, in the recent [AGK24], a proof of the Genericity Property for CbV is given, in a sense inspired from the one for the ordinary λ-calculus given in [BM20]. Once a good notion of CbV-Böhm trees (or similar) at hand, one should be able to directly adapt the latter proof to CbV and understand the relations between the two proof techniques. We see the Stability and Genericity Properties of CbV (once agreed on an established form of the calculus) as the first steps of the development of a "mathematical theory of CbV", in the same sense that we have for the ordinary one. A third work would be, for instance, to ask if CbV does enjoy the Perpendicular Lines Property and the Continuity Lemma (see [BM20]).

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