Multi-modalities and non-commutativity/associativity in functorial linear logic: a case study

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1 Introduction

Similar to modal connectives, the exponential ! in intuitionistic linear logic (ILL) is not *canonical* [2], in the sense that if $i \neq j$ then $!^i F \not\equiv !^j F$. Intuitively, this means that we can mark the exponential with *labels* taken from a set I organized in a pre-order $\preceq$ (*i.e.*, reflexive and transitive), obtaining (possibly infinitely-many) exponentials ($!^i$ for $i \in I$). Also as in multi-modal systems, the pre-order determines the provability relation: for a general formula F, $!^b F$ *implies* $!^a F$ iff $a \preceq b$.

The algebraic structure of subexponentials, combined with their intrinsic structural property allow for the proposal of rich linear logic based frameworks. This opened a venue for proposing different multi-modal substructural logical systems, that encountered a number of different applications.

There are, however, two main differences between multi-modalities in normal modal logics and subexponentials in linear logic.

i. substructural behaviour. Subexponentials carry the possibility of having different structural behaviors;

ii. nature of modalities. Normal modal logics start from the weakest version, assuming only axiom K, then extensions are considered, by adding other axioms. Exponentials in linear logic"take for granted" the behaviors expressed by axioms T and 4.

Regarding (i), originally [10], subexponentials could assume only weakening and contraction axioms, while in [4,5], non-commutative systems allowing commutative subexponentials were presented

$$\mathsf{C}: \quad !^i F \multimap !^i F \otimes !^i F \qquad \mathsf{W}: \quad !^i F \multimap 1 \qquad \mathsf{E}: \quad (!^i F) \otimes G \equiv G \otimes (!^i F)$$

Finally, in [1] associativity was added to the scene, where exponentials could assume or not the axioms

$$\mathsf{A1}: \quad !^i F \otimes (G \otimes H) \multimap (!^i F \otimes G) \otimes H \qquad \mathsf{A2}: \quad (G \otimes H) \otimes !^i F \multimap G \otimes (H \otimes !^i F)$$

Concerning (ii), Guerrini et al. [3] unified the modal and LL approaches, with the exponentials assuming only the linear version of K, with the possibility of adding modal extensions to it. This discussion was brought to multi-modal case in [7], where subexponentials consider not only the structural axioms for contraction and weakening, but also the subexponential version of axioms {K, 4, D, T}:

$$\mathsf{K}: \quad !^i(F \multimap G) \multimap !^i F \multimap !^i G \qquad \mathsf{D}: \quad !^i F \multimap ?^i F \qquad \mathsf{T}: \quad !^i F \multimap F \qquad \mathsf{4}\,; \quad !^i F \multimap !^i !^i F$$

In this work, we intend to join these two studies. This means that $?^i$ can behave classically or not, model associative and commutative systems or not, but also with exponential behaviors different from those in LL. Hence, by assigning different modal axioms one obtains, in a modular way, a class of different substructural modal logics.

2 Non-commutative, non-associative linear logic (acLL_Σ)

In the following, we will briefly describe the system acLL_Σ. For more details, please refer to [1].

Connectives. First of all, non-commutativity implies that the left residual $\multimapinv$ should be added to the language of (propositional intuitionistic) linear logic with subexponentials (SELL), which contains the connectives $\otimes, \&, \oplus, 1, 0, \multimap, !^l, ?^l$. The subexponentials are organized as follows.

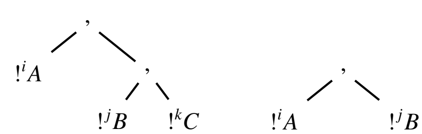
Definition 1 (Signature). *Let $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}\}$. A* subexponential signature *is given by a triple $\Sigma = (I, \preccurlyeq, f)$, where I is a set of indices, $(I, \preccurlyeq)$ is a pre-order, and f is a mapping from I to $2^{\mathcal{A}}$. Finally, every signature Σ is assumed to be upwardly closed w.r.t. $\preceq$, that is, if $i \preceq j$ then $f(i) \subseteq f(j)$ for all $i, j \in I$.*

Contexts and rules. Losing commutativity implies that contexts should be handled by lists instead of multisets, and the lack of associativity forces tree-shaped contexts. Rules then act deeply in these structures. For example, the promotion and exchange rules are

$$\frac{\Gamma^{\Uparrow(i)} \Rightarrow F}{\Gamma \Rightarrow !^i F}\ !^i R \qquad \frac{\Gamma\{(\Delta_2, !^e \Delta_1)\} \Rightarrow G}{\Gamma\{(!^e \Delta_1, \Delta_2)\} \Rightarrow G}\ \mathsf{E1} \qquad \frac{\Gamma\{(!^e \Delta_2, \Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1, !^e \Delta_2)\} \Rightarrow G}\ \mathsf{E2}$$

where $\uparrow(i)$ denotes the *upset* of the index i, *i.e.*, the set $\{j \in I : i \preccurlyeq j\}$.

Example 2. Let $\Gamma = (!^i A, (!^j B, !^k C))$ be represented below left, $i \preceq j$ but $i \npreceq k$, and $\mathsf{W} \in f(k)$. Then $\Gamma^{\Uparrow(i)} = (!^i A, !^j B)$ is depicted below right



Observe that, if $\mathsf{W} \notin f(k)$, then $\Gamma^{\Uparrow(i)}$ cannot be built. In this case, any derivation of $\Gamma \Rightarrow !^i(A \otimes B)$ cannot start with an application of the promotion rule $!^i R$ (similarly to how promotion in ILL cannot be applied in the presence of non-classical contexts). In this case, if A, B are atomic, this sequent would not be provable.

3 Linear nested systems

In this section we present a brief introduction to linear nested systems. For further details, refer to ([6,8]).

One of the main problems of using sequent systems as a framework is that sequents are often not adequate for expressing modal behaviors. In order to propose a better formulation, we need a tighter control of formulas in the context, something that sequents do not provide. Hence the need for extending the notion of sequent systems.

Definition 3 (LNS). *The set* LNS *of* linear nested sequents *is given recursively by:*
(i) if $\Gamma \Rightarrow \Delta$ *is a sequent then* $\Gamma \Rightarrow \Delta \in \mathsf{LNS}$
(ii) if $\Gamma \Rightarrow \Delta$ *is a sequent and* $\mathcal{G} \in \mathsf{LNS}$ *then* $\Gamma \Rightarrow \Delta \mathbin{/\!\!/} \mathcal{G} \in \mathsf{LNS}$.

We call each sequent in a linear nested sequent a component *and slightly abuse notation, abbreviating "linear nested sequent" to* LNS. *We shall denote by* $\mathsf{LNS}_{\mathcal{L}}$ *a linear nested sequent system for a logic* $\mathcal{L}$.

In words, a linear nested sequent is simply a finite list of sequents that matches exactly the *history* of a backward proof search in an ordinary sequent calculus ([6]).

A further advantage of this framework is that it is often possible to restrict the list of sequents in a LNS to the last 2 components, that we call *active*.

Definition 4 (End-active). *An application of a linear nested sequent rule is* end-active *if the rightmost components of the premises are active and the only active components (in premise and conclusion) are the two rightmost ones. The* end-active variant *of a* LNS *calculus is the calculus with the rules restricted to end-active applications.*

3.1 LNS for multi-modal LL

In the quest for locality, [3] proposed 2-sequents systems for LL variants, with separate rules for the exponentials. In [7] this work was revisited, establishing a lighter notation and extending the discussion to multi-modalities.

$\mathsf{LNS}_{\mathsf{LL}}$ ([7]) is an end-active, linear nested system for linear logic. In this system, the promotion rule is split into the following local rules:

$$\frac{\vdash \Gamma \Rightarrow \mathbin{/\!\!/} \Rightarrow F}{\mathcal{E} \mathbin{/\!\!/} \Gamma \Rightarrow !F}\ !R \qquad \frac{\Gamma \Rightarrow H \mathbin{/\!\!/} \Delta, !\,F \Rightarrow G}{\Gamma, !\,F \Rightarrow H \mathbin{/\!\!/} \Delta \Rightarrow G}\ !L$$

Observe that no checking must be done in the context in order to apply the $!\,L$ rule: The only checking is in the $!\,R$ rule, where $\mathcal{E}$ should be the empty sequent or an empty list of components.

More precisely, applying the ! rule enables the creation of the *future history*, in which the banged formula should be proved. The intended interpretation of a LNS in LL is

$$\begin{aligned} \iota(\Gamma \Rightarrow F) \quad &:= \bigotimes \Gamma \multimap F \\ \iota(\Gamma \Rightarrow F \mathbin{/\!\!/} \mathcal{H}) &:= \bigotimes(\Gamma \multimap F) \parr\ !\,\iota(\mathcal{H}) \end{aligned}$$

The notion of signatures is then enhanced. We say that $\Sigma = (I, \preccurlyeq, f)$ is a *functorial signature* if it is defined over $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}, \mathsf{K}, \mathsf{4}, \mathsf{T}, \mathsf{D}\}$ and $\mathsf{K} \in f(i),\ \forall i \in I$. If, moreover, $\mathsf{A1}, \mathsf{A2} \in f(i),\ \forall i \in I$, then the functorial signature is called *associative*.

The associative case. In the presence of associativity, contexts are lists of formulas, and the tree structure is not needed. Given an associative signature Σ, we introduce nesting operators $\mathbin{/\!\!/}^i$ and their unfinished versions $\mathbin{\backslash\!\!\backslash}^i$ for every $i \in I$, and change the interpretation so that they are interpreted by the corresponding modality:

$$\iota(\Gamma \Rightarrow F) := \bigotimes \Gamma \multimap F$$
$$\iota(\Gamma \Rightarrow F \mathbin{/\!\!/}^i \mathcal{H}) := \iota(\Gamma \Rightarrow F \mathbin{\backslash\!\!\backslash}^i \mathcal{H}) := (\bigotimes \Gamma \multimap F) \parr !^i\, \iota(\mathcal{H})$$

The operators $\backslash\!\!\backslash^i$ indicate that the standard sequent rule for the modality indexed by i has been partially processed as shown below.

The exponential rules for the end-active linear nested system $\mathsf{LNS}_{(I,\preccurlyeq,F)}$ is given by the following rules

$$\frac{\Gamma \Rightarrow G \backslash\!\!\backslash^i F, \varDelta \Rightarrow H}{\Gamma, ?^j F \Rightarrow G \backslash\!\!\backslash^i \varDelta \Rightarrow H}\ ?^i{}_{\mathsf{k}} \text{ (for } j \in \uparrow(i)) \qquad \frac{\Gamma \Rightarrow \backslash\!\!\backslash^i \Rightarrow F}{\Gamma \Rightarrow !^i F}\ !^i \qquad \frac{\Gamma \Rightarrow F}{\mathcal{E} \backslash\!\!\backslash^i \Gamma \Rightarrow F}\ \mathsf{r}$$

$$\frac{\Gamma \Rightarrow G \backslash\!\!\backslash^i ?^j F, \varDelta \Rightarrow H}{\Gamma, ?^j F \Rightarrow G \backslash\!\!\backslash^i \varDelta \Rightarrow H}\ ?^i{}_{4} \text{ (for } j \in \uparrow^4(i)) \qquad \frac{\Gamma \Rightarrow G \backslash\!\!\backslash^i F \Rightarrow}{\Gamma, ?^i F \Rightarrow G}\ ?^i{}_{\mathsf{d}} \text{ (for } \mathsf{D} \in F(i))$$

$$\frac{\Gamma\{F\} \Rightarrow G}{\Gamma\left\{!^i F\right\} \Rightarrow G}\ ?^i{}_{\mathsf{t}} \text{ (for } \mathsf{T} \in F(i))$$

Note that the order of formulas in the context is respected in the application of the rules.

3.2 The non-associative case

Since LNS do not preserve tree structures, the system just proposed is not adequate for the non-associative setting. In this case, we propose a skeleton that keeps track of the shape of the structure as follows.

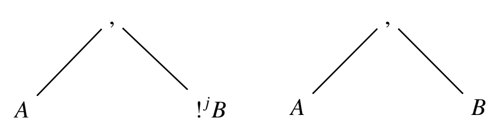
Given a structure Γ, we represent by Γ° its underlying tree structure, where the leaves of Γ are substituted by empty contexts $\{\ \}$ (holes). $\Gamma^\circ\{\}$ will then represent Γ° with a specific position highlighted, and the usual context substitution can be applied. That is, given an underlying structure Γ°, we write $\Gamma^\circ\{F\}$ for the context where some specific hole $\Gamma^\circ\{\ \}$ has been replaced by F. If $\Gamma^\circ = \emptyset$ the hole is removed.

The LNS rules are then adapted from the rules of the associative case, for example

$$\frac{\Gamma\{\ \} \Rightarrow G \backslash\!\!\backslash^i \Gamma^\circ\{F\} \Rightarrow H}{\Gamma\left\{?^j F\right\} \Rightarrow G \backslash\!\!\backslash^i \Gamma^\circ\{\ \} \Rightarrow H}\ ?^i{}_{\mathsf{k}} \text{ (for } j \in \uparrow(i)) \qquad \frac{\Gamma \Rightarrow \backslash\!\!\backslash^i \Gamma^\circ \Rightarrow F}{\Gamma \Rightarrow !^i F}\ !^i \qquad \frac{\Gamma \Rightarrow F}{\mathcal{E} \backslash\!\!\backslash^i \Gamma^\circ \Rightarrow F}\ \mathsf{r}$$

$$\frac{\Gamma \Rightarrow G \backslash\!\!\backslash^i \Gamma^\circ\{F\} \Rightarrow}{\Gamma\left\{?^i F\right\} \Rightarrow G}\ ?^i{}_{\mathsf{d}} \text{ (for } \mathsf{D} \in F(i))$$

Example 5. Let Γ be as in Example 2, and suppose that $j \in \uparrow^4(i), 4 \notin f(i)$ and $i \npreceq k$. Below are possible configurations of $\Gamma^{\leq i}$ after applying a $\mathsf{K4}_i$ rule



4 Conclusion

In this ongoing work, we propose a modular system for dealing with multi-modal LL systems, where substructural and modal axioms are taken into account.

The motivation for that is twofold: (1) to expand the role of LL as a framework for reasoning about systems, as in [9,12,14] and, mainly, (2) extending the computational interpretation of subexponentials [11,13] also to the non-associative/commutative case. We intend to pursue these directions in the near future.

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