How To Play The Accordion

On the (Non-)Conservativity of the Reduction Induced by the Taylor Approximation of λ-Terms

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The Taylor expansion, stemming from Linear Logic and its differential extensions and originally defined by Ehrhard and Regnier [ER08], provides an approximation framework for the λ-calculus (and many of its variants). This approximation enjoys a crucial *commutation* property: the normal form of the Taylor expansion of a term is exactly the Taylor expansion of its Böhm tree. In the usual λ-calculus, it can be refined into a *simulation* property giving account not only of the normalisation of the λ-terms, but also of their reduction: whenever $M \longrightarrow_{\beta}^{*} N$, the Taylor expansion $\mathscr{T}(M)$ reduces to $\mathscr{T}(N)$ [Vau17].

In our previous work [CV22], we extended the Taylor approximation to the infinitary λ-calculus Λ_{∞}^{001}. The infinitary λ-calculi feature possibly infinite terms and reductions and can be seen as metric completions [Ken+97] or as coinductive counterparts [EP13] of the usual, finitary λ-calculus. In particular, Λ_{∞}^{001} formalises the intuition that a λ-term infinitely reduces to its Böhm tree, hence its tight link to the Taylor expansion.

In that previous work, we suggested that, in addition to our main theorem:

Theorem [CV22, Th. 4.21]**.** For all $M, N \in \Lambda_{\infty}^{001}$, if $M \longrightarrow_{\beta}^{\infty} N$ then $\mathscr{T}(M) \mathrel{\widetilde{\longrightarrow}_{r}^{*}} \mathscr{T}(N)$.

it may be possible to prove a *conservativity* property, *i.e.* its converse:

Conjecture [CV22, p. 39]**.** For all $M, N \in \Lambda_{\infty}^{001}$, if $\mathscr{T}(M) \mathrel{\widetilde{\longrightarrow}_{r}^{*}} \mathscr{T}(N)$ then $M \longrightarrow_{\beta}^{\infty} N$.

As we now show, this is false in the general case (Theorem 2.3). However, the conservativity holds for the finitary λ-calculus (Theorem 1.4).

Setting and notations

The setting of this abstract is exactly the same as in our previous work [CV22], so we refer to it (or to the long version of this paper) for a thorough presentation. Let us only recall a few notations.

The set of the usual, finite λ-terms is denoted by Λ. The set Λ_∞^{001} of the 001-infinitary λ-terms is defined by the following inductive-coinductive system of rules:

$$\frac{}{x \in \Lambda_\infty^{001}} \qquad \frac{M \in \Lambda_\infty^{001}}{\lambda x.M \in \Lambda_\infty^{001}} \qquad \frac{M \in \Lambda_\infty^{001} \qquad \triangleright N}{(M)N \in \Lambda_\infty^{001}} \qquad \frac{M \in \Lambda_\infty^{001}}{\triangleright M}$$

This means that infinite branches must cross infinitely many times the right side of an application. We denote by $\longrightarrow_\beta$, $\longrightarrow_\beta^*$ and $\longrightarrow_\beta^\infty$ the β-reduction, its reflexive-transitive closure and its 001-infinitary closure. Informally, $M \longrightarrow_\beta^\infty N$ whenever there is an infinite sequence of β-reductions from M to N such that the applicative depth of the fired redexes tends to the inifinity.

The set of the resource λ-terms is denoted by Λ_r, and if X is a set then $X^!$ denotes the finite multisets of elements of X. The (unweighted, finite) sums of resource terms are the elements of $2\langle\Lambda_r\rangle$. The resource reduction, acting on $2\langle\Lambda_r\rangle$, is denoted by $\longrightarrow_r$. The Taylor expansion is the mapping $\Lambda \to \mathscr{P}(\Lambda_r)$ defined by induction by:

$$\mathscr{T}(x) := x \qquad \mathscr{T}(\lambda x.M) := \sum_{s \in \mathscr{T}(M)} \lambda x.s \qquad \mathscr{T}((M)N) := \sum_{\substack{s \in \mathscr{T}(M) \\ \bar{t} \in \mathscr{T}(N)^!}} \langle s \rangle \bar{t}$$

and the definition on Λ_∞^{001} can be considered as being exactly the same (even though formulated differently, because structural induction is not possible any more). Finally, given $\mathscr{S}, \mathscr{T} \in \mathscr{P}(\Lambda_r)$, we write $\mathscr{S} \mathrel{\widetilde{\longrightarrow_r^*}} \mathscr{T}$ whenever $\mathscr{T} = \sum_{s \in \mathscr{S}} T_s$ and $\forall s \in \mathscr{S}, s \longrightarrow_r^* T_s$.

1 Conservativity holds in the finitary case

Among the different possible definitions of the notion of conservativity of an ARS extending another ARS, we stick to the following one, which distinguishes between the conservativity of a reduction and the conservativity of the corresponding conversion.

Definition 1.1 (conservative extension)**.** Let $(A, \to_A)$ and $(B, \to_B)$ be two abstract rewriting systems. The latter is an *extension* of the former if:

1. there is an injection $i : A \hookrightarrow B$, (inclusion)
2. $\forall a, a' \in A$, if $a \to_A a'$ then $i(a) \to_B i(a')$, (simulation)

Furthermore, this extension is *conservative* if:

3. $\forall a, a' \in A$, if $i(a) \to_B i(a')$ then $a \to_A a'$. (conservativity)

In this setting, we consider $(\mathscr{P}(\Lambda_r), \widetilde{\longrightarrow_r^*})$ as an extension of $(\Lambda, \longrightarrow_\beta^*)$ through the injection $\mathscr{T}(-)$. The conservativity property is exactly what we want to prove. To do so, we use the

$$\frac{M \longrightarrow_\beta^* x}{M \vdash x} \qquad \frac{M \longrightarrow_\beta^* \lambda x.P \qquad P \vdash s}{M \vdash \lambda x.s} \qquad \frac{M \longrightarrow_\beta^* (P)Q \qquad P \vdash s \qquad Q \vdash^! \bar{t}}{M \vdash \langle s \rangle \bar{t}}$$

$$\frac{(M \vdash t_i)_{i=1}^n}{M \vdash^! [t_1, \dots, t_n]} \qquad \frac{(M \vdash s)_{s \in \mathcal{S}}}{M \,\widetilde{\vdash}\, \mathcal{S}}$$

Figure 1. — Rules for $\vdash$ and $\widetilde{\vdash}$

"mashup" technique designed by Kerinec and the second author to establish the converva-tivity of the algebraic λ-calculus wrt. the usual λ-calculus [KV23]. This technique relies on a relation $\vdash$ relating λ-terms to the approximants of their reducts.

Definition 1.2 (mashup relation)**.** The *mashup* relation $\vdash\ \subset \Lambda \times \Lambda_r$ is defined inductively by the first four rules of fig. 1. It is extended to $\widetilde{\vdash} \in \Lambda \times \mathscr{P}(\Lambda_r)$ by the last rule of fig. 1.

The proof is divided into five steps, from which the conservativity theorem follows immediately.

Lemma 1.3. For all $M, N \in \Lambda$, $x \in \mathscr{V}$, $s \in \Lambda_r$, $\bar{t} \in \Lambda_r^!$ and $\mathcal{S}, \mathcal{T} \in \mathscr{P}(\Lambda_r)$, the following lemmas hold:

1. $M \,\widetilde{\vdash}\, \mathcal{T}(M)$.
2. If $M \longrightarrow_\beta^* N$ and $N \,\widetilde{\vdash}\, \mathcal{S}$, then $M \,\widetilde{\vdash}\, \mathcal{S}$.
3. If $M \vdash s$ and $N \vdash^! \bar{t}$, then $\forall s' \in s\langle \bar{t}/x \rangle$, $M[N/x] \vdash s'$.
4. If $M \,\widetilde{\vdash}\, \mathcal{S}$ and $\mathcal{S} \,\widetilde{\longrightarrow}_r^*\, \mathcal{T}$, then $M \,\widetilde{\vdash}\, \mathcal{T}$.
5. If $M \,\widetilde{\vdash}\, \mathcal{T}(N)$, then $M \longrightarrow_\beta^* N$.

Proof. (1)–(4) are by easy inductions. (5) relies on the canonical injection $\lfloor - \rfloor : \Lambda \to \Lambda_r$ defined by $\lfloor x \rfloor := x$, $\lfloor \lambda x.P \rfloor := \lambda x. \lfloor P \rfloor$ and $\lfloor (P)Q \rfloor := \langle \lfloor P \rfloor \rangle [\lfloor Q \rfloor]$, such that for all $N \in \Lambda$, $\lfloor N \rfloor \in \mathcal{T}(N)$. Then from $M \vdash \lfloor N \rfloor$, we prove $M \longrightarrow_\beta^* N$ by induction on N. ◇

Theorem 1.4 (conservativity)**.** For all $M, N \in \Lambda$, if $\mathcal{T}(M) \,\widetilde{\longrightarrow}_r^*\, \mathcal{T}(N)$ then $M \longrightarrow_\beta^* N$.

Proof. By Lemma 1.3(1) we have $M \,\widetilde{\vdash}\, \mathcal{T}(M)$, and by assumption $\mathcal{T}(M) \,\widetilde{\longrightarrow}_r^*\, \mathcal{T}(N)$ so by Lemma 1.3(4) $M \,\widetilde{\vdash}\, \mathcal{T}(N)$. We can conlude with Lemma 1.3(5). ◇

2 Conservativity fails in the 001-infinitary case

The previous theorem was arguably expected, since the Taylor approximation of the λ-calculus has excellent properties: in particular, a single well-chosen term $\lfloor M \rfloor \in \mathcal{T}(M)$ is enough to characterise M, and a single sequence of resource reducts of some $s \in \mathcal{T}(M)$ suffices to characterise any sequence $M \longrightarrow_\beta^* N$. These properties are not true any more when considering more complicated settings, like infinitary λ-calculi: $M \in \Lambda_\infty^{001}$ is only characterised by a sequence of approximants [see CV22, Lem. 5.29]. This is enough not only to make the "mashup" proof technique fail (and in particular Lemma 1.3(5)), but even to make the extension of Theorem 1.4 to Λ^{001}_{∞} false — as we will show by exhibiting a counterexample, the "Accordion" λ-term.

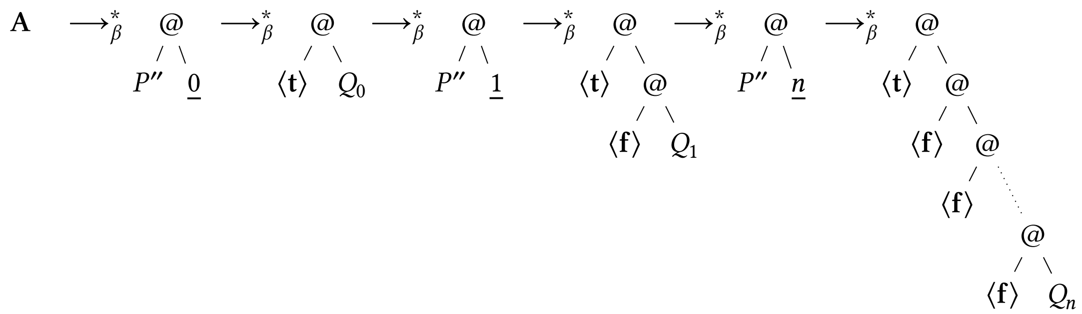
Notation 2.1. We denote the usual representation of booleans as $\mathbf{t}$ and $\mathbf{f}$, an "applicator" construction as $\langle M\rangle := \lambda b.(b)M$, and the Church encodings of integers and of the successor function as $\underline{n}$ and succ.

Definition 2.2 (the Accordion)**.** The *Accordion term* is defined as $\mathbf{A} := (P)\underline{0}$, where:

$$P := (\mathrm{Y})\,\lambda\phi.\lambda n.\,(\langle\mathbf{t}\rangle)\,((n)\,\langle\mathbf{f}\rangle)\,Q_{\phi,n} \qquad\qquad Q_{\phi,n} := (\mathrm{Y})\,\lambda\psi.\lambda b.\,((b)(\phi)(\mathrm{succ})n)\,\psi$$

We also define $\bar{\mathbf{A}} := (\langle\mathbf{t}\rangle)\,(\langle\mathbf{f}\rangle)^{\infty}$.

Let us show describe this term behaves (and why we named it the Accordion). There exist terms P'' and Q_n (for all $n \in \mathbb{N}$) such that the following reductions hold:



This means that:

1. for any $d \in \mathbb{N}$, $\mathbf{A}$ reduces to terms $\mathbf{A}_d$ that are similar to $\bar{\mathbf{A}}$ up to depth d (and, as a consequence, any finite approximant of $\bar{\mathbf{A}}$ if a reduct of approximants of $\mathbf{A}$);
2. but this is not a valid infinitary reduction because we *need* to reduce a redex at depth 0 to obtain $\mathbf{A}_d \longrightarrow^*_\beta \mathbf{A}_{d+1}$ (so the depth of the reduced redexes does not tend to the infinity, which is required in the definition of $\longrightarrow^\infty_\beta$).

This dynamics (the term $\mathbf{A}$ is "stretched" and "compressed" over and over) justifies the name "Accordion".

Theorem 2.3. $\mathcal{T}(\mathbf{A}) \widetilde{\longrightarrow^*_r} \mathcal{T}(\bar{\mathbf{A}})$ and $\neg\left(\mathbf{A} \longrightarrow^\infty_\beta \bar{\mathbf{A}}\right)$.

Proof sketch. The first part consists in observing that for all $d \in \mathbb{N}$, $\mathbf{A} \longrightarrow^*_\beta \bar{\mathbf{A}}_d := (\langle\mathbf{t}\rangle)(\langle\mathbf{f}\rangle)^d Q_d$, hence $\mathcal{T}(\mathbf{A}) \widetilde{\longrightarrow^*_r} \mathcal{T}(\bar{\mathbf{A}}_d)$. Then, using the fact that $\mathcal{T}(\bar{\mathbf{A}}) = \sum_{d\in\mathbb{N}} \mathcal{T}_{<d}(\bar{\mathbf{A}}) = \sum_{d\in\mathbb{N}} \mathcal{T}_{<d}(\bar{\mathbf{A}}_d)$, we can deduce $\mathcal{T}(\mathbf{A}) \widetilde{\longrightarrow^*_r} \mathcal{T}(\bar{\mathbf{A}})$.

The second part is quite technical. It heavily relies on the following well-known decomposition of the β-reduction: for all $M, N \in \Lambda$, if $M \longrightarrow^*_\beta N$ then there exists an $M' \in \Lambda$ such that $M \longrightarrow^*_h M' \longrightarrow^*_i N$, where $\longrightarrow_h$ and $\longrightarrow_i$ denote *head* and *internal* β-reductions.

Suppose there is a reduction $\mathbf{A} \longrightarrow^\infty_\beta \bar{\mathbf{A}}$. Thanks to [CV22, Lem. 4.11], for all $d \in \mathbb{N}$, there exist $\mathbf{A}_1, \dots, \mathbf{A}_d \in \Lambda$ such that $\mathbf{A} \longrightarrow^*_{\beta\geqslant 0} \mathbf{A}_0 \longrightarrow^*_{\beta\geqslant 1} \mathbf{A}_1 \longrightarrow^*_{\beta\geqslant 2} \dots \longrightarrow^*_{\beta\geqslant d} \mathbf{A}_d \longrightarrow^\infty_{\beta>d} \bar{\mathbf{A}}$ $(*)$. In addition, the head-internal decomposition ensures the existence of $\mathbf{A}'_0 \in \Lambda$ such that $\mathbf{A} \longrightarrow^*_h \mathbf{A}'_0 \longrightarrow^*_i \mathbf{A}_0$. Since there are only internal reductions from $\mathbf{A}'_0$ to $\bar{\mathbf{A}}$, the former must have the same "head structure" than the latter, *i.e.* have the shape $(\lambda b.M)N$ for some $M, N \in \Lambda$. An exhaustive head reduction of $\mathbf{A}$ shows that there are only four possible options for such a reduct $\mathbf{A}'_0$ of $\mathbf{A}$.

From $(*)$, it follows that $\mathbf{A}'_0 \longrightarrow_i^* \mathbf{A}_{n+3}$. One can show that this is impossible by exploring the four possible cases for $\mathbf{A}'_0$, by abundantly using the head-internal decomposition again. ⋄

Thus, the extension $(\mathscr{P}(\Lambda_r), \widetilde{\longrightarrow_r^*})$ of the reduction system $(\Lambda_\infty^{001}, \longrightarrow_\beta^\infty)$ is not conservative. However, let us underline as a consolation that a weaker result is available. Indeed, consider the λ⊥-calculus $\Lambda_{\infty\perp}^{001}$ — *i.e.* Λ_∞^{001} with a constant $\perp$ such that $\mathscr{T}(\perp) := 0$ — endowed with the usual β⊥-reduction — *i.e.* the reduction generated by contextually reducing all unsolvables to $\perp$, as well as all the terms $\lambda x.\perp$ and $(\perp)M$. Then, as a corollary of the infinitary Commutation theorem [CV22, Thm. 5.20], the following are equivalent:

- $M =_{\beta\perp}^\infty N$, where $=_{\beta\perp}^\infty$ is the conversion generated by $\longrightarrow_{\beta\perp}^\infty$ — which is also the same as the equivalence $=_{\mathscr{B}}$ generated by $M =_{\mathscr{B}} N$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$,
- $\mathscr{T}(M) \cong_r \mathscr{T}(N)$, where $\cong_r$ is the conversion generated by $\widetilde{\longrightarrow_r^*}$.

This can be reformulated as follows: $(\mathscr{P}(\Lambda_r), \cong_r)$ is a conservative extension of $(\Lambda_{\infty\perp}^{001}, =_{\beta\perp}^\infty)$. As a future investigation, a similar convervativity property could be looked for in other settings where the Taylor expansion enjoys fruitful simulation properties. In particular, a simulation theorem has been prooved by the second author for the algebraic calculus [Vau19, Cor. 7.7]; as far as we know, the question whether this yields a conversative extension is open and involves a major difficulty resulting from the non-uniformity of the algebraic setting.

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