Unifying Graded Linear Logic and Differential Operators

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1 Introduction

Linear logic (LL) [7] and its differential counterpart [4] give a framework to study resource usages of proofs and programs. These logics were invented by enriching the syntax of proofs with new constructions observed in denotational models of λ-calculus. The exponential connective ! introduces non-linearity in the context of linear proofs and encapsulate the notion of resource usage. This notion was refined into *parametrised exponentials* [8, 3, 5, 6], where exponential connectives are indexed by annotations specifying different behaviors. Our aim here is to follow Kerjean's former works [9] by indexing formulas of Linear Logic with Differential Operators. Thanks to the setting of Bounded Linear Logic, we formalize and deepen the connection between Differential Linear Logic and Differential Operators.

The fundamental linear decomposition of LL is the decomposition of the usual non-linear implication $\Rightarrow$ into a linear one $\multimap$ from a set of resources represented by the new connective !: $(A \Rightarrow B) \equiv (!A \multimap B)$. Bounded Linear Logic (BLL) [8] was introduced as the first attempt to use typing systems for complexity analysis. But our interest for this logic stems from the fact that it extends LL with several exponential connectives which are indexed by *polynomially bounded intervals*. Since then, some other indexations of LL have been developed for many purposes, for example IndLL [3] where the exponential modalities are indexed by some functions, or the graded logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ [2, 6, 11] where they are indexed by the elements of a semiring $\mathcal{S}$. This theoretical development finds applications in programming languages.

Differential linear logic [4] (DiLL) consists in an a priori distinct approach to linearity, and is based on the denotational semantics of linear proofs in terms of linear functions. In the syntax of LL, the dereliction rule states that if a proof is linear, one can then forget its linearity and consider it as non-linear. To capture differentiation, DiLL is based on a codereliction rule which is the syntactical opposite of the dereliction. It states that from a non-linear proof (or a non-linear function) one can extract a linear approximation of it, which, in terms of functions, is exactly the differential (one can notice that here, the analogy with resources does not work). Then, models of DiLL interpret the codereliction by different kinds of differentiation [1].

A first step towards merging the graded and the differential extension of LL was made by Kerjean in 2018 [9]. In this paper, she defines an extension of DiLL, named D-DiLL, in which the exponential connectives ? and ! are indexed with a *fixed* linear partial differential operator with constant coefficients (LPDOcc) D. There, formulas $!_D A$ and $?_D A$ are respectively interpreted in a denotational model as spaces of functions or distributions which are solutions of the differential equation induced by D. The dereliction and codereliction rules then represent respectively the resolution of a differential equation and the application of a differential operator. This is a significant step forward in our aim to make the theory of programming languages and functional analysis closer, with a Curry-Howard perspective. In this work, we will generalize D-DiLL to a logic indexed by a monoid of LPDOcc.

Contributions. This work considerably generalizes, corrects and consolidates the extention of DiLL to differential operators sketched in [9]. It extends D-DiLL in the sense that the logic is now able to deal with all LPDOcc and combine their action. It corrects D-DiLL as the denotational interpretation of indexed exponential $?_D$ and $!_D$ are changed, leaving the interpretation of *inference rules* unchanged but reversing their type in a way that is now compatible with graded logics. Finally, this work consolidates D-DiLL by proving a cut-elimination procedure in the graded case, making use of an algebraic property on the monoid of LPDOcc.

2 Linear logic and its extensions

Linear Logic refines Classical Logic by introducing a notion of linear proofs. Differentiation is then introduced through a "codereliction" rule $\bar{\mathsf{d}}$, which is symmetrical to d and allows to linearize a non-linear proof [4]. To express the cut-elimination with the promotion rule, other costructural rules are needed, which find a natural interpretation in terms of differential calculus.

$$\frac{}{\vdash !A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

Recently, Kerjean [9] gave an interpretation of the connective ? by a space of smooth scalar functions, while ! is interpreted as the space of linear maps acting on those functions, that is a space of *distributions*:

$$[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R}) \qquad [\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'.$$

Nicely, every exponential rule of DiLL has an interpretation in terms of functions and distributions.

A first advance in merging the graded and the differential extensions of LL was made by Kerjean in 2018 [9]. In this paper, she defines an extension of DiLL named D-DiLL. This logic is based on a *fixed single* linear partial differential operator D, which appears as a single index in exponential connectives $!_D$ and $?_D$.

Definition 2.1. *Let D be a LPDOcc. A* fundamental solution *of D is a distribution $\Phi_D \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $D(\Phi_D) = \delta_0$.*

Theorem 2.2 (Malgrange-Ehrenpreis)**.** *Every linear partial differential operator with constant coefficients admits exactly one fundamental solution.*

Using this result, D-DiLL gives new definitions for d and $\bar{\mathsf{d}}$, depending of a LPDOcc D:

$$\mathsf{d}_D : f \mapsto \Phi_D * f \qquad \bar{\mathsf{d}}_D : \phi \mapsto \phi \circ D.$$

Indexed linear logics: resources, effects and coeffects Since Girard's original BLL [8], several systems have implemented indexed exponentials to keep track of resource usage. More recently, several authors [6, 5, 2] have defined a modular (but a bit less expressive) version $\mathsf{B}_\mathcal{S}\mathsf{LL}$ where the exponentials are indexed (more specifically "graded", as in graded algebras) by elements of a given semiring $\mathcal{S}$. The rules of $\mathsf{B}_\mathcal{S}\mathsf{LL}$ are as follow.

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\ \mathsf{w} \qquad \frac{\Gamma, !_x A, !_y A \vdash B}{\Gamma, !_{x+y} A \vdash B}\ \mathsf{c} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\ \mathsf{d} \qquad \frac{!_{x_1} A_1, \dots, !_{x_n} A_n \vdash B}{!_{x_1 \times y} A_1, \dots, !_{x_n \times y} A_n \vdash !_y B}\ \mathsf{p}$$

Finally, a subtyping rule is also added, which uses the order of $\mathcal{S}$.

$$\frac{\Gamma, !_x A \vdash B \quad x \leq y}{\Gamma, !_y A \vdash B}\ \mathsf{d}_I$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_0 A}\ \mathsf{w} \qquad \frac{\vdash \Gamma, ?_x A, ?_y A}{\vdash \Gamma, ?_{x+y} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_x A \qquad x \leq y}{\vdash \Gamma, ?_y A}\ \mathsf{d}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \mathsf{d}$$

$$\frac{}{\vdash !_0 A}\ \bar{\mathsf{w}} \qquad \frac{\vdash \Gamma, !_x A \qquad \vdash \Delta, !_y A}{\vdash \Gamma, \Delta, !_{x+y} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_x A \qquad x \leq y}{\vdash \Gamma, !_y A}\ \bar{\mathsf{d}}_I \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ \bar{\mathsf{d}}$$

Figure 1: Exponential rules of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?_D A}\ \mathsf{w}_I \qquad \frac{\vdash \Gamma, ?_{D_1} A, ?_{D_2} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{c} \qquad \frac{\vdash \Gamma, ?_{D_1} A}{\vdash \Gamma, ?_{D_1 \circ D_2} A}\ \mathsf{d}_I$$

$$\frac{}{\vdash !_D A}\ \bar{\mathsf{w}}_I \qquad \frac{\vdash \Gamma, !_{D_1} A \qquad \vdash \Delta, !_{D_2} A}{\vdash \Gamma, \Delta, !_{D_1 \circ D_2} A}\ \bar{\mathsf{c}} \qquad \frac{\vdash \Gamma, !_{D_1} A}{\vdash \Gamma, !_{D_1 \circ D_2} A}\ \bar{\mathsf{d}}_I$$

Figure 2: Exponential rules of IDiLL

3 Unifying graded linear logic and differential operators

We define a differential version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ by extending its set of exponential rules. Here, we will restrict ourselves to a version without promotion, as it has been done for DiLL originally. Following the ideas behind DiLL, we add *costructural* exponential rules: a coweakening $\bar{\mathsf{w}}$, a cocontraction $\bar{\mathsf{c}}$, an indexed codereliction $\bar{\mathsf{d}}_I$ and a codereliction $\bar{\mathsf{d}}$. The set of exponential rules of our new logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is given in Figure 1. Note that by doing so we study a *classical* version of $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$, with an involutive linear duality.

Theorem 3.1. *The logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ has a cut elimination procedure when $\mathcal{S}$ is additive splitting.*

An indexed differential linear logic The logic $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ is a syntactical differentiation of BLL, as it uses the idea that differentiation is expressed through co-structural rules that mirror the structural rules of LL. Here we will take a semantical point of view: starting from differential linear logic, we will index it with LPDOcc into a logic named IDiLL, and then study the relation between $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ and IDiLL.

Kerjean generalized $\bar{\mathsf{d}}$ and d in previous work [9], with the idea that in DiLL, the codereliction corresponds to the application of the differential operator D_0 whereas the dereliction corresponds to the resolution of the differential equation associated to D_0. This led to a logic D-DiLL, where $\bar{\mathsf{d}}$ and d have the same effect but with a LPDOcc D instead of D_0, and where the exponential connectives are indexed by this operator D. We change the logic D-DiLL into a logic IDiLL, which is much closer to what is done in the graded setting. In this new framework, we will consider the composition of two LPDOcc as our monoidal operation. We describe the exponential rules of IDiLL in Figure 2.

The indexed rules d_D and $\bar{\mathsf{d}}_D$ are generalized to rules d_I and $\bar{\mathsf{d}}_I$ involving a variety of LPDOcc, while rules d and $\bar{\mathsf{d}}$ are ignored for now. The interpretations of $?_D A$ and $!_D A$, and hence the typing of d_I and $\bar{\mathsf{d}}_I$ are changed from what D-DiLL would have directly enforced (see remark 3.2). Our new interpretations for $?_D A$ and $!_D A$ are now compatible with the intuition that in graded logics, rules are supposed to add information.

$$[\![?_D A]\!] := \{g \mid \exists f \in [\![?A]\!],\ D(g) = f\} \qquad\qquad [\![!_D A]\!] := ([\![?_D A^{\perp}]\!])' = \hat{D}([\![!A]\!])$$

$$\mathsf{d}_I : [\![?_{D_1}A]\!] \to [\![?_{D_1\circ D_2}A]\!] \qquad\qquad \bar{\mathsf{d}}_I : [\![!_{D_1}A]\!] \to [\![!_{D_1\circ D_2}A]\!]$$

Remark 3.2. *Our definition for indexed connectives and thus for the types of* d_D *and* $\bar{\mathsf{d}}_D$ *differs from the original one in D-DiLL [9]. Kerjean gave types* $\mathsf{d}_D : ?_{D,old}E' \to ?E'$ *and* $\bar{\mathsf{d}}_D : !_{D,old}E \to !E$*. However, graded linear logic carries different intuitions: indices are here to keep track of the operations made through the inference rules. As such,* d_D *and* $\bar{\mathsf{d}}_D$ *should introduce indices* D *and not delete it. Compared with work in [9], we then change the interpretation of* $?_DA$ *and* $!_DA$*, and the types of* d_D *and* $\bar{\mathsf{d}}_D$*. Thanks to this change, we will see D-DiLL as a particular case of* $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$*.*

Theorem 3.3. *The set* $\mathcal{D}$ *of LPDOcc is an additive splitting monoid under composition, with the identity operator id as the identity element.*

Then, $\mathcal{D}$ induces a logic $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$. In this logic, since the order of the monoid is defined through the composition rule, for D_1 and D_2 in $\mathcal{D}$ we have

$$D_1 \leq D_2 \iff \exists D_3 \in \mathcal{D},\ D_2 = D_1 \circ D_3$$

From these results, we deduce the following theorem which expresses that both syntactical and semantical differentiations, based either on $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ or DiLL, lead to the same logic.

Theorem 3.4. *Each rule of IDiLL is admissible in* $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$*, and each rule of* $\mathsf{DB}_{\mathcal{D}}\mathsf{LL}$ *except* d *and* $\bar{\mathsf{d}}$ *is admissible in IDiLL.*

A concrete semantics for IDiLL Since we have changed the rules of D-DiLL to be closer to the graded point of view, the semantics of D-DiLL has to be modify while trying to define a semantics of IDiLL. The exponential types will be interpreted as follows.

$$\begin{aligned} [\![!_DA]\!] &:= (\{f \in \mathcal{C}^\infty([\![A]\!],\mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A]\!],\mathbb{R}),\ D(f) = g\})' = \hat{D}([\![!A]\!]) \\ [\![?_DA]\!] &:= \{f \in \mathcal{C}^\infty([\![A]\!]',\mathbb{R}) \mid \exists g \in \mathcal{C}^\infty([\![A]\!]',\mathbb{R}),\ D(f) = g\} = D^{-1}([\![?A]\!]) \end{aligned} \tag{1}$$

From this interpretation of the types, we can define an interpretation for each exponential rule.

Definition 3.5. *We define the interpretation of each exponential rule of IDiLL by:*

$$\mathsf{w}\colon \begin{cases} \mathbb{R} \to ?_{id}E \\ 1 \mapsto cst_1 \end{cases} \qquad \bar{\mathsf{w}}\colon \begin{cases} \mathbb{R} \to !_{id}E \\ 1 \mapsto \delta_0 \end{cases}$$

$$\mathsf{c}\colon \begin{cases} ?_{D_1}E \hat{\otimes} ?_{D_2}E \to ?_{D_1\circ D_2}E \\ \qquad f \otimes g \mapsto \Phi_{D_1\circ D_2} * (D_1(f).D_2(g)) \end{cases} \qquad \bar{\mathsf{c}}\colon \begin{cases} !_{D_1}E \hat{\otimes} !_{D_2}E \to !_{D_1\circ D_2}E \\ \qquad \psi \otimes \phi \mapsto \psi * \phi \end{cases}$$

$$\mathsf{d}_I\colon \begin{cases} ?_{D_1}E \to ?_{D_1\circ D_2}E \\ \qquad f \mapsto \Phi_{D_2} * f \end{cases} \qquad \bar{\mathsf{d}}_I\colon \begin{cases} !_{D_1}E \to !_{D_1\circ D_2}E \\ \qquad \psi \mapsto \psi \circ D_2 \end{cases}$$

Proposition 3.6. *Each morphism* $\mathsf{w}, \bar{\mathsf{w}}, \mathsf{c}, \bar{\mathsf{c}}, \mathsf{d}_I$ *and* $\bar{\mathsf{d}}_I$ *is well-typed, and compatible with the cut elimination procedure.*

4 Conclusion

We have defined a multi-operator version to D-DiLL, which turns out to be the finitary differential version of Graded Linear Logic. We describe the cut-elimination procedure and give a denotational model of this calculus in terms of differential operators. This provides a new and unexpected semantics for Graded Linear Logic, and tighten the links between Linear Logic and Functional Analysis.

There are several directions to explore now that the proof theory of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$ has been established. The obvious missing piece in our work is the *categorical axiomatization* of our model. In a version with promotion, that would consist in a differential version of bounded linear exponentials [2]. A first study based on with differential categories was recently done by Pacaud-Lemay and Vienney [10]. Beware that our logic does not yet extend to higher-order and that without a concrete higher-model it might be difficult to design elegant categorical axioms.

Another line of research would consist in introducing more complex differential operators as indices of exponential connectives. Equations involving LPDOcc are extremely simple to manipulate as they are solved in a one step computation (by applying a convolution product with their fundamental solution). The vast majority of differential equations are difficult if not impossible to solve. One could introduce fixpoint operators within the theory of $\mathsf{DB}_{\mathcal{S}}\mathsf{LL}$, to try and modelize the resolution of differential equation by fixed point.

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