Towards Unifying (Co)induction and Structural Control

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- Goal: uniform treatment of guarded (co)recursion & bounded linearity

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- Why? Concise substructural logical framework w/ (co)induction
- How? Hybrid logic/labeled deduction

Outline

- Infinitary proof system

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- Sneak preview of its metatheory
- Separately, controlling structural rules
- Ongoing work

MALL

$$\frac{}{\vdash \mathbf{1}}\ \mathbf{1} \qquad \frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \bot, \Delta}\ \bot$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \mathbin{⅋} B, \Delta}\ ⅋$$

$$\frac{}{\vdash \Gamma, \top, \Delta}\ \top$$

(no rule for $\mathbf{0}$)

$$\frac{\vdash \Gamma, A, \Delta \quad \vdash \Gamma, B, \Delta}{\vdash \Gamma, A \mathbin{\&} B, \Delta}\ \& \qquad \frac{\vdash \Gamma, A_i, \Delta}{\vdash \Gamma, A_1 \oplus A_2, \Delta}\ \oplus_i, i \in \{1, 2\}$$

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Modalities

$$\dfrac{\begin{array}{c}[u < w] \\ \vdots \\ \vdash \Gamma, A \mathbin{@} u, \Delta\end{array}}{\vdash \Gamma, \bullet A \mathbin{@} w, \Delta}\ \bullet \qquad \dfrac{\vdash \Gamma, A \mathbin{@} u, \Delta}{\vdash \Gamma, \circ A \mathbin{@} w, \Delta}\ \circ, u < w$$

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- Vezzosi's (?) idea: use $\bigcirc$ for guarded induction

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1} \,@\, 0}\ \mathbf{1}}{\vdash (\mathbf{1} \oplus \mathrm{gnat}) \,@\, 0}\ \oplus_1}{\vdash \mathrm{gnat} \,@\, 1}\ \bigcirc}{\vdash (\mathbf{1} \oplus \mathrm{gnat}) \,@\, 1}\ \oplus_2}{\vdash \mathrm{gnat} \,@\, 2}\ \bigcirc$$

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$$\text{eat} = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1} \mathbin{@} v}\ \mathbf{1}}{\vdash \bot \mathbin{@} u, \mathbf{1} \mathbin{@} v}\ \bot \qquad [u/w]\text{eat} \vdash \text{gnat}^{\perp} \mathbin{@} u, \mathbf{1} \mathbin{@} v}{\vdash \bot \mathbin{\&} \text{gnat}^{\perp} \mathbin{@} u, \mathbf{1} \mathbin{@} v}\ \&}{\vdash \bullet(\bot \mathbin{\&} \text{gnat}^{\perp}) \mathbin{@} w, \mathbf{1} \mathbin{@} v}\ \bullet$$

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$$\mathrm{ones} = \dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1} \mathbin{@} u}\ \mathbf{1} \qquad [u/w]\,\mathrm{ones} \vdash \mathrm{gstr}_{\mathbf{1}} \mathbin{@} u}{\vdash \mathbf{1} \otimes \mathrm{gstr}_{\mathbf{1}} \mathbin{@} u}\ \otimes}{\vdash \bullet(\mathbf{1} \otimes \mathrm{gstr}_{\mathbf{1}}) \mathbin{@} w}\ \bullet$$

Cut Admissibility

Theorem

If $D \vdash \Gamma, A \mathbin{@} w$ *and* $E \vdash A^{\perp} \mathbin{@} w, \Delta$*, then* $\mathrm{cut}(w, A, D, E) \vdash \Gamma, \Delta$.

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Proof.

By lexicographic induction on (w, A, D, E).

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Proof.

By lexicographic induction on (w, A, D, E).

$$\mathrm{cut}\left(\frac{D' \vdash \Gamma, B \mathbin{@} u}{\vdash \Gamma, \bullet B \mathbin{@} w} \bullet, \frac{E' \vdash B^{\perp} \mathbin{@} v}{\vdash \circ B^{\perp} \mathbin{@} w, \Delta} \circ \right) = \mathrm{cut}\,(v, B, [v/u]D', E')$$

□

Bounded Linearity

$$\frac{\vdash \Gamma, A \mathbin{@} u, \Delta}{\vdash \Gamma, A \,\mathrm{at}\, u \mathbin{@} w, \Delta}\ \mathrm{at} \qquad \frac{\vdash \Gamma, A(w) \mathbin{@} w, \Delta}{\vdash \Gamma, {\downarrow} u.\, A(u) \mathbin{@} w, \Delta}\ {\downarrow}$$

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$$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma, A \mathbin{@} 0, \Delta}\ \mathsf{W} \qquad \frac{\vdash \Gamma, A \mathbin{@} u, A \mathbin{@} w, \Delta}{\vdash \Gamma, A \mathbin{@} u + w, \Delta}\ \mathsf{C}$$

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$$\frac{\vdash \Gamma, A \mathbin{@} 1 \quad \vdash B \mathbin{@} 1, \Delta}{\vdash \Gamma, A \otimes B \mathbin{@} 1, \Delta}\ \otimes$$

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$$\frac{\vdash \Gamma, A \mathbin{@} 1 \quad \vdash B \mathbin{@} 1, \Delta}{\vdash \Gamma, A \otimes B \mathbin{@} 1, \Delta}\ \otimes$$

- Contraction: $A \,\mathrm{at}\, 2 \multimap (A \otimes A) \,\mathrm{at}\, 1$
- Weakening: $A \,\mathrm{at}\, 0 \multimap \mathbf{1}$

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An aspect of modularity in BLL is that the notion of size of data is given by their type. For example, the data type of tally natural numbers of size at most x is:

$$\mathbf{N}_x \equiv \forall\alpha\, !_{y<x}(\alpha(y) \multimap \alpha(y+1)\ ) \multimap (\alpha(0) \multimap \alpha(x)\ )$$

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- But Church encodings don't scale. What can be done?