Proof nets

A tutorial

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TLLA 2021, *e*-Roma
27–28 June 2021

A (partial, biased) tutorial on the basics of proof nets

- from the distant 80's

A (partial, biased) tutorial on the basics of proof nets

- from the distant 80's to the dazzling 90's

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- only prerequisites: MELL sequent calculus + a bit of graph theory (paths, cycles)

A (partial, biased) tutorial on the basics of proof nets

- from the distant 80's to the dazzling 90's

- only prerequisites: MELL sequent calculus + a bit of graph theory (paths, cycles)
- we should have a book for this

MLL

$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B$$

$$X^{\perp\perp} \coloneqq X \quad (A \otimes B)^\perp \coloneqq A^\perp \parr B^\perp \quad (A \parr B)^\perp \coloneqq A^\perp \otimes B^\perp$$

$$\frac{}{A^\perp, A}\ (ax) \qquad\qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\ (cut)$$

$$\frac{\Gamma, A, B}{\Gamma, A \parr B}\ (\parr) \qquad\qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ (\otimes)$$

MLL

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$$\frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta}\ (ex)$$

An example

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{X^\perp, X}\,(ax) \qquad \dfrac{}{Y^\perp, Y}\,(ax)}{X^\perp, Y^\perp, X \otimes Y}\,(\otimes)}{X^\perp \parr Y^\perp, Z^\perp, X \otimes Y, Z}\,(\parr) \qquad \dfrac{}{Z^\perp, Z}\,(ax)}{\dfrac{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)}\,(\otimes)
$$

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$$

$$\simeq$$

$$(X \otimes Y) \otimes Z \multimap (X \otimes Y) \otimes Z$$

$(A \multimap B := A^\perp \parr B)$

An example

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$$

$$\simeq$$

$$(X \otimes Y) \otimes Z \multimap (X \otimes Y) \otimes Z$$

$(A \multimap B := A^\perp \parr B)$

Some rules permute

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\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$X^\perp, X$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$Y^\perp, Y$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$X^\perp, Y^\perp, X \otimes Y$}
\AxiomC{}
\RightLabel{(ax)}
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\BinaryInfC{$X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z$}
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\UnaryInfC{$X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z$}
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\UnaryInfC{$(X^\perp \parr Y^\perp) \underline{\parr} Z^\perp, (X \otimes Y) \otimes Z$}
\RightLabel{(cut)}
\BinaryInfC{$(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z$}
\end{prooftree}
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Cut elimination requires non-logical steps

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\UnaryInfC{$(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z$}
\end{prooftree}
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Cut elimination requires non-logical steps

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\dfrac{}{X^\perp, X}\,(ax)
\qquad
\dfrac{}{Y^\perp, Y}\,(ax)
}{X^\perp, Y^\perp, X \otimes Y}\,(\otimes)
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\dfrac{}{Z^\perp, Z}\,(ax)
}{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \underline{\otimes} Z}\,(\otimes)
\qquad
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\dfrac{}{X^\perp \parr Y^\perp, X \otimes Y}\,(ax)
\qquad
\dfrac{}{Z^\perp, Z}\,(ax)
}{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\,(\otimes)
}{(X^\perp \parr Y^\perp) \underline{\parr} Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
}{
\dfrac{
\dfrac{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
}\,(cut)
$$

Cut elimination requires non-logical steps

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  \dfrac{
    \dfrac{
      \dfrac{}{X^\perp, X}\,(ax) \qquad \dfrac{}{Y^\perp, Y}\,(ax)
    }{X^\perp, Y^\perp, X \underline{\otimes} Y}\,(\otimes)
    \qquad
    \dfrac{
      \dfrac{}{Z^\perp, \underline{Z}}\,(ax) \qquad
      \dfrac{
        \dfrac{}{X^\perp \parr Y^\perp, X \otimes Y}\,(ax) \qquad \dfrac{}{Z^\perp, Z}\,(ax)
      }{X^\perp \parr Y^\perp, \underline{Z^\perp}, (X \otimes Y) \otimes Z}\,(\otimes)
    }{Z^\perp, X^\perp \underline{\parr} Y^\perp, (X \otimes Y) \otimes Z}\,(cut)
  }{
    \dfrac{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
  }\,(cut)
}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
$$

Cut elimination requires non-logical steps

or maybe:

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\dfrac{}{Z^\perp, \underline{Z}}\;(ax)
\qquad
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\dfrac{\dfrac{}{X^\perp, X}\;(ax) \quad \dfrac{}{Y^\perp, Y}\;(ax)}{X^\perp, Y^\perp, X \underline{\otimes} Y}\;(\otimes)
\quad
\dfrac{\dfrac{}{X^\perp \parr Y^\perp, X \otimes Y}\;(ax) \quad \dfrac{}{Z^\perp, Z}\;(ax)}{X^\perp \underline{\parr} Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\;(\otimes)
}{X^\perp, Y^\perp, \underline{Z^\perp}, (X \otimes Y) \otimes Z}\;(cut)
}{
\dfrac{Z^\perp, X^\perp \underline{\parr} Y^\perp, (X \otimes Y) \otimes Z}{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\;(\parr)
}\;(cut)
}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\;(\parr)
$$

A multiplicative proof

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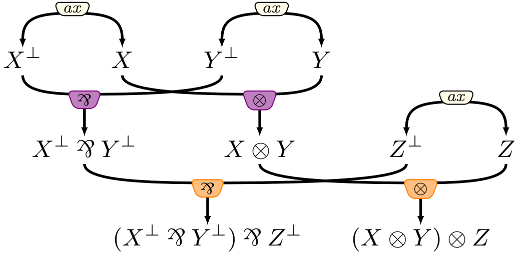
  \dfrac{
    \dfrac{
      \dfrac{}{X^\perp, X}\,(ax)
      \quad
      \dfrac{}{Y^\perp, Y}\,(ax)
    }{X^\perp, Y^\perp, X \otimes Y}\,(\otimes)
  }{X^\perp \parr Y^\perp, Z^\perp, X \otimes Y, Z}\,(\parr)
  \quad
  \dfrac{}{Z^\perp, Z}\,(ax)
}{
  \dfrac{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
}\,(\otimes)
$$

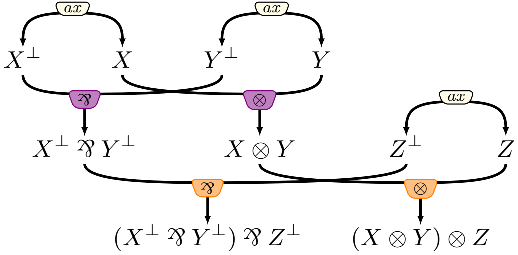
A multiplicative proof net

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X^\perp, X}\,(ax) \qquad \dfrac{}{Y^\perp, Y}\,(ax)}{X^\perp, Y^\perp, X \otimes Y}\,(\otimes)}{X^\perp \parr Y^\perp, Z^\perp, X \otimes Y, Z}\,(\parr) \qquad \dfrac{}{Z^\perp, Z}\,(ax)}{\dfrac{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)}\,(\otimes) \quad \mapsto$$



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X^\perp, X}\,(ax) \qquad \dfrac{}{Y^\perp, Y}\,(ax)}{X^\perp, Y^\perp, X \otimes Y}\,(\otimes) \qquad \dfrac{}{Z^\perp, Z}\,(ax)}{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\,(\otimes)}{X^\perp \parr Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)}{(X^\perp \parr Y^\perp) \parr Z^\perp, (X \otimes Y) \otimes Z}\,(\parr)
$$

$\mapsto$



Cut

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\begin{prooftree}
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\UnaryInfC{$X^\perp, X$}
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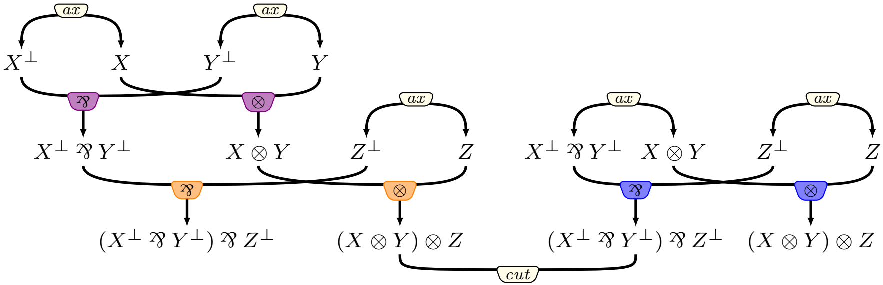
\end{prooftree}
$$

Cut


ax
ax
$X^\perp$
X
$Y^\perp$
Y
⅋
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ax
ax
ax
$X^\perp ⅋ Y^\perp$
$X \otimes Y$
$Z^\perp$
Z
$X^\perp ⅋ Y^\perp$
$X \otimes Y$
$Z^\perp$
Z
⅋
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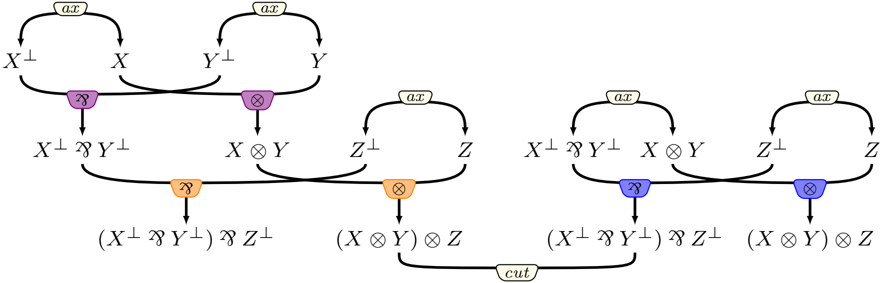
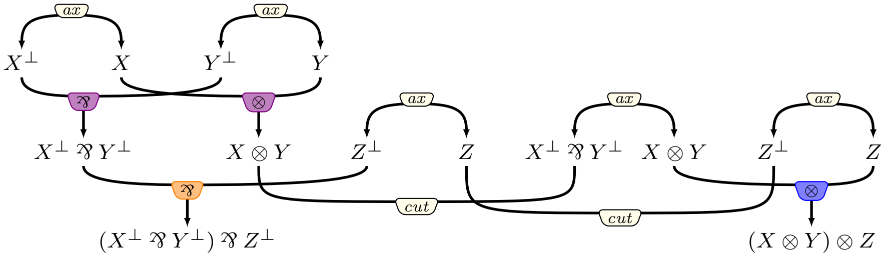
⅋
⊗
$(X^\perp ⅋ Y^\perp) ⅋ Z^\perp$
$(X \otimes Y) \otimes Z$
$(X^\perp ⅋ Y^\perp) ⅋ Z^\perp$
$(X \otimes Y) \otimes Z$
cut

Cut elimination becomes local



Cut elimination becomes local



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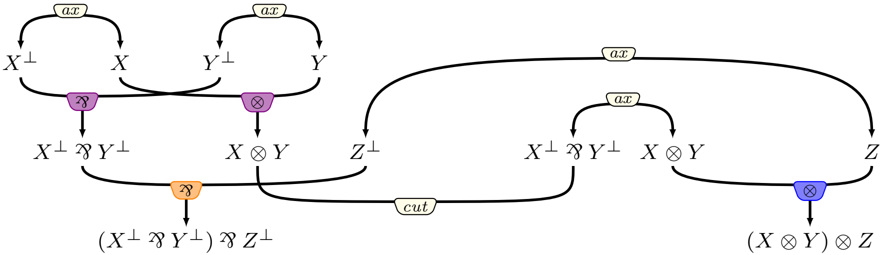
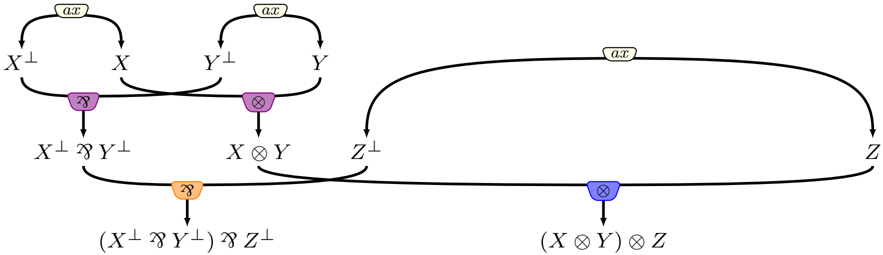
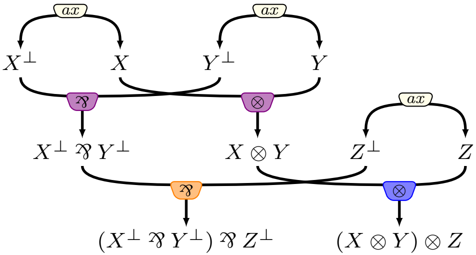


Cut elimination becomes local



Cut elimination becomes local

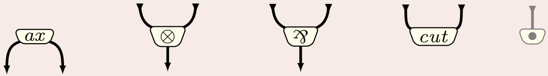


Multiplicative proof structures

Definition

A **multiplicative proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes



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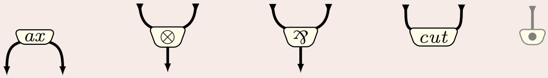
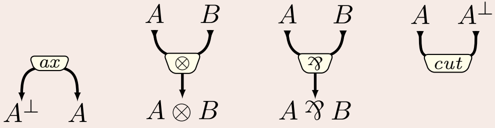


- a labelling of edges with MLL formulas, compatible with typing rules



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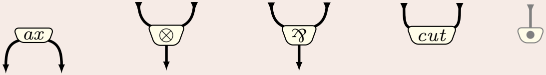
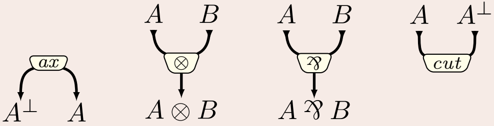
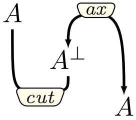


together with an ordering of the two input edges of each $\otimes$ and $\parr$ (one left, one right)

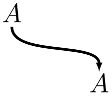
- a labelling of edges with MLL formulas, compatible with typing rules

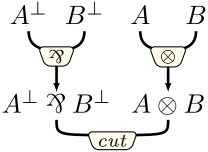


Cut elimination in multiplicative proof structures

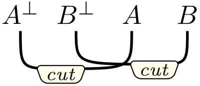


$\rightsquigarrow$

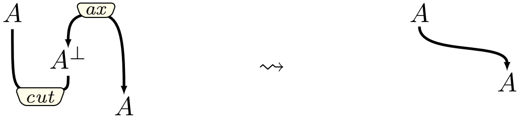




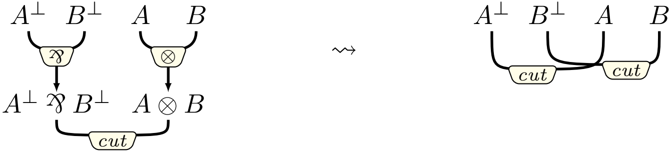
$\rightsquigarrow$



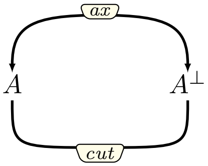
Cut elimination in multiplicative proof structures



(provided the in/out edges do not coincide)

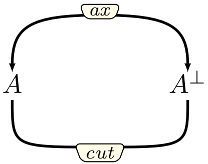


Cycles



This cut cannot be eliminated!

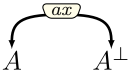
Cycles



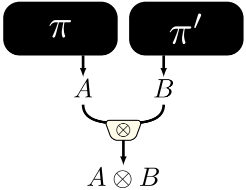
This cut cannot be eliminated!
But this is not the translation of a proof.

Translation of proofs

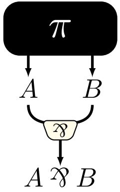
$$\frac{}{A, A^{\perp}}\ (ax) \quad \mapsto$$



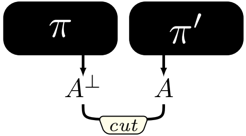
$$\frac{\overset{\pi}{\Gamma, A} \qquad \overset{\pi'}{\Delta, B}}{\Gamma, \Delta, A \otimes B}\ (\otimes) \quad \mapsto$$



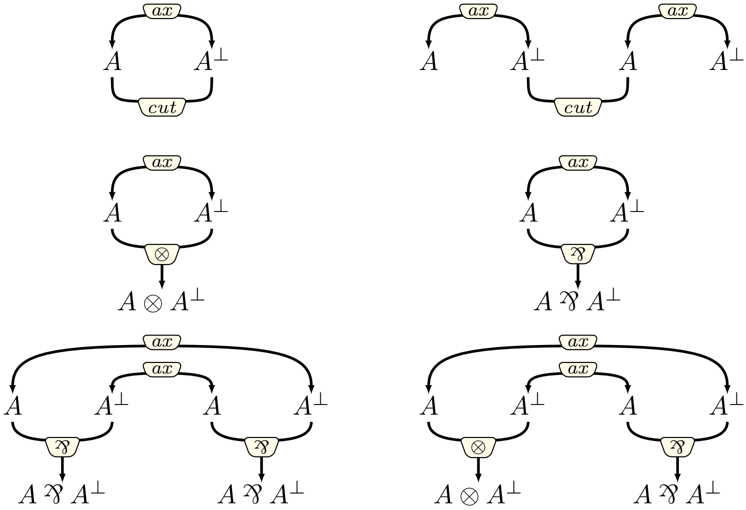
$$\frac{\overset{\pi}{\Gamma, A, B}}{\Gamma, A \parr B}\ (\parr) \quad \mapsto$$

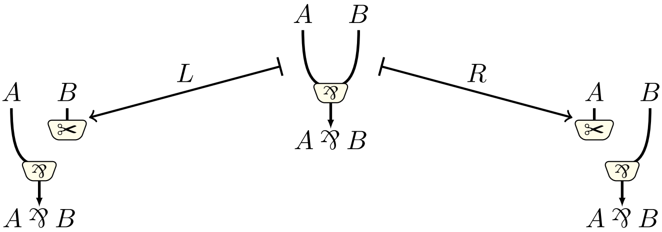


$$\frac{\overset{\pi}{\Gamma, A} \qquad \overset{\pi'}{\Delta, A^{\perp}}}{\Gamma, \Delta}\ (cut) \quad \mapsto$$



Some proof structures





Definition

- a **switching** σ of a structure S, is the choice of L or R for each ⅋ of S
- the **switching graph** S_σ is obtained by replacing each ⅋ accordingly

Multiplicative proof nets

Definition

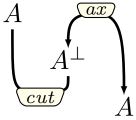
- A (multiplicative) **proof net** is a (multiplicative) proof structure whose switching graphs are all (undirected) acyclic.
- A **connected proof net** is a proof net whose switching graphs have exactly one (undirected) connected component.

In proof nets, each cut can be eliminated.

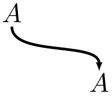
Theorem

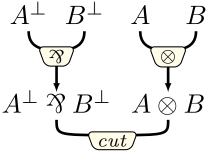
Proof nets (resp. connected proof nets) are stable under cut elimination.

Cut elimination in multiplicative proof structures

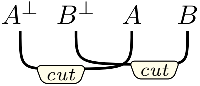


$\rightsquigarrow$





$\rightsquigarrow$



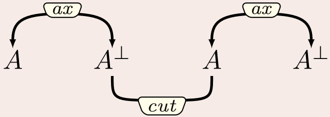
Properties of cut elimination

Lemma

Cut elimination is strongly confluent (up to graph isomorphism).

Proof.

The only critical pair is



□

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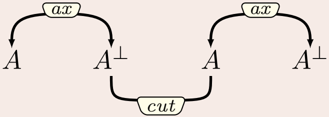
The only critical pair is 

□

Lemma

Cut elimination is strongly normalizing.

Proof.

The size of nets decreases when eliminating cuts. □

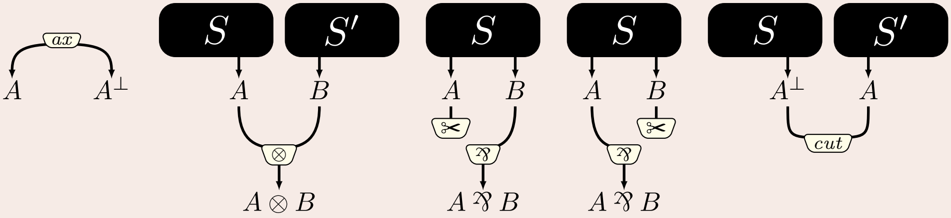
Correctness of the translation

Lemma

The translation of a proof is always a connected proof net.

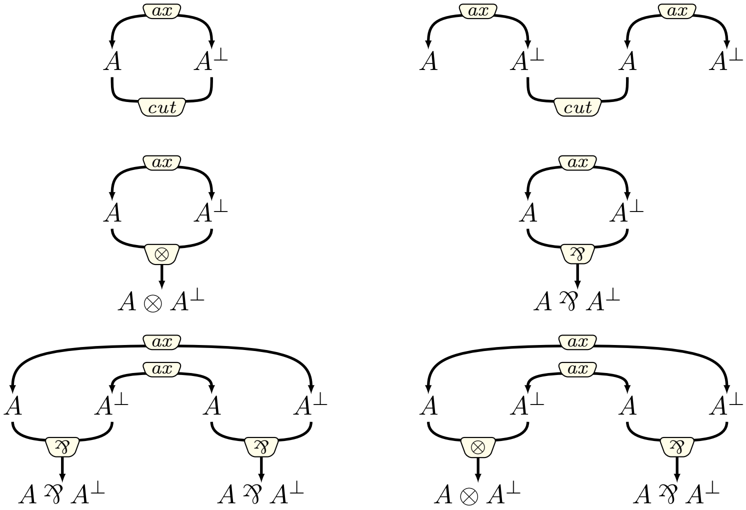
Proof.

The following graphs are (undirected) connected and acyclic as soon as S and S' are:



□

Some proof structures



Sequentialization

Theorem

A proof structure is a connected proof net iff it is the translation of some proof.

Proof.

By induction on the size of the proof structure S:

- replace each $\boxed{\mathit{cut}}$ with $\boxed{\otimes}$
- erase each $\boxed{\parr}$ above conclusions
- then:
  - either S is just an axiom
  - or it "suffices" to find a $\boxed{\otimes}$ above a conclusion, splitting S in two, and conclude by induction hypothesis

□

Units

MLL with units:

$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid \mathbf{1} \mid \perp$$

$$\mathbf{1}^\perp := \perp \quad \perp^\perp := \mathbf{1}$$

Units

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$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \mathbin{\rotatebox[origin=c]{180}{\&}} B \mid \mathbf{1} \mid \perp$$

$$\mathbf{1}^\perp := \perp \qquad \perp^\perp := \mathbf{1}$$

$$\frac{}{\mathbf{1}}\ (\mathbf{1}) \qquad\qquad \frac{\Gamma}{\Gamma, \perp}\ (\perp)$$

Units

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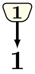
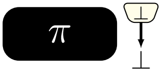


$$\frac{\begin{array}{c}\pi \\ \Gamma\end{array}}{\Gamma, \perp}\ (\perp) \quad \mapsto$$



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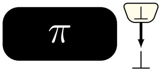

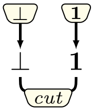

$$\rightsquigarrow \qquad \emptyset$$

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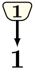
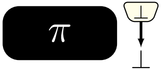
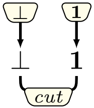


$$\frac{\begin{matrix}\pi \\ \Gamma\end{matrix}}{\Gamma, \perp}\ (\perp) \quad \mapsto$$



Breaks connectedness!



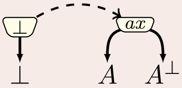
$$\leadsto \qquad \emptyset$$

Option 1: restoring connectedness

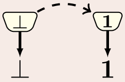
Definition

A *multiplicative proof structure with jumps* is:

- a proof structure with units
- a jump from each $\perp$ to some initial node:

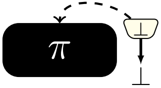


or



- Change the translation:

$$\frac{\overset{\pi}{\Gamma}}{\Gamma, \perp}\ (\perp) \quad \mapsto$$



- Switching graphs include jumps as edges.

Jumps are not canonical

Theorem

A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

Jumps are not canonical

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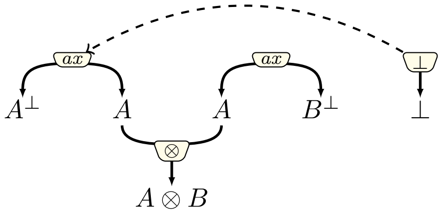
A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

But:

- There is no canonical translation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A^\perp, A}\,(ax) \qquad \dfrac{}{B, B^\perp}\,(ax)}{A^\perp, A \otimes B, B^\perp}\,(\otimes)}{A^\perp, A \otimes B, B^\perp, \perp}\,(\perp)}{} \quad \mapsto$$



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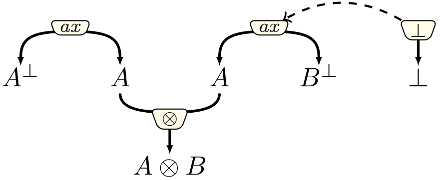


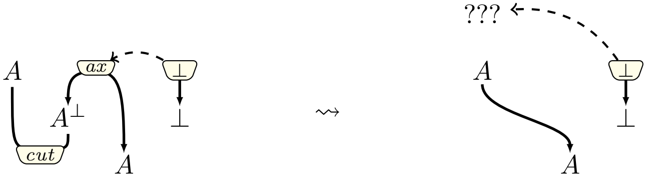
Jumps are not canonical

Theorem

A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

But:

- Cut elimination needs some care:



Option 2: acyclicity only

The mix rules:

$$\frac{}{\varepsilon}\,(mix_0) \quad\mapsto\quad \emptyset \qquad\qquad \frac{\overset{\pi}{\Gamma} \qquad \overset{\pi'}{\Delta}}{\Gamma, \Delta}\,(mix) \quad\mapsto\quad \boxed{\pi}\;\boxed{\pi'}$$

Theorem

A proof structure is a proof net iff it is the translation of a proof of $\mathsf{MLL} + (mix_0) + (mix)$.

Same switching graphs, same cut elimination with or without units.

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Stumbling block

Identifying proofs of MLL with units up to rule permutations is hard.

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Stumbling block

Identifying proofs of MLL with units up to rule permutations is hard.

We stick to option 2.

Exponentials

MELL:

$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

$$!A^\perp := ?A^\perp \quad ?A^\perp := !A^\perp$$

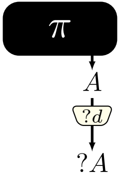
Exponentials

MELL:

$$A, B, C, \ldots ::= X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

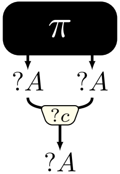
$$!A^{\perp} := ?A^{\perp} \quad ?A^{\perp} := !A^{\perp}$$

$$\dfrac{\overset{\pi}{\Gamma, A}}{\Gamma, ?A}\ (?d) \quad \mapsto$$

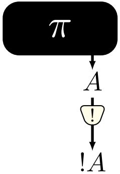


$$\dfrac{\overset{\pi}{\Gamma}}{\Gamma, ?A}\ (?w) \quad \mapsto$$

$$\dfrac{\overset{\pi}{\Gamma, ?A, ?A}}{\Gamma, ?A}\ (?c) \quad \mapsto$$

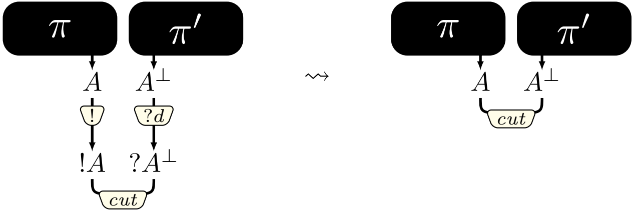


$$\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\ (!) \quad \mapsto$$



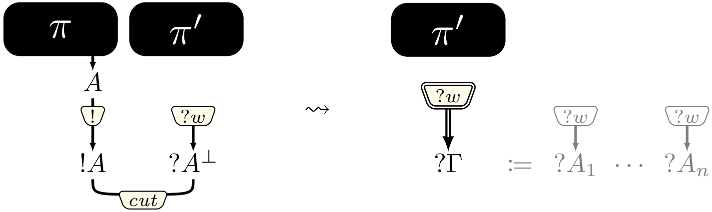
Cut elimination in MELL

$$\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\overset{\pi'}{A^{\perp}, \Delta}}{?A^{\perp}, \Delta}\,(?d)}{?\Gamma, \Delta}\,(cut) \qquad \rightsquigarrow \qquad \dfrac{\overset{\pi}{?\Gamma, A} \qquad \overset{\pi'}{A^{\perp}, \Delta}}{?\Gamma, \Delta}\,(cut)$$



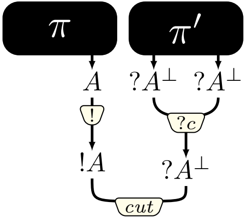
Cut elimination in MELL

$$\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\overset{\pi'}{\Delta}}{?A^\perp, \Delta}\,(?w)}{?\Gamma, \Delta}\,(cut) \quad \rightsquigarrow \quad \dfrac{\overset{\pi'}{\Delta}}{\overline{?\Gamma, \Delta}}\,(?w)$$

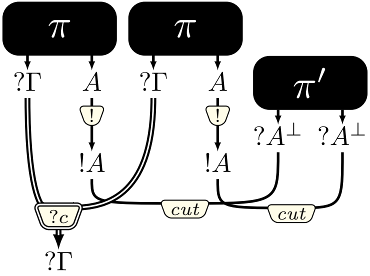


Cut elimination in MELL

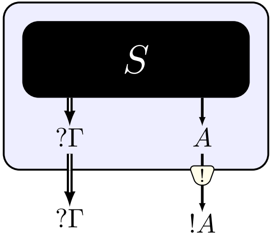
$$\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\overset{\pi'}{?A^{\perp}, ?A^{\perp}, \Delta}}{?A^{\perp}, \Delta}\,(?c)}{?\Gamma, \Delta}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \overset{\pi'}{?A^{\perp}, ?A^{\perp}, \Delta}}{?\Gamma, ?A^{\perp}, \Delta}\,(cut)}{?\Gamma, ?\Gamma, \Delta}\,(cut)}{?\Gamma, \Delta}\,(?c)$$



$\rightsquigarrow$



Promotion boxes

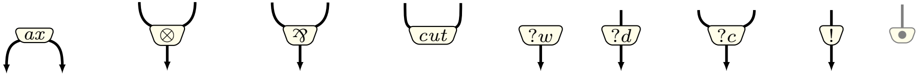


- amounts to every node of S (except for ?-conclusions) jumping to !
- boxes must be either disjoint or nested
- allows to treat the box as one initial node

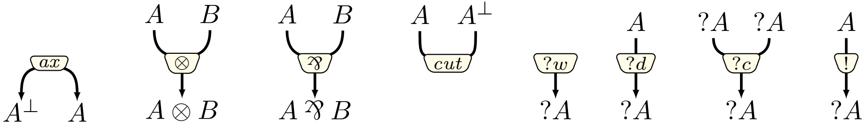
MELL proof structures

An **MELL proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

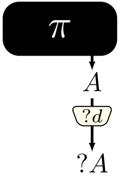


  together with an ordering of the two input edges of each $\otimes$ and $⅋$ (one left, one right)
- a labelling of edges with MELL formulas, compatible with typing rules



- a tree order on $!$ nodes (+ a root for top level) together with a map from all nodes:
  - edges are monotone and conclusions are at top level
  - the only edges that change box depth are ?-typed (crossing any number of boxes) or the premisses of $!$ nodes (crossing exactly one box)

$$\dfrac{\overset{\pi}{\Gamma, A}}{\Gamma, ?A}\ (?d) \quad \mapsto$$



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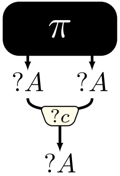

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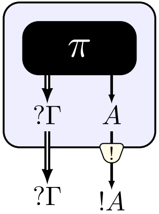
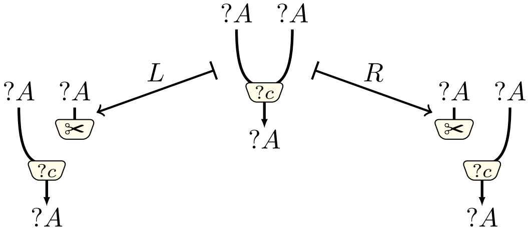
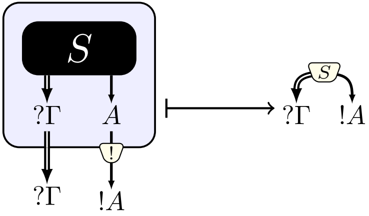


Switchings





Definition

- a **switching** σ of a structure S, is the choice of L or R for each ⅋ or $?c$ of S
- the **switching graph** S_σ is obtained by replacing each ⅋ or $?c$ accordingly, and each box with an initial node

MELL proof nets

Definition

A **proof net** is a proof structure such that:

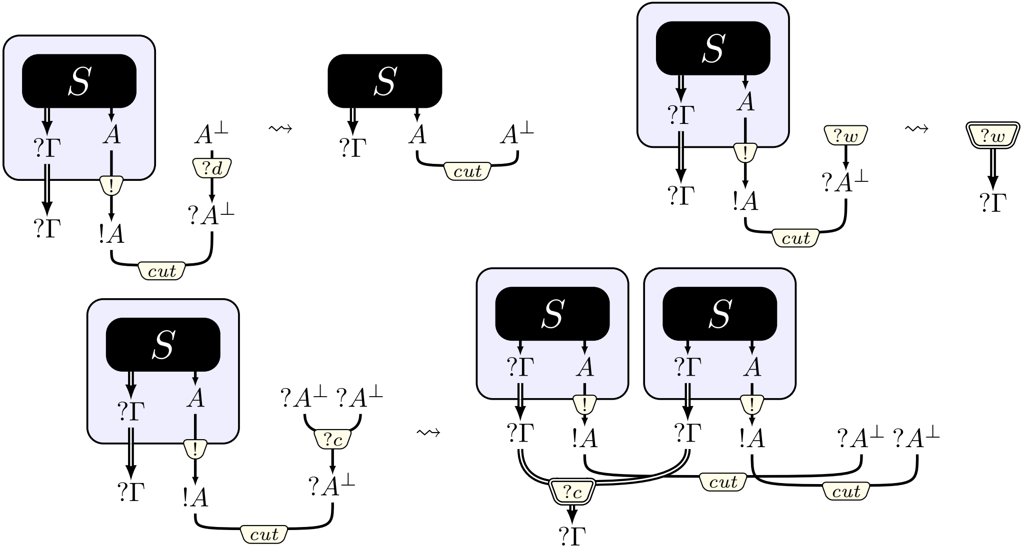
- all switching graphs are (undirected) acyclic,
- the content of each box is a proof net, inductively.

In proof nets, each cut can be eliminated.

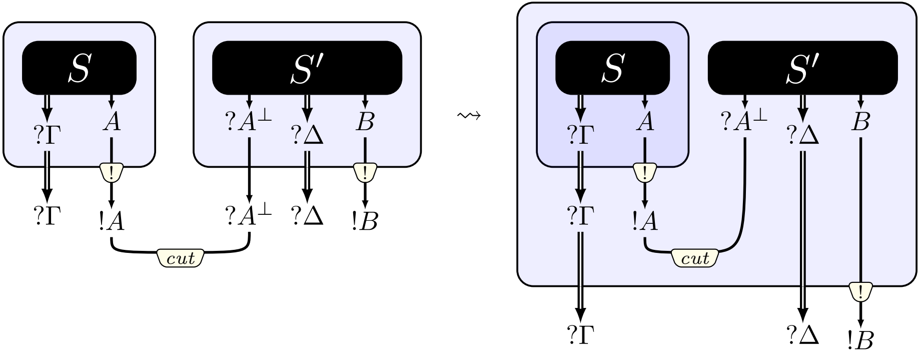
Theorem

Proof nets are stable under cut elimination.

Cut elimination in MELL, with boxes



Cut elimination in MELL, with boxes



Properties of cut elimination

Theorem

Cut elimination is confluent and strongly normalizing.

Proof.

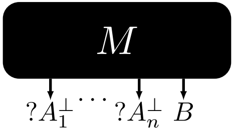
There are some papers and theses... $\square$

Translating the λ-calculus

One famous translation:

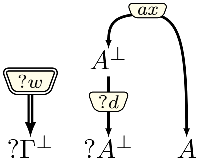
$$(A \to B) \mapsto (!A \multimap B)$$

Given $\Gamma \vdash M : B$ with $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ we construct

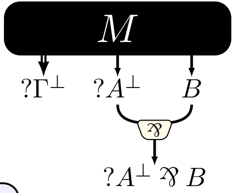


Translating the λ-calculus

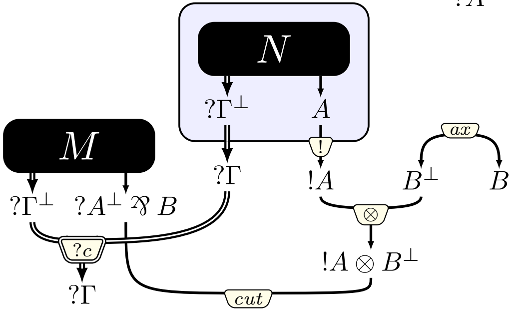
$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \mapsto$$



$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (abs) \mapsto$$

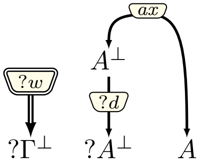


$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (app) \quad \mapsto$$

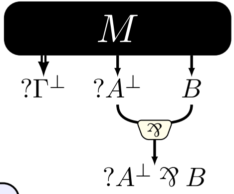


Translating the λ-calculus: $\Lambda \to \mathsf{NJ} \to \mathsf{MELL}$

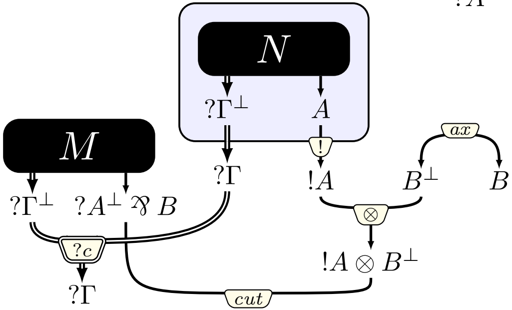
$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \mapsto$$



$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (abs) \mapsto$$

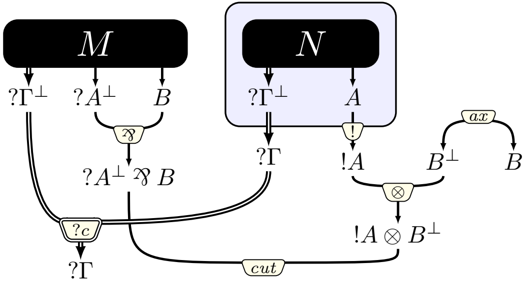


$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (app) \quad \mapsto$$

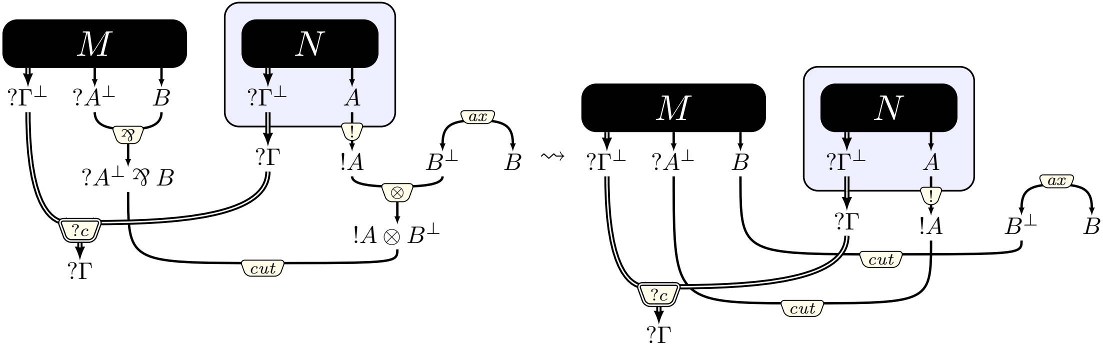


Simulating β-reduction

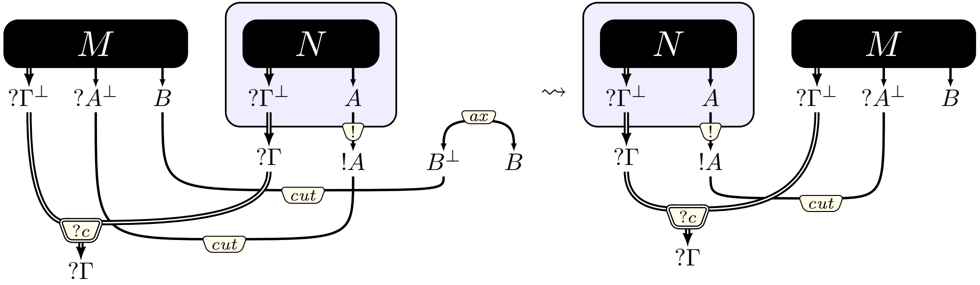
$$\dfrac{\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\ (abs) \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x.M)N : B}\ (app) \mapsto$$



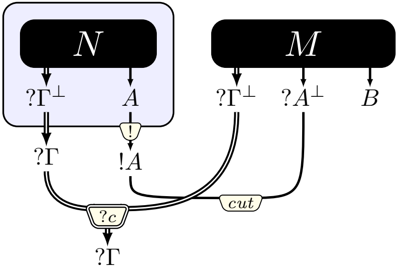
Simulating β-reduction



Simulating β-reduction



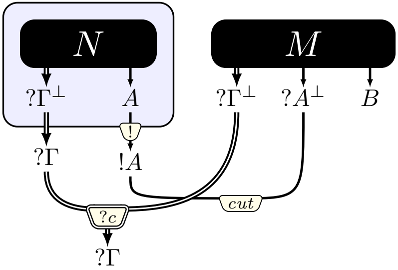
Simulating β-reduction



Lemma

Iterating exponential cut elimination in the above net yields the translation of $M[N/x]$.

Simulating β-reduction



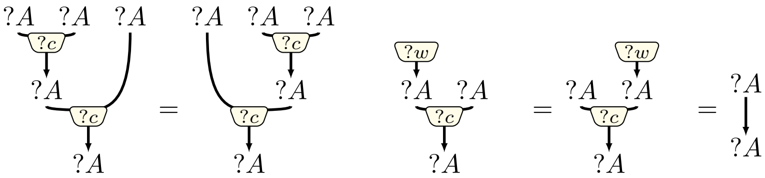
Lemma

Iterating exponential cut elimination in the above net yields the translation of $M[N/x]$.

In fact, not quite…

Rétoré conversions and ?-canonical nets

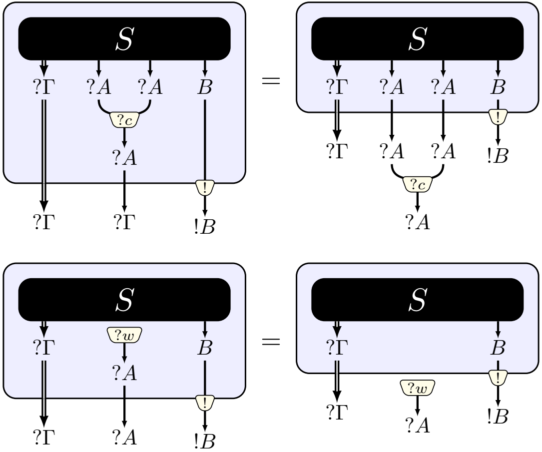
Contraction and weakening form a commutative monoid:



We should thus consider n-ary contractions.

Rétoré conversions and ?-canonical nets

Contraction and weakening cross the border of boxes:



Rétoré conversions and ?-canonical nets

- up to these identities, cut elimination refines β-reduction
- with n-ary contractions, normal forms for the natural orientation of these identities are **?-canonical nets** (the old name for this is *nouvelle syntaxe*)
- cut elimination can be defined directly on ?-canonical nets

Conclusion: what was not in this tutorial?

- correctness criteria with better complexity
- an explanation of why there are no satisfactory proof nets with weakening
- additives, quantifiers,
- pure types for simulating the untyped λ-calculus
- polarized / intuitionistic variants
- differential nets and Taylor expansion
- explicit substitutions
- denotational semantics (experiments)
- geometry of interaction
- *etc.*

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Have a nice TLLA 2021!