Proof nets
A tutorial

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A (partial, biased) tutorial on the basics of proof nets

from the distant 80’s
A (partial, biased) tutorial on the basics of proof nets

- from the distant 80’s to the dazzling 90’s
A (partial, biased) tutorial on the basics of proof nets

▶ from the distant 80’s to the dazzling 90’s

▶ only prerequisites: MELL sequent calculus + a bit of graph theory (paths, cycles)
A (partial, biased) tutorial on the basics of proof nets

- from the distant 80’s to the dazzling 90’s

- only prerequisites: MELL sequent calculus + a bit of graph theory (paths, cycles)
- we should have a book for this
\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \]

\[ X^{\perp \perp} ::= X \quad (A \otimes B)^\perp ::= A^\perp \parr B^\perp \quad (A \parr B)^\perp ::= A^\perp \otimes B^\perp \]

\[ \frac{}{\Gamma, A \quad \Delta, A^\perp} (\text{cut}) \]

\[ \frac{\Gamma, A \quad B}{} (\text{ax}) \]

\[ \frac{\Gamma, A \quad B}{} (\parr) \]

\[ \frac{\Gamma, A \quad \Delta, B}{} (\otimes) \]
\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \& B \]

\[ X^\perp \perp ::= X \quad (A \otimes B)^\perp ::= A^\perp \& B^\perp \quad (A \& B)^\perp ::= A^\perp \otimes B^\perp \]

\[
\begin{array}{c}
\frac{}{A^\perp, A} \quad (ax) \\
\frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \quad (cut)
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma, A, B}{\Gamma, A \& B} \quad (\&) \\
\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \quad (\otimes)
\end{array}
\]

\[
\frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \quad (ex)
\]
An example

\[
\begin{align*}
X^\perp, X & \quad (ax) \\
Y^\perp, Y & \quad (ax) \\
\overline{X^\perp, Y^\perp, X \otimes Y} & \quad (\otimes) \\
\overline{X^\perp, Y^\perp, Z, X \otimes Y, Z} & \quad (\wedge) \\
\overline{Z^\perp, Z} & \quad (ax) \\
\overline{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} & \quad (\otimes) \\
\overline{(X^\perp, Y^\perp) \wedge Z^\perp, (X \otimes Y) \otimes Z} & \quad (\wedge)
\end{align*}
\]
An example

\[
\frac{X^\perp, X}{Y^\perp, Y} \quad (ax) \quad \frac{X^\perp, Y^\perp, X \otimes Y}{(\otimes)}
\]

\[
\frac{X^\perp \otimes Y^\perp, Z^\perp, X \otimes Y, Z}{(\otimes)} \quad \frac{Z^\perp, Z}{(ax)}
\]

\[
\frac{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(\otimes)} \quad \frac{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z}{(\otimes)}
\]

\[
(A \rightarrow B := A^\perp \otimes \otimes B)
\]
An example

\[
\begin{align*}
\frac{X \perp X}{X, X} & \quad (ax) & \frac{Y \perp Y}{Y, Y} & \quad (ax) \\
\frac{X \perp Y, X \otimes Y}{X, Y, X \otimes Y} & \quad (\otimes) \\
\frac{X \perp Z, X \otimes Y, Z}{X \perp Y, Z, X \otimes Y, Z} & \quad (\supset) \\
\frac{Z \perp Z, Z \otimes Z}{(X \otimes Y) \otimes Z} & \quad (\otimes) \\
\frac{(X \otimes Y) \otimes Z}{(X \otimes Y) \otimes Z} & \quad (\supset) \\
\end{align*}
\]

\((A \rightarrow B := A \perp \supset B)\)
Some rules permute

\[
\begin{array}{c}
X^\perp, X \quad Y^\perp, Y \\
\hline
X^\perp, Y^\perp, X \otimes Y \quad Z^\perp, Z
\end{array}
\]

\[\otimes\]

\[
\begin{array}{c}
X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z \\
\hline
X^\perp \nabla Y^\perp, Z^\perp, (X \otimes Y) \otimes Z
\end{array}
\]

\[\nabla\]

\[
\begin{array}{c}
X^\perp \nabla Y^\perp \nabla Z^\perp, (X \otimes Y) \otimes Z
\end{array}
\]

\[\nabla\]

\[
\begin{array}{c}
X^\perp, X \quad Y^\perp, Y \\
\hline
X^\perp, Y^\perp, X \otimes Y \quad Z^\perp, Z
\end{array}
\]

\[\otimes\]

\[
\begin{array}{c}
X^\perp \nabla Y^\perp, Z^\perp, (X \otimes Y) \otimes Z \\
\hline
(X^\perp \nabla Y^\perp) \nabla Z^\perp, (X \otimes Y) \otimes Z
\end{array}
\]

\[\nabla\]
Cut elimination requires non-logical steps

\[
\frac{X^\perp, X}{X^\perp, Y, X \otimes Y} \quad \frac{Y^\perp, Y}{Z^\perp, Z} \\
\quad \frac{X^\perp, Y^\perp, X \otimes Y \otimes Z}{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad \frac{X^\perp \otimes Y^\perp, X \otimes Y \otimes Z}{Z^\perp, Z} \\
\quad \frac{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z} \quad \frac{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z}
\]
Cut elimination requires non-logical steps

\[
\begin{array}{c}
\frac{X \perp, X}{X \perp, Y \perp, X \otimes Y} \quad (ax) \\
\frac{Y \perp, Y}{Z \perp, Z} \quad (\otimes) \\
\frac{X \perp, Y \perp, Z \perp, (X \otimes Y) \otimes Z}{X \perp \otimes Y \perp, Z \perp, (X \otimes Y) \otimes Z} \quad (\otimes) \\
\frac{(X \perp \otimes Y \perp) \otimes Z \perp, (X \otimes Y) \otimes Z}{(X \perp \otimes Y \perp) \otimes Z \perp, (X \otimes Y) \otimes Z} \quad (\otimes) \\
\frac{X \perp \otimes Y \perp, Z \perp, (X \otimes Y) \otimes Z}{(X \perp \otimes Y \perp) \otimes Z \perp, (X \otimes Y) \otimes Z} \quad (\otimes) \\
\end{array}
\]
Cut elimination requires non-logical steps

\[
\begin{align*}
\frac{X^\perp, X}{X^\perp, Y^\perp, X \otimes Y} & \quad \frac{Y^\perp, Y}{Z^\perp, Z} \\
& \frac{X^\perp, Y^\perp, X \otimes Y}{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \\
& \frac{Z^\perp, Z}{(X \otimes Y)^{\perp}} \\
& \frac{(X \otimes Y)^{\perp}}{(X \otimes Y)^{\perp}, (X \otimes Y) \otimes Z} \\
\end{align*}
\]
Cut elimination requires non-logical steps

\[
\frac{X^\perp, X}{X^\perp, X^\perp, X \otimes Y} \quad (ax) \quad \frac{Y^\perp, Y}{Y^\perp, Y^\perp, Y \otimes Y} \quad (ax) \quad \frac{Z^\perp, Z}{Z^\perp, Z^\perp, Z \otimes Z} \quad (ax)
\]

\[
\frac{X^\perp \otimes Y^\perp, X \otimes Y}{X^\perp \otimes Y^\perp, Z \otimes Z, (X \otimes Y) \otimes Z} \quad (cut) \quad \frac{Z^\perp, Z}{Z^\perp, Z^\perp, Z \otimes Z} \quad (cut)
\]

\[
\frac{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\]

\[
\frac{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\]

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Cut elimination requires non-logical steps

or maybe:

\[
\begin{array}{c}
\frac{X^\perp, X}{X^\perp, Y^\perp, X \otimes Y} \quad (ax) \\
\frac{Y^\perp, Y}{X^\perp, Y^\perp, X \otimes Y} \quad (ax) \\
\frac{Z^\perp, Z}{X^\perp, Y^\perp, X \otimes Y} \quad (ax)
\end{array}
\]

\[
\frac{X^\perp \land Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \land Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\]

\[
\frac{X^\perp \land Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \land Y^\perp) \land Z^\perp, (X \otimes Y) \otimes Z} \quad (\land)
\]

\[
\frac{(X^\perp \land Y^\perp) \land Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \land Y^\perp) \land Z^\perp, (X \otimes Y) \otimes Z} \quad (\land)
\]
A multiplicative proof

\[
\begin{array}{c}
\begin{array}{c}
\frac{X^\perp, X}{X^\perp, Y^\perp, X \otimes Y} \quad (\text{ax}) \\
\frac{X^\perp, Y^\perp, X \otimes Y}{X^\perp, \forall Y^\perp, Z^\perp, X \otimes Y, Z} \quad (\forall) \\
\frac{X^\perp, \forall Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp, \forall Y^\perp) \forall Z^\perp, (X \otimes Y) \otimes Z} \quad (\forall)
\end{array}
\end{array}
\]
A multiplicative proof net

\[
\begin{array}{c}
\frac{X^\perp}{X^\perp, X} \quad \frac{Y^\perp}{Y^\perp, Y} \\
\frac{X^\perp, Y^\perp, X \otimes Y}{X^\perp \otimes Y^\perp, X \otimes Y, Z} \\
\frac{X^\perp \otimes Y^\perp, Z^\perp, X \otimes Y, Z}{(X^\perp \otimes Y^\perp) \otimes Z, (X \otimes Y) \otimes Z}
\end{array}
\]

\[
\begin{array}{c}
\frac{\gamma}{\gamma} \\
\frac{\gamma}{\gamma} \\
\frac{\gamma}{\gamma}
\end{array}
\]

\[\mapsto\]

\[
\begin{array}{c}
\frac{X^\perp}{X^\perp} \quad \frac{Y^\perp}{Y^\perp} \\
\frac{X^\perp \otimes Y^\perp}{X \otimes Y} \\
\frac{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \otimes Y^\perp) \otimes Z, (X \otimes Y) \otimes Z}
\end{array}
\]

\[
\begin{array}{c}
\frac{\gamma}{\gamma} \\
\frac{\gamma}{\gamma} \\
\frac{\gamma}{\gamma}
\end{array}
\]

\[\mapsto\]

\[
\begin{array}{c}
\frac{X^\perp \otimes Y^\perp}{X \otimes Y} \\
\frac{(X^\perp \otimes Y^\perp) \otimes Z^\perp}{(X \otimes Y) \otimes Z}
\end{array}
\]

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Proof nets

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A multiplicative proof net

\[
\begin{align*}
\frac{X^\perp, X}{X^\perp, Y, Y} \quad (\text{ax})
\end{align*}
\]

\[
\begin{align*}
\frac{Y^\perp, Y}{X^\perp, Y^\perp, X \otimes Y} \quad (\otimes)
\end{align*}
\]

\[
\begin{align*}
\frac{Z^\perp, Z}{X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\end{align*}
\]

\[
\begin{align*}
\frac{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z}{(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\end{align*}
\]

\[
\begin{align*}
\frac{X^\perp \otimes Y^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\end{align*}
\]

\[
\begin{align*}
\frac{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z}{X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z} \quad (\otimes)
\end{align*}
\]
Cut

\[
\begin{array}{c}
X^\perp, X \quad (\text{ax}) \\
\overline{\; Y^\perp, Y \; (\text{ax})}
\end{array} \quad \begin{array}{c}
X^\perp, Y^\perp, X \otimes Y \\
\overline{\; \otimes \; (\text{\otimes})}
\end{array} \quad \begin{array}{c}
Z^\perp, Z \quad (\text{ax})
\end{array}
\]

\[
\begin{array}{c}
X^\perp, Y^\perp, Z^\perp, (X \otimes Y) \otimes Z \\
\overline{\; \text{\otimes} \; (\text{\otimes})}
\end{array} \quad \begin{array}{c}
X^\perp \otimes Y^\perp, X \otimes Y \\
\overline{\; Z^\perp, Z \; (\text{ax})}
\end{array} \quad \begin{array}{c}
\text{cut}
\end{array}
\]

\[
\begin{array}{c}
X^\perp \otimes Y^\perp, Z^\perp, (X \otimes Y) \otimes Z \\
\overline{\; \text{\otimes} \; (\text{\otimes})}
\end{array} \quad \begin{array}{c}
\overline{\; \text{\otimes} \; (\text{\otimes})}
\end{array}
\]

\[
\begin{array}{c}
(X^\perp \otimes Y^\perp) \otimes Z^\perp, (X \otimes Y) \otimes Z \\
\overline{\; \text{cut} \; (\text{\otimes})}
\end{array}
\]

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Cut elimination becomes local

\[ X \perp Y \perp X \]

\[ X \perp Y \perp X \otimes Y \]

\[ X \perp Y \perp X \otimes Y \otimes Z \]

\[ (X \perp Y \perp Z) \perp Z \perp (X \otimes Y) \otimes Z \]

\[ (X \perp Y \perp Z) \perp Z \perp (X \otimes Y) \otimes Z \]

\[ (X \perp Y \perp Z) \perp Z \perp (X \otimes Y) \otimes Z \]
Cut elimination becomes local
Cut elimination becomes local

\[ X \perp \otimes Y \perp \]
Cut elimination becomes local

\[ X \perp \& Y \perp \rightarrow X \& Y \rightarrow X \& Y \triangleleft Z \rightarrow (X \& Y) \triangleleft Z \]

\[ X \& Y \rightarrow X \& Y \triangleleft Z \rightarrow (X \& Y) \triangleleft Z \]

\[ X \& Y \rightarrow X \& Y \triangleleft Z \rightarrow (X \& Y) \triangleleft Z \]

\[ X \& Y \rightarrow X \& Y \triangleleft Z \rightarrow (X \& Y) \triangleleft Z \]

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Cut elimination becomes local

\[ X \perp \otimes Y \perp \]

\[ (X \perp \otimes Y \perp) \otimes Z \perp \]

\[ (X \otimes Y) \otimes Z \]
Cut elimination becomes local

\[
\begin{align*}
X &\perp X \\
\alpha x &
\end{align*}
\]

\[
\begin{align*}
Y &\perp Y \\
\alpha x &
\end{align*}
\]

\[
\begin{align*}
X &\otimes Y \\
\alpha x &
\end{align*}
\]

\[
\begin{align*}
Z &\perp Z \\
\alpha x &
\end{align*}
\]

\[
\begin{align*}
(X \perp \gamma Y \perp) &\otimes Z \perp \\
\alpha x &
\end{align*}
\]

\[
\begin{align*}
(X \otimes Y) &\otimes Z \\
\alpha x &
\end{align*}
\]
Definition

A **multiplicative proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

\[\begin{align*}
\text{ax} & \quad \times & \quad \neg & \quad \text{cut} & \quad \text{end}
\end{align*}\]
A **multiplicative proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

- a labelling of edges with **MLL** formulas, compatible with typing rules
A **multiplicative proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

- together with an ordering of the two input edges of each $\otimes$ and $\multimap$ (one left, one right)

- a labelling of edges with MLL formulas, compatible with typing rules
Cut elimination in multiplicative proof structures

\[
\begin{align*}
A & \quad \text{\textit{cut}} \quad A \quad \text{\textit{cut}} \quad A \\
A & \quad \text{\textit{ax}} \quad A \quad \text{\textit{cut}} \quad A
\end{align*}
\]

\[
\begin{align*}
A & \quad \text{\textit{cut}} \quad A \\
A & \quad \text{\textit{cut}} \quad A \\
A & \quad \text{\textit{cut}} \quad A
\end{align*}
\]
Cut elimination in multiplicative proof structures

(provided the in/out edges do not coincide)
This cut cannot be eliminated!
This cut cannot be eliminated!
But this is not the translation of a proof.
Translation of proofs

\[
\frac{A, A \bot}{ax}\quad \rightarrow\quad \frac{\pi}{\Gamma, A} \frac{\pi'}{\Delta, B} \frac{\otimes}{\Gamma, \Delta, A \otimes B} (\otimes) \quad \rightarrow
\]

\[
\frac{\pi}{\Gamma, A, B} \frac{\gamma}{\Gamma, A \gamma B} (\gamma) \quad \rightarrow\quad \frac{\pi}{A} \frac{\pi'}{B} \frac{\gamma}{A \gamma B} \frac{\otimes}{A \otimes B} (cut) \quad \rightarrow
\]

\[
\frac{\pi}{\Gamma, A} \frac{\pi'}{\Delta, A \bot} (cut) \quad \rightarrow
\]
Some proof structures
Definition

- a **switching** $\sigma$ of a structure $S$, is the choice of $L$ or $R$ for each $\otimes$ of $S$
- the **switching graph** $S_\sigma$ is obtained by replacing each $\otimes$ accordingly
Definition

- A (multiplicative) proof net is a (multiplicative) proof structure whose switching graphs are all (undirected) acyclic.
- A connected proof net is a proof net whose switching graphs have exactly one (undirected) connected component.

In proof nets, each cut can be eliminated.

Theorem

Proof nets (resp. connected proof nets) are stable under cut elimination.
Cut elimination in multiplicative proof structures

\[
A \vdash \bot \Rightarrow A
\]

\[
A \otimes B \Rightarrow A \otimes B
\]

\[
A \vdash \bot \Rightarrow B \vdash \bot \Rightarrow A \otimes B
\]

\[
A \vdash \bot \Rightarrow B \vdash \bot \Rightarrow A \otimes B
\]

\[
A \vdash \bot \Rightarrow B \vdash \bot \Rightarrow A \otimes B
\]

\[
A \vdash \bot \Rightarrow B \vdash \bot \Rightarrow A \otimes B
\]

\[
A \vdash \bot \Rightarrow B \vdash \bot \Rightarrow A \otimes B
\]
Properties of cut elimination

Lemma

*Cut elimination is strongly confluent (up to graph isomorphism).*

Proof.

The only critical pair is $A \vdash A \perp, A \perp \vdash A$. 

\[
\begin{array}{c}
A \\
\downarrow \text{ax}
\end{array} \quad \begin{array}{c}
A \perp \\
\downarrow \text{cut}
\end{array} \\
\begin{array}{c}
A \\
\downarrow \text{ax}
\end{array} \quad \begin{array}{c}
A \perp \\
\downarrow \text{cut}
\end{array}
\]
Properties of cut elimination

Lemma

Cut elimination is strongly confluent (up to graph isomorphism).

Proof.
The only critical pair is

\[
\begin{array}{c}
A \\
\text{at} \\
\text{cut} \\
A \perp \\
A \\
\text{at} \\
A \perp
\end{array}
\]

Lemma

Cut elimination is strongly normalizing.

Proof.
The size of nets decreases when eliminating cuts.
Correctness of the translation

Lemma

The translation of a proof is always a connected proof net.

Proof.

The following graphs are (undirected) connected and acyclic as soon as $S$ and $S'$ are:
Some proof structures

- Proof structure 1
  - \( A \vdash A \perp \)
  - \( \text{cut} \)
- Proof structure 2
  - \( A \vdash A \perp \)
  - \( \text{ax} \)
- Proof structure 3
  - \( A \otimes A \perp \)
  - \( \text{ax} \)
  - \( \otimes \)
- Proof structure 4
  - \( A \vdash A \perp \)
  - \( \text{ax} \)
  - \( \otimes \)

- Proof structure 5
  - \( A \vdash A \perp \)
  - \( \text{ax} \)
  - \( \text{cut} \)
- Proof structure 6
  - \( A \vdash A \perp \)
  - \( \text{ax} \)
  - \( \text{cut} \)
- Proof structure 7
  - \( A \otimes A \perp \)
  - \( \text{ax} \)
  - \( \otimes \)
- Proof structure 8
  - \( A \vdash A \perp \)
  - \( \text{ax} \)
  - \( \otimes \)
**Theorem**

A proof structure is a connected proof net iff it is the translation of some proof.

**Proof.**

By induction on the size of the proof structure $S$:

- replace each $\text{cut}$ with $\otimes$
- erase each $\Rightarrow$ above conclusions
- then:
  - either $S$ is just an axiom
  - or it "suffices" to find a $\otimes$ above a conclusion, splitting $S$ in two, and conclude by induction hypothesis
Units

MLL with units:

\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid 1 \mid \bot \]

\[ 1^\perp ::= \bot \quad \bot^\perp ::= 1 \]
Units

MLL with units:

\[ A, B, C, \ldots ::= X | X^\perp | A \otimes B | A \land B | 1 | \bot \]

\[ 1^\perp ::= \bot \quad \bot^\perp ::= 1 \]

\[ \overline{1} \quad (1) \]

\[ \frac{\Gamma}{\Gamma, \bot} \quad (\bot) \]
Units

MLL with units:

\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \& B \mid 1 \mid \bot \]

\[ 1^\perp ::= \bot \quad \bot^\perp ::= 1 \]

\[ \overline{1} (1) \quad \Rightarrow \quad \begin{array}{c}
\pi \\
\Gamma, \bot \quad (\bot)
\end{array} \quad \Rightarrow \quad \boxed{\pi} \]

Breaks connectedness!
Units

MLL with units:

\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid 1 \mid \bot \]

\[ 1^\perp ::= \bot \quad \bot^\perp ::= 1 \]

\[ \to \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Gamma \quad 1
\end{array}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Gamma, \bot
\end{array}
\end{array}
\end{array} \]

\[ \Rightarrow \]

\[ \pi \]

\[ \downarrow \]

\[ \Rightarrow \]

\[ \emptyset \]

\[ \text{(cut)} \]
Units

MLL with units:

\[
A, B, C, \ldots ::= X \ | \ X^\perp \ | \ A \otimes B \ | \ A \Rightarrow B \ | \ 1 \ | \bot
\]

\[
1^\perp ::= \bot \quad \bot^\perp ::= 1
\]

Breaks connectedness!
Definition

A **multiplicative proof structure with jumps** is:

- a proof structure with units
- a jump from each $\bot$ to some initial node:

$\bot$  $A$ $A\bot$ or $\bot$ $\bot$ $1$

- Change the translation:

$$\frac{\pi \Gamma}{\Gamma, \bot} (\bot) \quad \mapsto \quad \pi \bot$$

- Switching graphs include jumps as edges.
Jumps are not canonical

**Theorem**

*A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.*
Jumps are not canonical

Theorem

A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

But:

- There is no canonical translation:

\[
\begin{align*}
A & \perp, A \\
A & \perp, A \otimes B, B & \perp \\
A & \perp, A \otimes B, B & \perp, \bot \\
A & \perp, A \otimes B, B & \perp, \bot \\
\end{align*}
\]

\[
\begin{align*}
A & \perp, A \\
A & \perp, A \otimes B, B & \perp \\
A & \perp, A \otimes B, B & \perp, \bot \\
A \otimes B & \\
A \otimes B & \\
A & \perp \\
A & \perp \\
A & \perp \\
A & \perp \\
B & \perp \\
B & \perp \\
B & \perp \\
\bot & \\
\bot & \\
\bot & \\
\bot & \\
\end{align*}
\]
Jumps are not canonical

**Theorem**

A proof structure with jumps is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

But:

- There is no canonical translation:

\[
\begin{align*}
A^\perp, A &\quad (\text{ax}) \\
B, B^\perp &\quad (\text{ax}) \\
\hline \\
A^\perp, A \otimes B, B^\perp &\quad (\otimes) \\
\hline \\
A^\perp, A \otimes B, B^\perp, \bot &\quad (\bot)
\end{align*}
\]

\[
\begin{array}{c}
\quad A^\perp \\
\quad A \\
\quad B^\perp \\
\quad \bot \\
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \\
\quad \otimes \\
\quad B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A^\perp \\
\quad A \\
\quad B^\perp \\
\quad \bot \\
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \\
\quad \otimes \\
\quad B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A^\perp \\
\quad A \\
\quad B^\perp \\
\quad \bot \\
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \\
\quad \otimes \\
\quad B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A^\perp \\
\quad A \\
\quad B^\perp \\
\quad \bot \\
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \\
\quad \otimes \\
\quad B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad A \otimes B \\
\quad \bot
\end{array}
\]

\[
\begin{array}{c}
\quad \bot
\end{array}
\]
Jumps are not canonical

**Theorem**

*Proof structure with jumps* is the translation of a proof iff all its switching graphs are (undirected) connected and acyclic.

**But:**

- Cut elimination needs some care:
Option 2: acyclicity only

The mix rules:

\[ \varepsilon (\text{mix}_0) \mapsto \emptyset \quad \frac{\pi \quad \pi'}{\Gamma, \Delta} (\text{mix}) \mapsto \pi \quad \pi' \]

**Theorem**

*A proof structure is a proof net iff it is the translation of a proof of* $\text{MLL} + (\text{mix}_0) + (\text{mix})$.

Same switching graphs, same cut elimination with or without units.
Option 2: acyclicity only

The mix rules:

\[ \bar{\varepsilon} \ (mix_0) \rightarrow \emptyset \]

\[ \frac{\pi \quad \pi'}{\Delta} \ (mix) \rightarrow \pi \quad \pi' \]

**Theorem**

A proof structure is a proof net iff it is the translation of a proof of \(\text{MLL} + (mix_0) + (mix)\).

Same switching graphs, same cut elimination with or without units. But:

- may identify some proofs that are not equal up to rule permutations
- introduces a proof of the empty sequent
Option 2: acyclicity only

The mix rules:

\[ \varepsilon (\text{mix}_0) \quad \Rightarrow \quad \emptyset \quad \frac{\pi}{\Gamma, \Delta} (\text{mix}) \quad \Rightarrow \quad \pi \quad \pi' \]

Theorem

A proof structure is a proof net iff it is the translation of a proof of \( MLL + (\text{mix}_0) + (\text{mix}) \).

Same switching graphs, same cut elimination with or without units. But:

- may identify some proofs that are not equal up to rule permutations
- introduces a proof of the empty sequent

Stumbling block

Identifying proofs of \( MLL \) with units up to rule permutations is hard.
Option 2: acyclicity only

The mix rules:

\[ \overline{\varepsilon} \ (mix_0) \Rightarrow \emptyset \]
\[ \frac{\pi \quad \pi'}{\Gamma, \Delta} (mix) \Rightarrow \pi \quad \pi' \]

Theorem

A proof structure is a proof net iff it is the translation of a proof of \( \text{MLL} + (mix_0) + (mix) \).

Same switching graphs, same cut elimination with or without units. But:

- may identify some proofs that are not equal up to rule permutations
- introduces a proof of the empty sequent

Stumbling block

Identifying proofs of \( \text{MLL} \) with units up to rule permutations is hard.

We stick to option 2.
Exponentials

MELL:

\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \oslash B \mid !A \mid ?A \]

\[ !A^\perp ::= ?A^\perp \quad ?A^\perp ::= !A^\perp \]
Exponentials

MELL:

\[ A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \Rightarrow B \mid !A \mid ?A \]

\[ !A^\perp ::= ?A^\perp \quad ?A^\perp ::= !A^\perp \]

\[
\begin{array}{c}
\pi \\
\frac{\Gamma, A}{\Gamma, ?A} (\text{?d}) \Rightarrow \pi \\
\end{array}
\]

\[
\begin{array}{c}
\pi \\
\frac{\Gamma, ?A}{\Gamma, ?A} (\text{?w}) \Rightarrow \pi \\
\end{array}
\]

\[
\begin{array}{c}
\pi \\
\frac{\Gamma, ?A, ?A}{\Gamma, ?A} (\text{?c}) \Rightarrow \pi \\
\end{array}
\]

\[
\begin{array}{c}
\pi \\
\frac{?\Gamma, A}{?\Gamma, !A} (!) \Rightarrow \pi \\
\end{array}
\]

L. Vaux Auclair (I2M)

Proof nets

TLLA 2021 25 / 41
Cut elimination in MELL

\[
\frac{\pi}{\Gamma, A} \quad \frac{\pi'}{A^\bot, \Delta} \quad (d) \quad \leadsto \quad \frac{\pi}{\Gamma, \Delta} \quad \frac{\pi'}{A^\bot, \Delta} \quad (cut)
\]

\[
\frac{!\Gamma, \neg A}{?\Gamma, !A} \quad (!) \quad \frac{\neg A^\bot, \Delta}{\neg A^\bot, \Delta} \quad (cut)
\]

\[
\pi \quad \pi' \\
\pi \quad \pi'
\]

L. Vaux Auclair (I2M) Proof nets TLLA 2021 26 / 41
Cut elimination in MELL

\[
\begin{align*}
\pi & \quad \frac{\pi'}{\Delta} \\
?\Gamma, A & \quad (\!) \\
?\Gamma, !A & \quad (?w) \\
\hline \\
?\Gamma, \Delta & \quad (cut) \\
?\Gamma, \Delta & \quad (cut) \\
\end{align*}
\]

\[
\pi' \\
\pi' \\
\pi'
\]

\[
\begin{align*}
\pi & \quad \frac{\pi'}{\Delta} \\
?\Gamma, A & \quad (\!) \\
?\Gamma, !A & \quad (?w) \\
\hline \\
?\Gamma, \Delta & \quad (cut) \\
?\Gamma, \Delta & \quad (cut) \\
\end{align*}
\]

\[
\pi' \\
\pi' \\
\pi'
\]

\[
\begin{align*}
\pi' & \quad \frac{\pi'}{\Delta} \\
?\Gamma & \quad (?w) \\
?\Gamma & \quad (?w) \\
\hline \\
?\Gamma & \quad (?w) \\
?\Gamma & \quad (?w) \\
\end{align*}
\]

:= \quad ?A_1 \ldots \quad ?A_n

L. Vaux Auclair (I2M)
Cut elimination in MELL

\[
\begin{align*}
\frac{?\Gamma, A (\,!)}{?\Gamma, !A} \quad & \quad \frac{?\Delta, ?A^\perp, \Delta (??c)}{?\Gamma, !A} \quad \frac{?\Gamma, A (\,!)}{?\Gamma, !A} \\
& \quad \frac{?\Gamma, !A}{?\Gamma, !A} \quad \frac{?\Delta, ?A^\perp, \Delta (cut)}{?\Gamma, !A} \\
& \quad \frac{?\Gamma, !A}{?\Gamma, !A} \quad \frac{?\Gamma, ?A^\perp, \Delta (cut)}{?\Gamma, ?A^\perp, \Delta (cut)} \\
& \quad \frac{?\Gamma, ?A^\perp, \Delta (cut)}{?\Gamma, ?A^\perp, \Delta (cut)} \\
\end{align*}
\]

\[
\begin{align*}
\pi & \quad \pi' \\
A & \quad ?A^\perp \quad ?A^\perp \\
!A & \quad ?A^\perp \quad ?c \quad ?A^\perp \\
\frac{?\Gamma, A (\,!)}{?\Gamma, !A} \quad \frac{?\Delta, ?A^\perp, \Delta (??c)}{?\Gamma, !A} \quad \frac{?\Gamma, A (\,!)}{?\Gamma, !A} \\
& \quad \frac{?\Gamma, !A}{?\Gamma, !A} \quad \frac{?\Delta, ?A^\perp, \Delta (cut)}{?\Gamma, !A} \\
& \quad \frac{?\Gamma, !A}{?\Gamma, !A} \quad \frac{?\Gamma, ?A^\perp, \Delta (cut)}{?\Gamma, ?A^\perp, \Delta (cut)} \\
& \quad \frac{?\Gamma, ?A^\perp, \Delta (cut)}{?\Gamma, ?A^\perp, \Delta (cut)} \\
\end{align*}
\]
Promotion boxes

- amounts to every node of $S$ (except for $?\text{-conclusions}$) jumping to $!$
- boxes must be either disjoint or nested
- allows to treat the box as one initial node
**MELL proof structures**

An **MELL proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

  ![Graph](image)

  together with an ordering of the two input edges of each \( \otimes \) and \( \otimes \) (one left, one right)

- a labelling of edges with **MELL** formulas, compatible with typing rules

  ![Labels](image)

- a tree order on \( \Uparrow \) nodes (+ a root for top level) together with a map from all nodes:
  - edges are monotone and conclusions are at top level
  - the only edges that change box depth are \( ? \)-typed (crossing any number of boxes) or the premisses of \( \Uparrow \) nodes (crossing exactly one box)
Translation of proofs

\[
\frac{\pi}{\Gamma, A} \quad \frac{\pi}{\Gamma, ?A} (\text{?d}) \quad \frac{\pi}{\Gamma, ?A} (\text{?w}) \\
\frac{\pi}{\Gamma, A, ?A} \quad \frac{\pi}{?\Gamma, !A} (\text{!})
\]
Definition

- a **switching** \( \sigma \) of a structure \( S \), is the choice of \( L \) or \( R \) for each \( \preceq \) or \( ?c \) of \( S \)
- the **switching graph** \( S_\sigma \) is obtained by replacing each \( \preceq \) or \( ?c \) accordingly, and each box with an initial node
Definition

A proof net is a proof structure such that:

▶ all switching graphs are (undirected) acyclic,
▶ the content of each box is a proof net, inductively.

In proof nets, each cut can be eliminated.

Theorem

Proof nets are stable under cut elimination.
Cut elimination in MELL, with boxes
Cut elimination in MELL, with boxes

\[ \begin{array}{c}
\vdash \Gamma \quad A \\
\vdash \Delta \quad B \\
\vdash \Gamma \quad A \\
\vdash \Delta \quad B \\
\end{array} \]

\[ \begin{array}{c}
\vdash \Gamma \quad !A \\
\vdash \Delta \quad !B \\
\vdash \Gamma \quad !A \\
\vdash \Delta \quad !B \\
\end{array} \]

\[ \vdash \Gamma \quad A \]

\[ \vdash \Delta \quad B \\
\vdash \Gamma \quad !A \\
\vdash \Delta \quad !B \\
\end{array} \]

\[ \vdash \Gamma \quad A \]

\[ \vdash \Delta \quad B \\
\vdash \Gamma \quad !A \\
\vdash \Delta \quad !B \\
\end{array} \]

\[ \sim \]

\[ \begin{array}{c}
\vdash \Gamma \quad A \\
\vdash \Delta \quad B \\
\vdash \Gamma \quad A \\
\vdash \Delta \quad B \\
\vdash \Gamma \quad A \\
\vdash \Delta \quad B \\
\end{array} \]
Properties of cut elimination

**Theorem**

*Cut elimination is confluent and strongly normalizing.*

**Proof.**

There are some papers and theses…
One famous translation:

$$(A \to B) \mapsto (!A \leadsto B)$$

Given $\Gamma \vdash M : B$ with $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ we construct

![Proof net diagram](image)
Translating the $\lambda$-calculus

$$\Gamma, x : A \vdash x : A \quad (var) \Rightarrow \quad ?w \quad A \perp \quad ax \quad \Gamma \perp \quad ?A \perp \quad \Rightarrow \quad A$$

$$\Gamma, x : A \vdash M : B \quad \Gamma \vdash \lambda x. M : A \to B \quad (abs) \Rightarrow \quad ?\Gamma \perp \quad ?A \perp \quad \Rightarrow \quad \Gamma \perp \quad ?A \perp\gamma B$$

$$\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A \quad (app) \Rightarrow \quad ?\Gamma \perp \quad ?A \perp \gamma B \quad \Rightarrow \quad ?\Gamma \perp \quad ?A \perp$$

$$\Gamma \vdash MN : B \quad \Rightarrow \quad \Gamma \perp \quad ?A \perp \gamma B$$

$$\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A \quad (app) \Rightarrow \quad ?\Gamma \perp \quad ?A \perp \gamma B \quad \Rightarrow \quad ?\Gamma \perp \quad ?A \perp$$

$$\Gamma \vdash MN : B \quad \Rightarrow \quad \Gamma \perp \quad ?A \perp \gamma B$$
Translating the λ-calculus: $\Lambda \rightarrow NJ \rightarrow MELL$

\[
\Gamma, x : A \vdash x : A \quad (\text{var}) \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\text{ax} \\
?w \\
?d \\
\varphi \\
\Gamma \perp \\
?A \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
A \perp \\
\end{array}
\end{array}
\]

\[
\Gamma, x : A \vdash M : B \\
\Gamma \vdash \lambda x. M : A \rightarrow B \quad (\text{abs}) \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\text{cut} \\
\text{ax} \\
\varphi \\
\Gamma \perp \\
?A \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\varphi \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
A \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
B \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
!A \times B \perp \\
\end{array}
\end{array}
\]

\[
\Gamma \vdash M : A \rightarrow B \\
\Gamma \vdash N : A \quad (\text{app}) \quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\text{cut} \\
\text{ax} \\
\varphi \\
\Gamma \perp \\
?A \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\varphi \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
A \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
B \perp \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
!A \times B \perp \\
\end{array}
\end{array}
\]

\[
\Gamma \vdash MN : B
\]
Simulating $\beta$-reduction

\[
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \to B} \quad (\text{abs}) \quad \frac{\Gamma \vdash N : A}{\Gamma \vdash (\lambda x. M) N : B} \quad (\text{app}) \quad \Rightarrow
\]

Lemma
Iterating exponential cut elimination in the above net yields the translation of $M[N/x]$.

In fact, not quite...
Simulating $\beta$-reduction

Lemma

Iterating exponential cut elimination in the above net yields the translation of $M[N/x]$.

In fact, not quite...

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Simulating $\beta$-reduction

Lemma
Iterating exponential cut elimination in the above net yields the translation of $M\left[\frac{N}{x}\right]$.
In fact, not quite...
Lemma

*Iterating exponential cut elimination in the above net yields the translation of* $M[N/x]$. 

*Simulating $\beta$-reduction*
Lemma

*Iterating exponential cut elimination in the above net yields the translation of* \( M[N/x] \).

In fact, not quite…
Rétoré conversions and $\Box$-canonical nets

Contraction and weakening form a commutative monoid:

We should thus consider $n$-ary contractions.
Rétoré conversions and \( ? \)-canonical nets

Contraction and weakening cross the border of boxes:
Rétoré conversions and $\cdot$-canonical nets

- up to these identities, cut elimination refines $\beta$-reduction
- with $n$-ary contractions, normal forms for the natural orientation of these identities are $\cdot$-canonical nets (the old name for this is *nouvelle syntaxe*)
- cut elimination can be defined directly on $\cdot$-canonical nets
Conclusion: what was not in this tutorial?

- correctness criteria with better complexity
- an explanation of why there are no satisfactory proof nets with weakening
- additives, quantifiers,
- pure types for simulating the untyped $\lambda$-calculus
- polarized / intuitionistic variants
- differential nets and Taylor expansion
- explicit substitutions
- denotational semantics (experiments)
- geometry of interaction
- etc.
Conclusion: what was not in this tutorial?

- correctness criteria with better complexity
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- etc.

Have a nice TLLA 2021!