Geometry of Interaction for ZX-Diagrams

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- Larger systems: $q_0 \otimes q_1$, $|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$
- Operation are *linear maps*

- $H := \frac{1}{\sqrt{2}} \begin{matrix} & |0\rangle & |1\rangle \\ |0\rangle & 1 & 1 \\ |1\rangle & 1 & -1 \end{matrix} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$ is unitary

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- $\text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\neg x\rangle \end{cases}$

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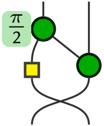

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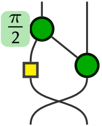

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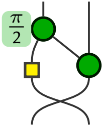
- Manipulates string diagrams e.g. 

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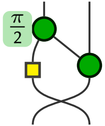
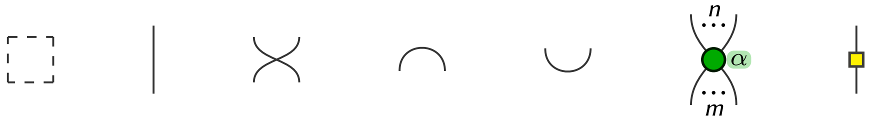


- Relaxes unitarity
- Is Universal (can encode any linear map)
- Lack a direct operational interpretation (this talk !)

Generators



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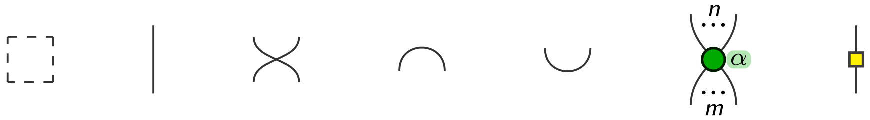
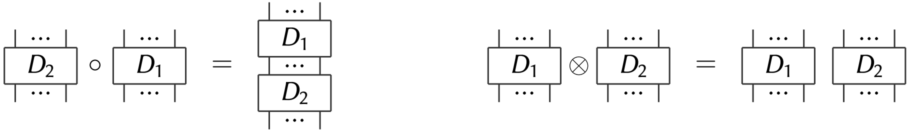


Compositions



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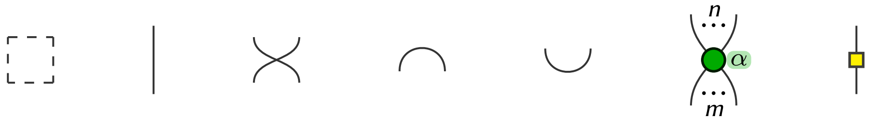

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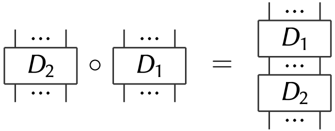
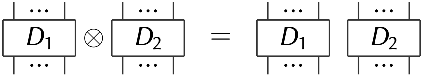
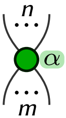


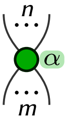


Standard Interpretation

$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

- A spider: 

$$:: \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0\dots0\rangle & \mapsto \quad |0\dots0\rangle \\ |1\dots1\rangle & \mapsto \quad e^{i\alpha}\,|1\dots1\rangle \\ \_ & \mapsto 0 \end{cases}$$

- A spider: 

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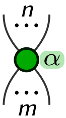
- A spider: 

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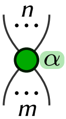
- A spider: 

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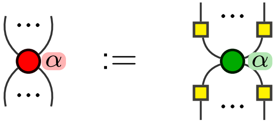
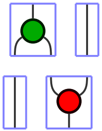
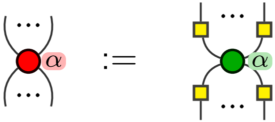
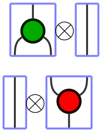
-  $:: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha \mapsto \alpha|00\rangle + \alpha|11\rangle : \mathbb{C} \to \mathbb{C}^2$

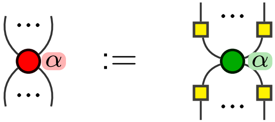
-  $:: \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = |xy\rangle \mapsto \delta_{x=y} : \mathbb{C}^2 \to \mathbb{C}$

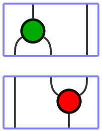
We define a new spider : 

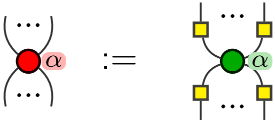
CNOT = 

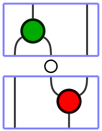
We define a new spider : 

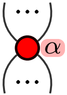
CNOT = 

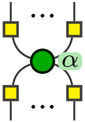
We define a new spider : 

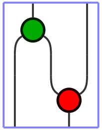
CNOT = 

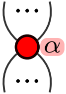
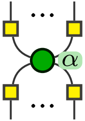
We define a new spider : 

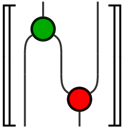
CNOT = 

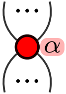
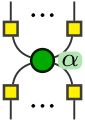
We define a new spider : 

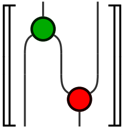
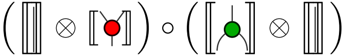
:= 

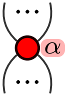
CNOT = 

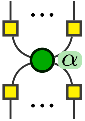
We define a new spider :  := 

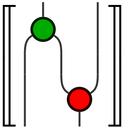
CNOT =  =

We define a new spider :  := 

CNOT =  = 

We define a new spider : 

$:=$ 

CNOT $=$ 

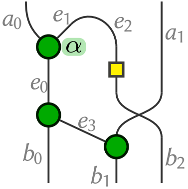
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- GoI as a **Token Machine**
- Proof Nets / ZX seen as graphs
- Tokens travel through the graph
- Proof Nets $\Rightarrow$ Capture computational content of the proof
- Our Token Machine : superposition of tokens, multiple tokens, collisions, ...

$\Rightarrow$ Bring Operational Semantic on ZX-Diagrams.

Already existing work: “The geometry of parallelism. classical, probabilistic, and quantum effects” [Dal Lago, Faggian, Valiron, Yoshimizu]

	Dal Lago et al.	This Work
Control	Classic	Quantum
Superposition	✗	✓
Asynchronisation	✗	✓
Types	✓	✗

$D =$ 

- Inputs : $\mathcal{I}(D) = (a_0, a_1)$
- Outputs : $\mathcal{O}(D) = (b_0, b_1, b_2)$
- Edges : $\mathcal{E}(D) = \{e_0, e_1, e_2, e_3\} \cup \mathcal{I}(D) \cup \mathcal{O}(D)$

Token

3-tuple $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$ where:

- e is an edge of the ZX-Diagram D
- d is a direction
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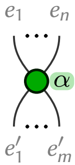
e.g : 

with $(e_1 \downarrow 0)(e'_3 \uparrow 1) + (e_1 \downarrow 1)$

Token State

A *token state* is a **sum** of **products** of tokens with complex coefficients.

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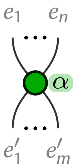

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- The tokens modify the global token state as they move.

- Tokens represent information  $\downarrow 0$ and  $\downarrow 1$

The Token Machine

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- Multiple tokens on a diagram

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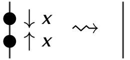
- Collisions : 

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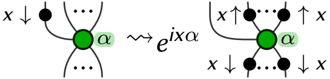
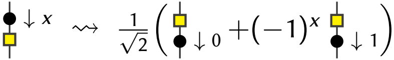
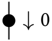
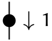
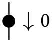
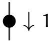
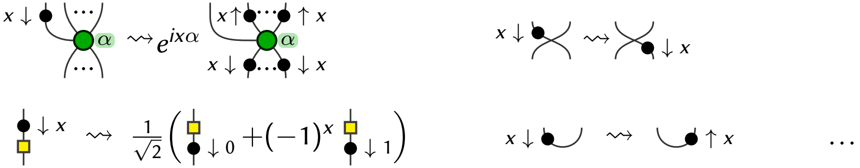
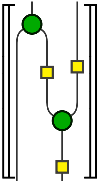




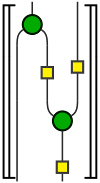
- Tokens represent information  and 
- Multiple tokens on a diagram
- Weighted sum of these diagrams : superposition (e.g. α  $+\ \beta$ )
- Collisions : 

- Diffusions : 



$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example



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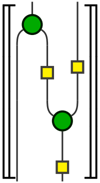

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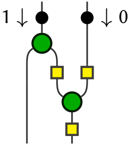

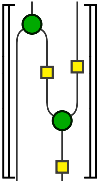

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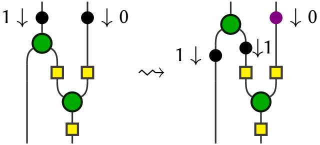

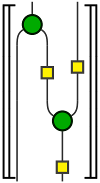

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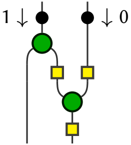
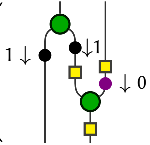


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \Bigg($$



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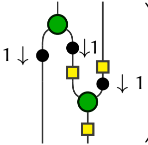

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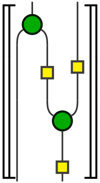

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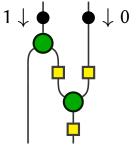

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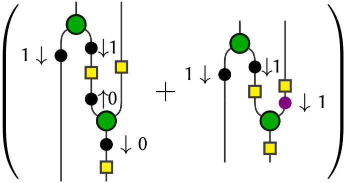

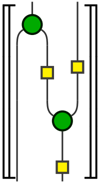

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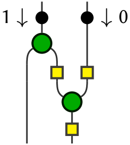

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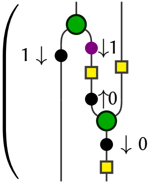

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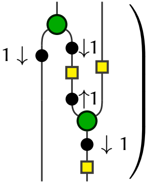

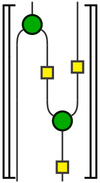

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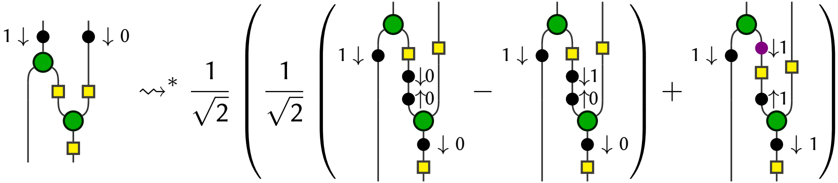
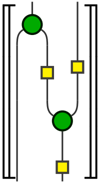
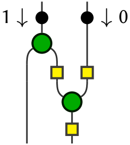


$[\![$  $]\!]$ $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$



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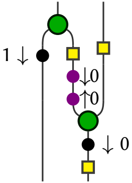

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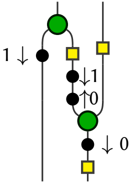

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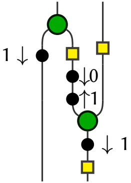

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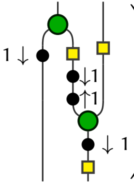

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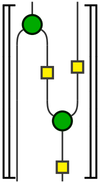

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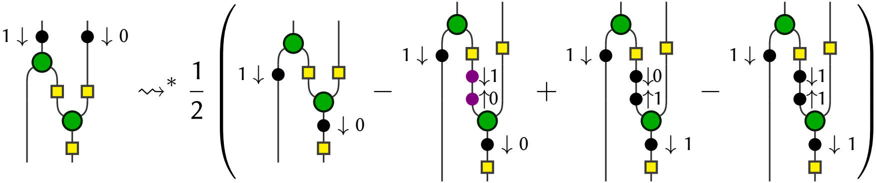

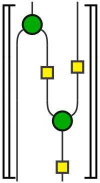

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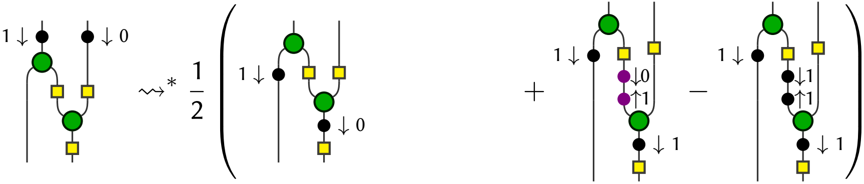

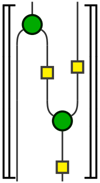

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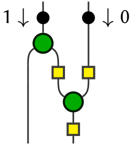

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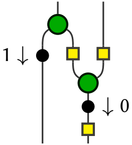

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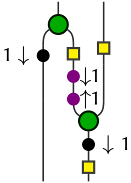

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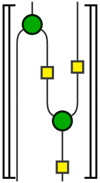

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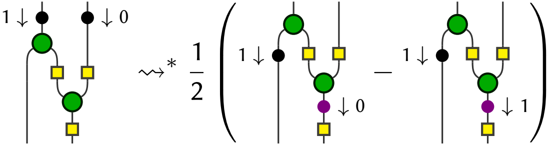

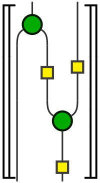

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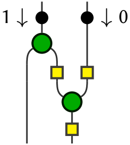

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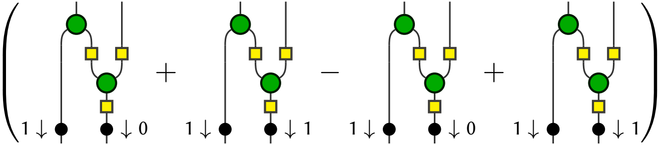

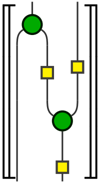

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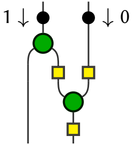

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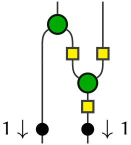
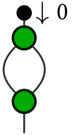
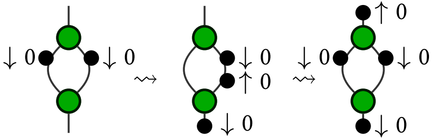


Take 

With arbitrary rewriting order:



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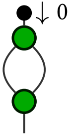

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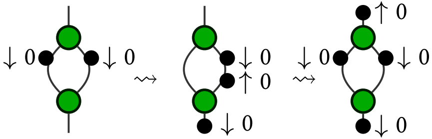
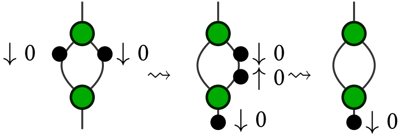


With good rewriting order:



Rewriting System

We define a transition system $\rightsquigarrow$ as *exactly* one **diffusion rule** rule followed by all possible **collision** rules until none apply

Want to avoid:

- Having multiple tokens on the same edge that don't collide (i.e $(a \downarrow x)(a \downarrow x)$)
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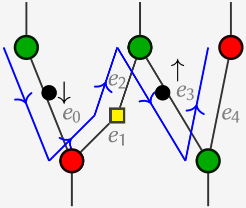
Two invariants:

- **Well-Formedness** : Avoid bad configuration
- **Cycle-Balancedness** : Termination, Confluence

Polarity in a Path

$p = (e_0, e_1, e_2, e_3, e_4)$ is an oriented path.



- If a token follows the path $+1$
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity 0

Well-Formed Token State

We say that a ZX-Diagram D is **Well-Formed** if for every path p its Polarity $\in \{-1, 0, 1\}$

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Theorem (Invariance of Well-Formedness)

Well-Formedness is preserved under $\rightsquigarrow$.

Theorem (Characterisation of Well-Formedness)

Well-formed states cannot reach “bad configurations”.

Cycle-Balanced Token State

We say that a ZX-Diagram D is **Cycle-Balanced** if for every cycle c its Polarity $= 0$

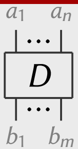
Cycle-Balanced Token State

We say that a ZX-Diagram D is **Cycle-Balanced** if for every cycle c its Polarity $= 0$

Theorem (Termination of well-formed, cycle-balanced token state)

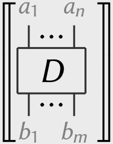
Let D be a ZX-Diagram, and s a well-formed, cycle-balanced token state, then s terminates.

Theorem (Simulation of Standard Interpretation)

Let 

a ZX-Diagram and let $t = (e \downarrow 0)(e \uparrow 0) + (e \downarrow 1)(e \uparrow 1)$.

t is a well-formed, cycle balanced token state.

We have that $t \rightsquigarrow^*$ 

$\Rightarrow$ we recover the **standard interpretation**.

Conclusion

- The Token Machine gives us a very general framework to study ZX-Calculus.
- More operational approach to ZX-Calculus.
- Can be easily adapted to extensions of ZX-Calculus.
- Already extended to : SOP, Mixed-Processes.

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Future Work

- Stronger relation with Linear Logic and Proof Nets (WIP)
- Hope it can help in defining new extensions of ZX-Calculus such as recursion (Future Work).