

Geometry of Interaction for ZX-Diagrams

Kostia Chardonnet, Benoît Valiron, Renaud Vilmart

Univ. Paris Saclay, LMF

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- Operations are *linear maps*

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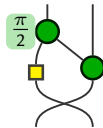
- $\text{CNOT} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{cases} |0x\rangle \mapsto |0x\rangle \\ |1x\rangle \mapsto |1\neg x\rangle \end{cases}$

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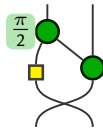
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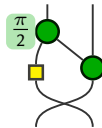
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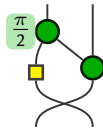
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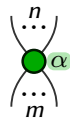
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- Relaxes unitarity
- Is Universal (can encode any linear map)
- Lack a direct operational interpretation (this talk !)

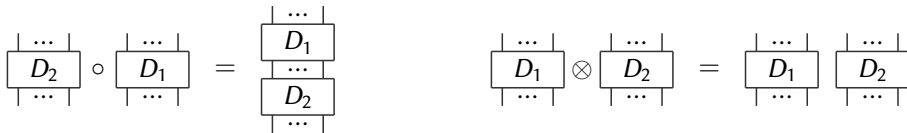
Generators



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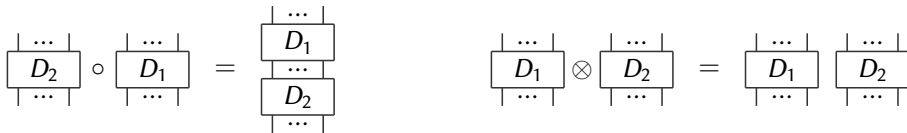
Compositions



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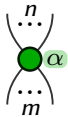



Standard Interpretation

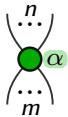

$$[\![\cdot]\!] : \mathbf{ZX} \rightarrow \mathcal{M}(\mathbb{C})$$

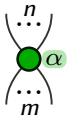
• A spider:


$$\begin{array}{c}
 \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \\
 \text{---} \\
 \begin{array}{c} \text{---} \\ \vdots \\ \text{---} \end{array} \\
 \text{---}
 \end{array}
 \begin{array}{c}
 n \\
 \dots \\
 m
 \end{array}
 \alpha
 \quad :: \quad
 \begin{pmatrix}
 1 & 0 & \dots & \dots & 0 \\
 0 & 0 & & & \vdots \\
 \vdots & & \ddots & & \vdots \\
 \vdots & & & 0 & 0 \\
 0 & \dots & \dots & 0 & e^{i\alpha}
 \end{pmatrix}
 = \begin{cases}
 |0 \dots 0\rangle \mapsto |0 \dots 0\rangle \\
 |1 \dots 1\rangle \mapsto e^{i\alpha} |1 \dots 1\rangle \\
 - \mapsto 0
 \end{cases}$$

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• A change of basis:  $::$ $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{cases} |0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$


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- Wires: $\begin{vmatrix} | \\ | \\ | \\ | \end{vmatrix} :: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |x\rangle \mapsto |x\rangle$, $\begin{matrix} \diagup & \diagdown \\ \diagdown & \diagup \end{matrix} :: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = |xy\rangle \mapsto |yx\rangle$

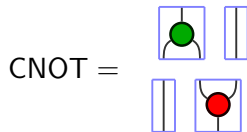
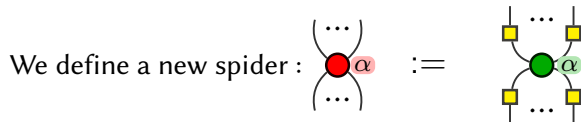
• A spider:  $\::$ $\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} = \begin{cases} |0 \dots 0\rangle \mapsto |0 \dots 0\rangle \\ |1 \dots 1\rangle \mapsto e^{i\alpha} |1 \dots 1\rangle \\ - \mapsto 0 \end{cases}$

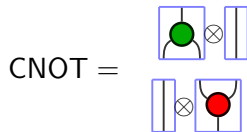
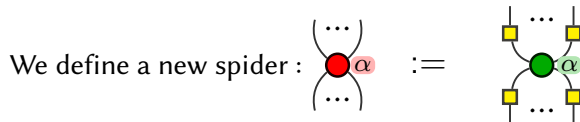
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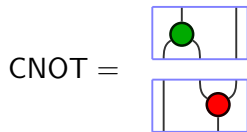
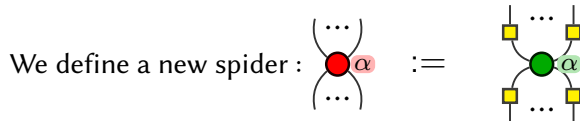
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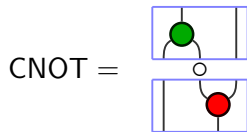
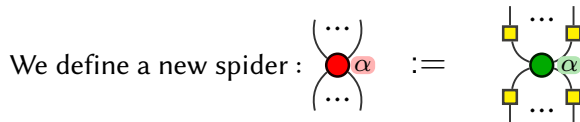
•  $\::$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha \mapsto \alpha |00\rangle + \alpha |11\rangle : \mathbb{C} \rightarrow \mathbb{C}^2$

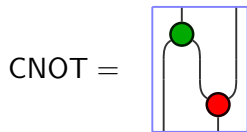
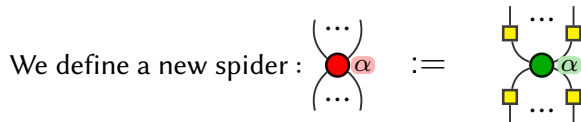
•  $\::$ $(1 \ 0 \ 0 \ 1) = |xy\rangle \mapsto \delta_{x=y} : \mathbb{C}^2 \rightarrow \mathbb{C}$

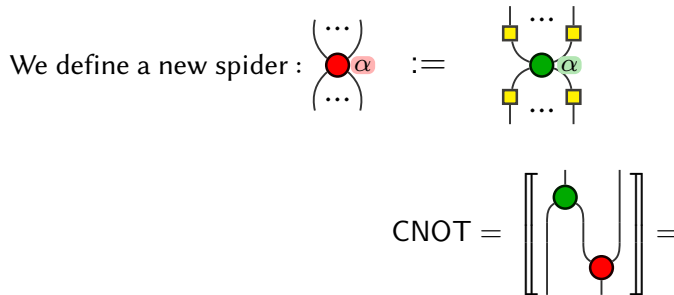



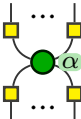










We define a new spider :  $:=$ 

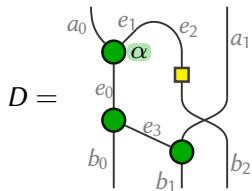
$$\text{CNOT} = \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Gol as a **Token Machine**
- Proof Nets / ZX seen as graphs
- Tokens travel through the graph
- Proof Nets \Rightarrow Capture computational content of the proof
- Our Token Machine : superposition of tokens, multiple tokens, collisions, ...

\Rightarrow Bring Operational Semantic on ZX-Diagrams.

Already existing work: “The geometry of parallelism. classical, probabilistic, and quantum effects” [Dal Lago, Faggian, Valiron, Yoshimizu]

	Dal Lago et al.	This Work
Control	Classic	Quantum
Superposition	✗	✓
Asynchronisation	✗	✓
Types	✓	✗



- Inputs : $\mathcal{I}(D) = (a_0, a_1)$
- Outputs : $\mathcal{O}(D) = (b_0, b_1, b_2)$
- Edges : $\mathcal{E}(D) = \{e_0, e_1, e_2, e_3\} \cup \mathcal{I}(D) \cup \mathcal{O}(D)$

Token

3-tuple $(e, d, b) \in \mathcal{E}(D) \times \{\uparrow, \downarrow\} \times \{0, 1\}$

where:

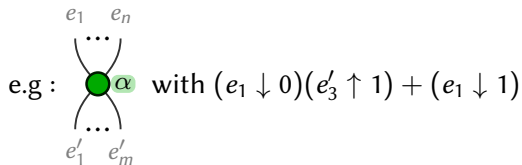
- e is an edge of the ZX-Diagram D
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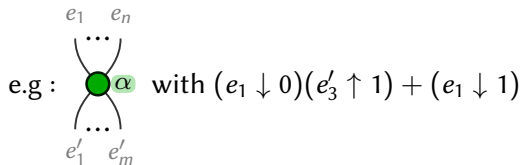
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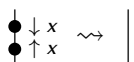
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- The tokens modify the global token state as they move.

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• Diffusions : $\begin{array}{c} \bullet \downarrow x \\ \dots \\ \bullet \end{array} \rightsquigarrow e^{ix\alpha} \begin{array}{c} \bullet \uparrow x \dots \bullet \uparrow x \\ \bullet \downarrow x \dots \bullet \downarrow x \end{array}$

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$$\begin{array}{c} \bullet \downarrow x \\ \square \end{array} \rightsquigarrow \frac{1}{\sqrt{2}} \left(\begin{array}{c} \square \\ \bullet \downarrow 0 \end{array} + (-1)^x \begin{array}{c} \square \\ \bullet \downarrow 1 \end{array} \right)$$

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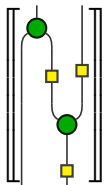
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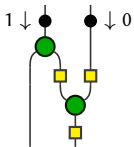
$$x \downarrow \bullet \rightsquigarrow \bullet \uparrow x \quad \dots$$

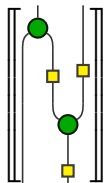
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{|l} \hline \\ \hline \\ \hline \\ \hline \end{array}
 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

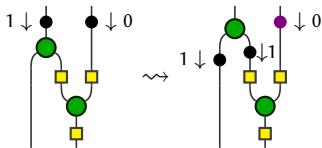


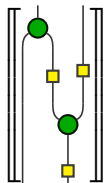
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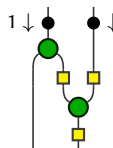


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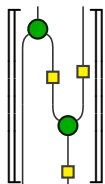


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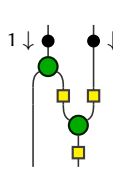


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

The diagram shows a superposition of two states. The first state has a black dot with a down arrow on the top wire of the left circle and a black dot with a down arrow on the top wire of the right circle. The second state has a black dot with a down arrow on the top wire of the left circle, a black dot with a down arrow on the top wire of the right circle, and a purple dot with a down arrow on the bottom wire of the right circle.

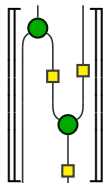


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

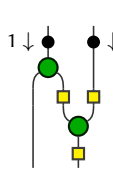


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

The diagram is a sum of two terms inside large parentheses, scaled by $\frac{1}{\sqrt{2}}$. The first term is a ZX-diagram with two vertical wires. The left wire has a black dot with a downward arrow labeled '1'. The right wire has a black dot with a downward arrow labeled '0'. The diagram contains two green circles and two yellow squares. The second term is a similar ZX-diagram with two vertical wires. The left wire has a black dot with a downward arrow labeled '1'. The right wire has a black dot with a downward arrow labeled '1' and a purple dot with a downward arrow labeled '1'. The diagram contains two green circles and two yellow squares.

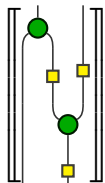


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

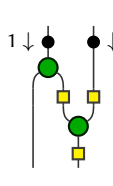


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

The diagram is a sum of two terms inside large parentheses, each multiplied by $\frac{1}{\sqrt{2}}$. The first term is a ZX-diagram with two vertical wires. The left wire has a black dot at the top with a '1' and a downward arrow, and a green circle at the bottom. The right wire has a black dot at the top with a '0' and a downward arrow, and a green circle at the bottom. They are connected by two yellow squares in the middle. The second term is a ZX-diagram with two vertical wires. The left wire has a black dot at the top with a '1' and a downward arrow, and a green circle at the bottom. The right wire has a black dot at the top with a '1' and a downward arrow, and a green circle at the bottom. They are connected by two yellow squares in the middle. A purple dot is on the left wire of the second term, and a black dot is on the right wire of the second term.

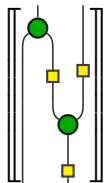


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

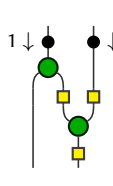


$$\rightsquigarrow^* \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) - \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) + \begin{array}{c} \text{Diagram 5} \end{array}$$

The diagram shows a transformation of a ZX-diagram. On the left is a diagram with two green circles, two yellow squares, and two black dots with arrows labeled '1' and '0'. This is transformed into a sum of three terms, each enclosed in large parentheses. The first term is a diagram with two green circles, two yellow squares, and two black dots with arrows labeled '0' and '0'. The second term is a diagram with two green circles, two yellow squares, and two black dots with arrows labeled '1' and '0'. The third term is a diagram with two green circles, two yellow squares, and two black dots with arrows labeled '1' and '1'.



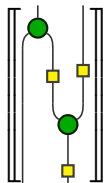
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



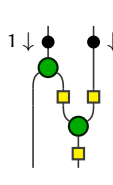
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The four diagrams in the sum are:

- Diagram 1: Same as the original diagram, but with two purple dots on the right wire, one above and one below the top yellow square. Labels: 0 down, 0 up.
- Diagram 2: Same as the original diagram, but with two black dots on the right wire, one above and one below the top yellow square. Labels: 1 down, 0 up.
- Diagram 3: Same as the original diagram, but with two black dots on the right wire, one above and one below the top yellow square. Labels: 0 down, 1 up.
- Diagram 4: Same as the original diagram, but with two black dots on the right wire, one above and one below the top yellow square. Labels: 1 down, 1 up.

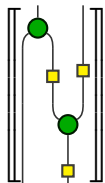


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

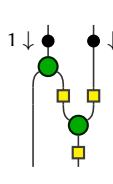


$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The diagram shows a decomposition of the CNOT gate into four terms, each with a different configuration of black dots and yellow squares. The terms are separated by minus signs, and the entire expression is enclosed in large parentheses.



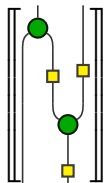
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



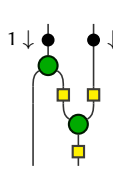
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)$$

The diagram shows a decomposition of a CNOT gate into a sum of four terms, each with a weight of 1/2. The terms are:

- Diagram 1: A CNOT gate with a green node on the control and a green node on the target. The control has a black node with a downward arrow labeled '1', and the target has a black node with a downward arrow labeled '0'.
- Diagram 2: A CNOT gate with a green node on the control and a green node on the target. The control has a black node with a downward arrow labeled '1', and the target has a black node with a downward arrow labeled '0'.
- Diagram 3: A CNOT gate with a green node on the control and a green node on the target. The control has a black node with a downward arrow labeled '1', and the target has a black node with a downward arrow labeled '1'.
- Diagram 4: A CNOT gate with a green node on the control and a green node on the target. The control has a black node with a downward arrow labeled '1', and the target has a black node with a downward arrow labeled '1'.

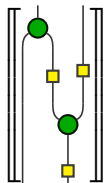


$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

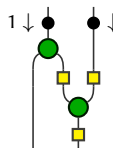


$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ - \text{Diagram 3} \end{array} \right)$$

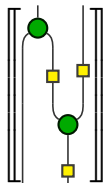
The diagram shows a transformation of a ZX-diagram. On the left, a diagram with two vertical lines, two green circles, two yellow squares, and two black dots with arrows (labeled 1 and 0) is transformed into a sum of three diagrams. The first diagram has a green circle at the top, a black dot with arrow 1, and a yellow square at the bottom. The second diagram has a green circle at the bottom, a black dot with arrow 0, and a yellow square at the bottom. The third diagram has a green circle at the bottom, two purple circles with arrows 1 and 1, a black dot with arrow 1, and a yellow square at the bottom. The entire sum is multiplied by 1/2.



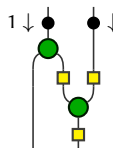
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$



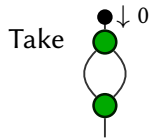
$$\rightsquigarrow^* \frac{1}{2} \left(\begin{array}{c} 1 \downarrow \bullet \quad \bullet \downarrow 0 \\ \text{Diagram 1} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} - \begin{array}{c} 1 \downarrow \bullet \quad \bullet \downarrow 0 \\ \text{Diagram 2} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} \right)$$



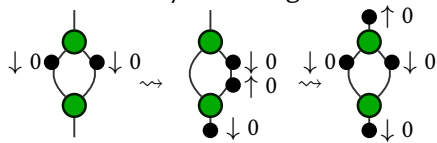
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies |10\rangle \mapsto \frac{1}{\sqrt{2}} |11\rangle$$

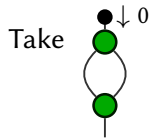


$$\rightsquigarrow^* \frac{1}{2\sqrt{2}} \left(\begin{array}{c} \text{Diagram 1} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{Diagram 2} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} - \begin{array}{c} \text{Diagram 3} \\ 1 \downarrow \bullet \quad \bullet \downarrow 0 \end{array} + \begin{array}{c} \text{Diagram 4} \\ 1 \downarrow \bullet \quad \bullet \downarrow 1 \end{array} \right)$$

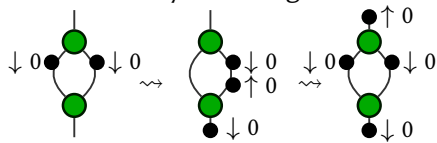


With arbitrary rewriting order:

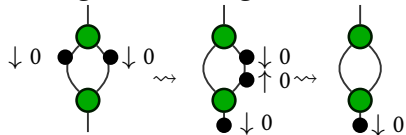




With arbitrary rewriting order:



With good rewriting order:



Rewriting System

We define a transition system \rightsquigarrow as *exactly* one **diffusion rule** rule followed by all possible **collision** rules until none apply

Want to avoid:

- Having multiple tokens on the same edge that don't collide (i.e $(a \downarrow x)(a \downarrow x)$)
- Non-termination

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Two invariants:

- **Well-Formedness** : Avoid bad configuration
- **Cycle-Balancedness** : Termination, Confluence

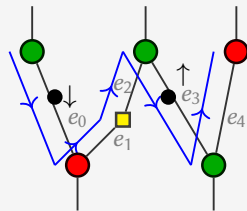
Polarity in a Path

$p = (e_0, e_1, e_2, e_3, e_4)$ is an oriented path.

- If a token follows the path +1
- If it goes against it -1
- If it is not on the path 0

Example:

- Here, polarity 0



Well-Formed Token State

We say that a ZX-Diagram D is **Well-Formed** if for every path p its Polarity $\in \{-1, 0, 1\}$

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Theorem (Invariance of Well-Formedness)

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Theorem (Characterisation of Well-Formedness)

Well-formed states cannot reach “bad configurations”.

Cycle-Balanced Token State

We say that a ZX-Diagram D is **Cycle-Balanced** if for every cycle c its Polarity = 0

Cycle-Balanced Token State

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Theorem (Termination of well-formed, cycle-balanced token state)

Let D be a ZX-Diagram, and s a well-formed, cycle-balanced token state, then s terminates.

Theorem (Simulation of Standard Interpretation)

Let $\begin{array}{c} a_1 \quad a_n \\ | \quad | \\ \dots \\ | \quad | \\ \boxed{D} \\ | \quad | \\ \dots \\ | \quad | \\ b_1 \quad b_m \end{array}$ a ZX-Diagram and let $t = (e \downarrow 0)(e \uparrow 0) + (e \downarrow 1)(e \uparrow 1)$.

t is a well-formed, cycle balanced token state.

We have that $t \rightsquigarrow^* \left[\begin{array}{c} a_1 \quad a_n \\ | \quad | \\ \dots \\ | \quad | \\ \boxed{D} \\ | \quad | \\ \dots \\ | \quad | \\ b_1 \quad b_m \end{array} \right]$

\Rightarrow we recover the **standard interpretation**.

Conclusion

- The Token Machine gives us a very general framework to study ZX-Calculus.
- More operational approach to ZX-Calculus.
- Can be easily adapted to extensions of ZX-Calculus.
- Already extended to : SOP, Mixed-Processes.

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Future Work

- Stronger relation with Linear Logic and Proof Nets (WIP)
- Hope it can help in defining new extensions of ZX-Calculus such as recursion (Future Work).