Towards a Curry-Howard Correspondence for Quantum Computation

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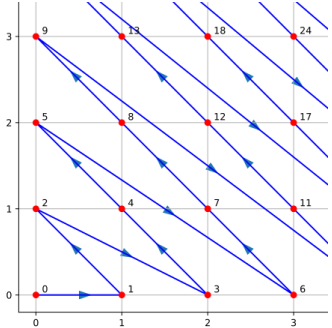
Origins

- Landauer and Bennet, 1961 : Reversible Computation and Energy Dissipation
- Functions f are reversible: $\exists f^{-1}$, st $f \circ f^{-1} = id = f^{-1} \circ f$
- Multiple approach : Turing Machines, imperative / functional programming languages, ...

Example

$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$



Reversible Programming Languages

Programming languages:

- RFun : Untyped language
- Theseus : Typed, Linear language, based on Pattern-Matching

Uses

- Reversible Computation
- Quantum Computation

Our Case

Work by [Sabry, Valiron, Vizzotto]:

- Subset of Theseus
- Linear, Reversible language with Lists
- Require terminating recursion
- Extension to the quantum case
- No general recursive type
- No expressivity theorem
- No relation with logic

} What we are trying to extend

Previous Work - Logic

- Logic $\mu\mathsf{MALL}^{\infty}$: Linear Logic with Least and Greatest Fixpoint
- Allow inductive and co-inductive statement
- We only look at the inductive fragment.
- Formulas $A, B ::= 1 \mid A \otimes B \mid A \oplus B \mid \mu X.A \mid X$

Examples:

- $nat = \mu X.\mathbb{1} \oplus X$
- $lists(A) = [A] = \mu X.\ \mathbb{1} \oplus (A \otimes X)$
- BinaryTree$(A) = \mu X.1 \oplus (A \otimes X \otimes X)$

Linear Logic with Induction ($\mu X.A$)

$$\frac{A \vdash B[X \leftarrow \mu X.B]}{A \vdash \mu X.B}\ \mu_R \qquad \frac{A[X \leftarrow \mu X.A] \vdash B}{\mu X.A \vdash B}\ \mu_L$$

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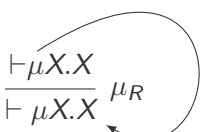

Infinite derivations represented as graphs

Beware : can derive any formula F

$$\dfrac{\dfrac{\dfrac{\vdots}{\mu X.X \vdash \mathtt{F}}\,\mu_L \qquad \dfrac{\vdots}{\vdash \mu X.X}\,\mu_R}{\vdash \mathtt{F}}\,\mathtt{cut}}$$

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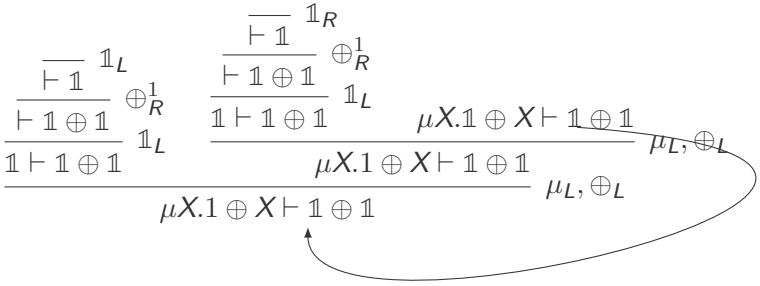
A derivation is valid if in every infinite branch:

- Infinity of rules μ_L

Example : Parity function



Our Work

Based on [Sabry, Valiron, Vizzotto] and [Baelde, Doumane, Saurin]

	Sabry et al.	Baelde et al.	This Work
Linear	✓	✓	✓
Reversible	✓	✗	✓
Induction	✗	✓	✓
Curry-Howard	✗	✗	WIP

Syntax

(Base types) $A, B ::= \mathbb{1} \mid X \mid A \oplus B \mid A \otimes B \mid \mu X.A$

(Isos, first-order) $T ::= A \leftrightarrow B$

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$$\mu f. \left\{ \begin{array}{lll} [\,] & \leftrightarrow & [\,] \\ h :: t & \leftrightarrow & \mathtt{let}\ x = \omega\ h\ \mathtt{in} \\ & & \mathtt{let}\ y = f\ t\ \mathtt{in}\ x :: y \end{array} \right\} : [A] \leftrightarrow [B]$$

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Properties

Syntax

- Language comes with a rewriting system and a type system.
  $\Rightarrow$ Confluence, Subject Reduction
- Ensuring exhaustivity and non-overlapping of clauses.
- Ensuring productivity.
  $\Rightarrow$ Similar to Coq, recursive argument have to be smaller.
- Can encode any Primitive Recursive Function

Semantic

- Isos denote computations from $A \to B$ and $B \to A$.

Syntax - Example 2

$$\mathtt{map}(\omega) = \mu \mathtt{f}. \left\{ \begin{array}{lcl} [\,] & \leftrightarrow & [\,] \\ h :: t & \leftrightarrow & \mathtt{let}\, x = \omega\ h\ \mathtt{in} \\ & & \mathtt{let}\, y = f\ t\ \mathtt{in}\ x :: y \end{array} \right\} : [\mathtt{A}] \leftrightarrow [\mathtt{B}]$$

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From Type Derivation To Proofs

Typed Terms $\rightsquigarrow$ Proofs

$$\omega : A \leftrightarrow B \rightsquigarrow \pi : A \vdash B$$

$$\omega^{\perp} : B \leftrightarrow A \rightsquigarrow \pi^{\perp} : B \vdash A$$

From Derivations To Proofs - Example

Let us take the recursive identity on lists

$$\mu f. \left\{ \begin{array}{ll} [\,] & \leftrightarrow [\,] \\ h :: t & \leftrightarrow \texttt{let}\ t' = f\, t\ \texttt{in} \\ & \quad\ h :: t' \end{array} \right\} : [A] \leftrightarrow [A]$$

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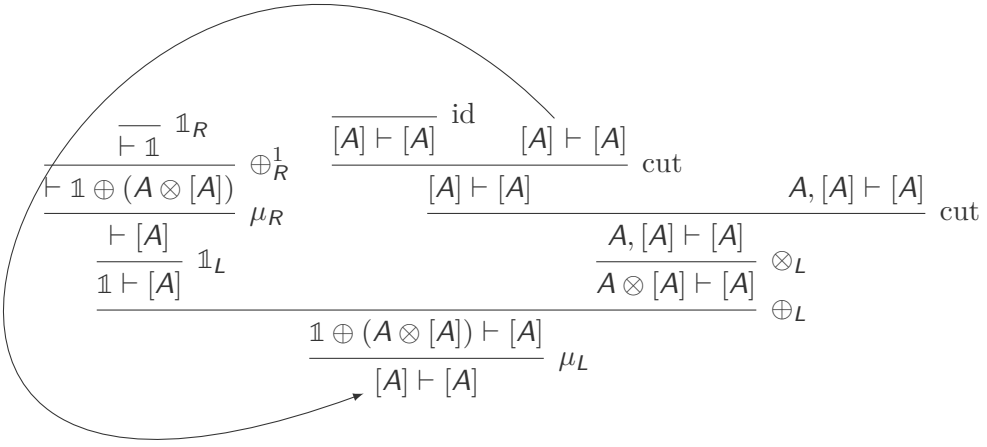

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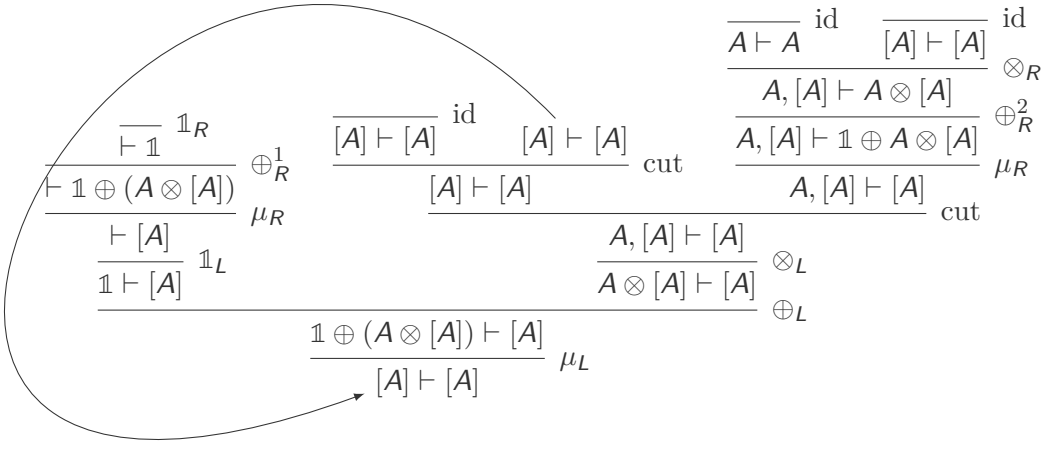

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Conclusion

Results

- Linear and Reversible language
- Curry-Howard Correspondence

In Progress

- Finish some proofs
- Consider the quantum case

Future Work

- Consider co-induction
- Categorical Semantic
- Term language for ZX-Calculus
- What makes a proof reversible