MELL proof-nets in the category of graphs

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Outline

1. MELL proof-nets as graphs

2. What for? MELL proof-nets in the category of graphs

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Boxes in MELL

Multiplicative exponential linear logic (MELL):

$$A, B ::= X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid 1 \mid \perp \mid !A \mid ?A$$

MELL sequent calculus $\rightsquigarrow$ MELL proof-nets ($\approx$ labeled directed graphs).

The promotion rule is the only contextual rule in MELL sequent calculus.

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad \Rightarrow$$

In MELL proof-nets, boxes are reminiscent of the sequentiality of promotion.

A box marks a subproof that can be erased or duplicated at will by cut-elimination.
$\rightsquigarrow$ Boxes encode crucial information in MELL proof-nets.

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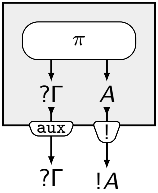

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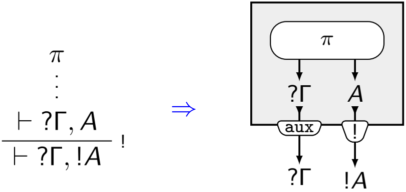

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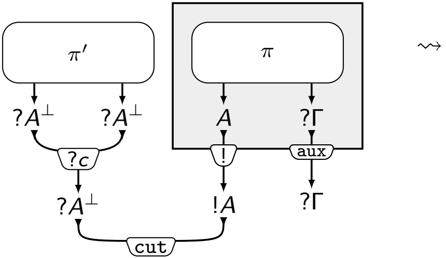
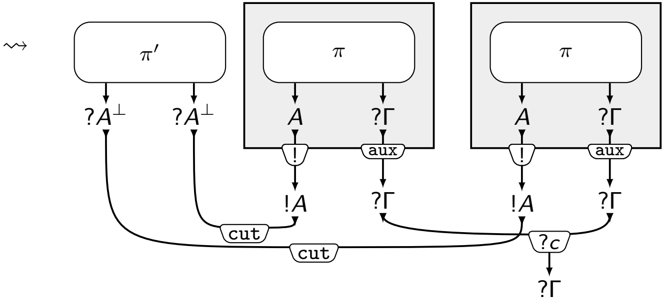
Exponential cut-elimination steps

contraction/promotion: duplication of a resource





Exponential cut-elimination steps

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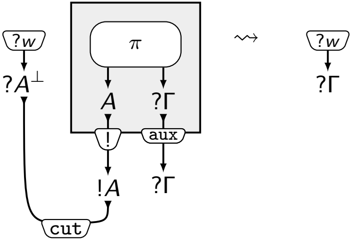

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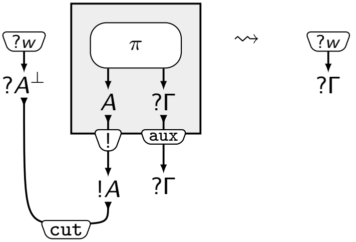

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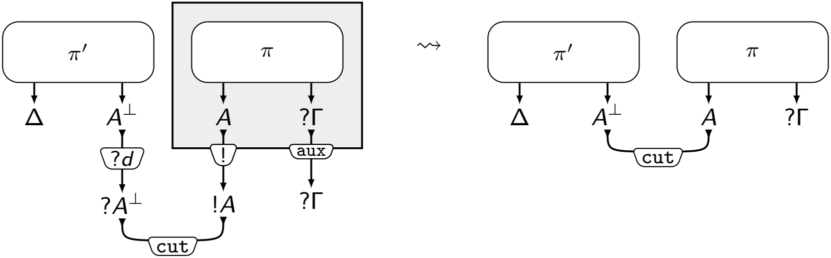
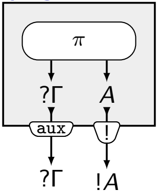


MELL proof-nets are really graphs?



How do we identify boxes? We have to get out of standard graph representations.

In the literature, there are several solutions:

1. A MELL proof-net is a directed graph + something to identify boxes
   - something can be provided informally ⇝ not rigorous, hand-waving;
   - something can be provided formally ⇝ *ad hoc* and tricky, not handy.
2. A MELL proof-net is an inductive directed graph, where with any node !

is associated another inductive directed graph (the content of the box).
⇝ Girard's idea of proofs as graphs in watered-down.

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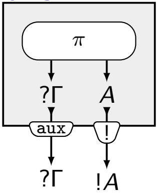

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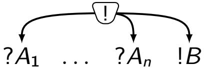

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Our goal

Small goal:
Can we recover the border of a boxes using only usual notions from graph theory?

Big goal:
Having a canonical and not *ad-hoc* syntax for MELL proof-nets is important!

1. To answer the questions: What is a proof? When are two proofs identical?
2. To have proofs about properties of proof-nets that are not hand-waving.
3. To define sophisticated operation on proof-nets.
4. To hope to formalize proof-nets in proof-assistants.

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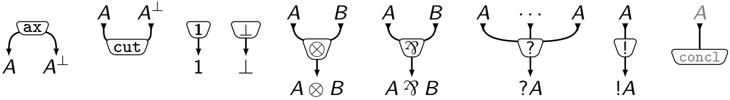
MELL proof-nets as graphs

Definition: An MELL proof-net is a triple $R = (|R|, \mathcal{A}, \text{box})$ such that:

1. $|R|$ is a directed graph (without boxes) labeled by MELL formulas, such that



   and the premises of each $\parr$- or $\otimes$-node are ordered;

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Remark: The last two conditions for *box* express the fact that

1. the rooted tree $\mathcal{A}$ represents the nesting of the boxes of R,
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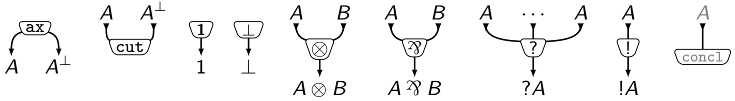

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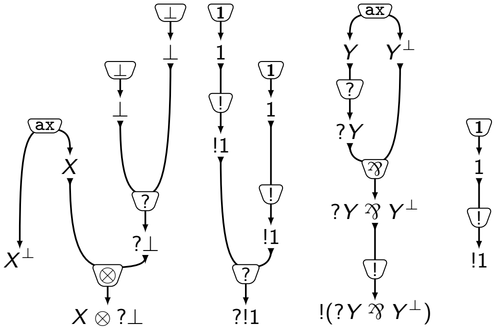

(a) The underlying graph $|R|$ (the information about boxes is extracted from box).

(b) The box-tree $\mathcal{A}$ (without dotted lines) and the reflexive-transitive closure $\mathcal{A}^{\circlearrowleft}$ of $\mathcal{A}$ (with also dotted lines).

There is no explicit constructor for boxes in $|R|$!

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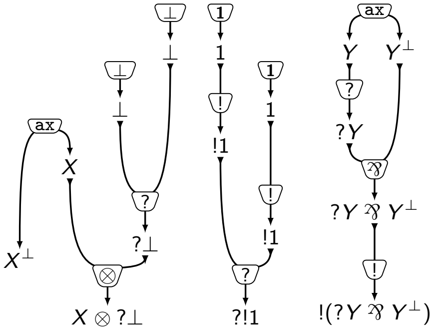

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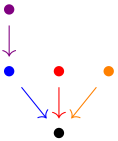

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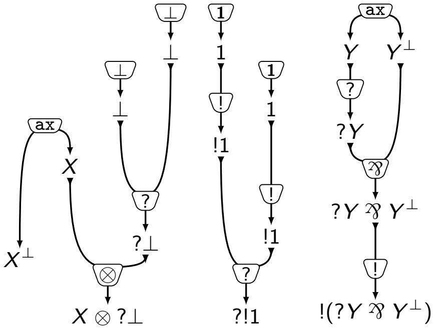

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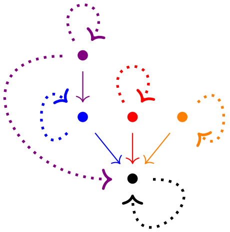

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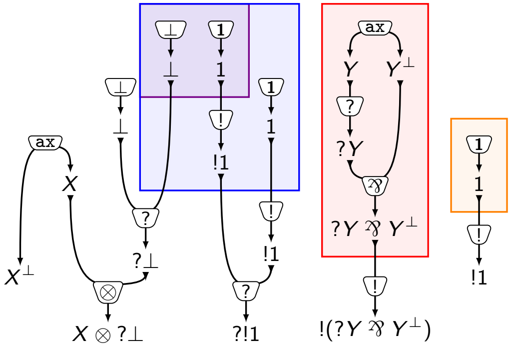

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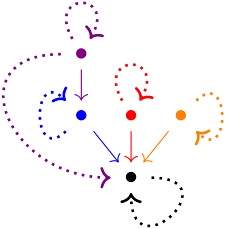

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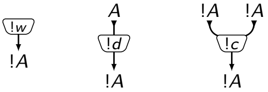
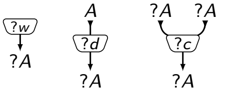
Differential Linear Logic: In addition to promotion, three more rules introduce the modality ! (perfectly symmetric to structural rules) $\Rightarrow$ DiLL extends MELL.



(the duals of



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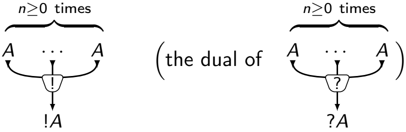

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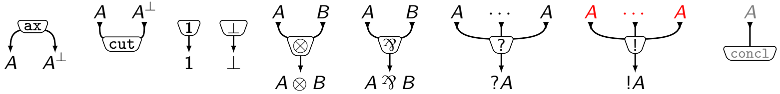

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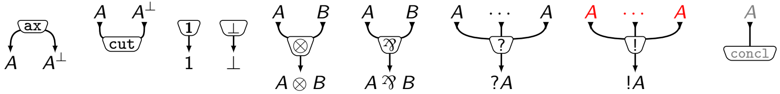

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When are two DiLL proof-nets identical?

Remark: Our definition of proof-nets relies on standard notions from graph theory.

Pros of this purely graphical approach: definitions are

- not *ad hoc*, more “canonical”,
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Example: Identity of proof-nets = graph isomorphism that commutes with *box*.

Definition: Let $R = (|R|, \mathcal{A}, box)$ and $R' = (|R'|, \mathcal{A}', box')$ be DiLL proof-nets. An isomorphism from R to R' is a couple $(f_{|\cdot|}, f_{\text{tree}})$ where:

- $f_{|\cdot|} \colon |R| \to |R'|$ is an isomorphism of labeled directed graphs,
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Exploiting the box-tree of DiLL proof-nets: thick subtrees

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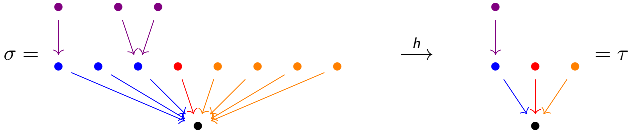
Example:



Idea: Let $R = (|R|, \mathcal{A}, box)$ be a DiLL proof-net. A thick subtree $(\mathcal{A}', h)$ of $\mathcal{A}$

- says how many copies to take for each box, iteratively;
- "generates" a new DiLL proof-net $R' = (|R'|, \mathcal{A}', box')$.

Which one? What are $|R'|$ and box'?

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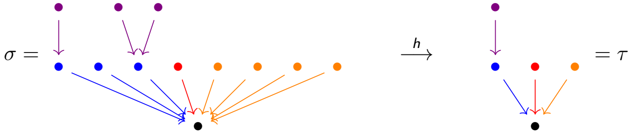

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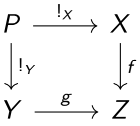
Pullbacks in category theory

Definition: Let $\mathcal{C}$ be a category, $X, Y, Z \in \mathcal{C}$ and $f\colon X \to Z$ and $g\colon Y \to Z$.
A pullback of f and g is a triple $(P, !_X, !_Y)$ such that (1) commutes and, for any $(Q, h : Q \to X, k : Q \to Y)$ making (1) commute, there is a unique arrow $u : Q \to P$ factorizing h and k, i.e. such that diagram (2) commutes.



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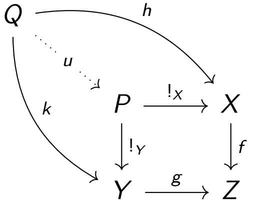


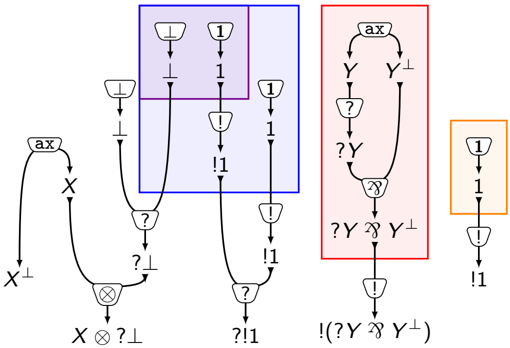
(2)

Theorem: The category of directed graphs has all pullbacks!

Idea: The pullback P of two graph morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ is the "maximal" directed graph compatible with X and Y.

Pullback in DiLL proof-nets

An MELL proof-structure $R = (|R|, \mathcal{A}, box)$: *box* is given by colors.



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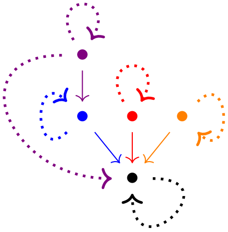

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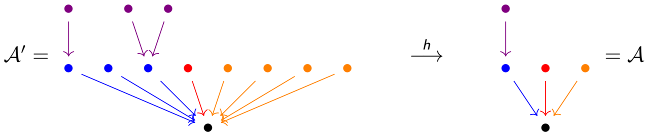

Remark: The graph homomorphism $h\colon \mathcal{A}' \to \mathcal{A}$ lifts to the trans.-refl. closure:

$$h^{\circlearrowleft}\colon {\mathcal{A}'}^{\circlearrowleft} \to \mathcal{A}^{\circlearrowleft}$$

Pullback in DiLL proof-nets

We have two graph homomorphisms $h^{\circlearrowleft} \colon {\mathcal{A}'}^{\circlearrowleft} \to \mathcal{A}^{\circlearrowleft}$ and $box \colon |R| \to \mathcal{A}^{\circlearrowleft}$.

Question: What is the pullback $|R'|$ of $h^{\circlearrowleft}$ and box?

$$\begin{array}{ccc} & & {\mathcal{A}'}^{\circlearrowleft} \\ & & \big\downarrow h^{\circlearrowleft} \\ |R| & \xrightarrow{box} & \mathcal{A}^{\circlearrowleft} \end{array}$$

$|R'| =$

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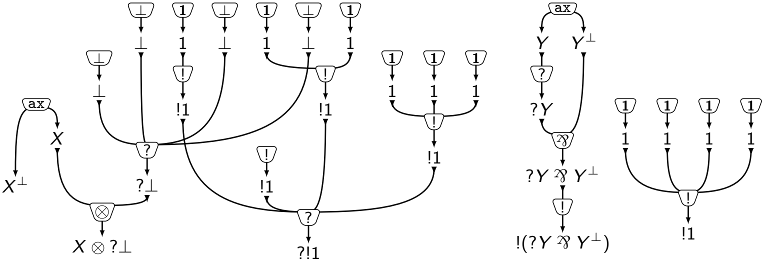
Remark: This is possible because all objects live in the category of directed graphs

Pullback in DiLL proof-nets

We have two graph homomorphisms $h^{\circlearrowleft} : \mathcal{A}'^{\circlearrowleft} \to \mathcal{A}^{\circlearrowleft}$ and $box : |R| \to \mathcal{A}^{\circlearrowleft}$.
Question: What is the pullback $|R'|$ of $h^{\circlearrowleft}$ and box?

$$\begin{array}{ccc} |R'| & \xrightarrow{box'} & \mathcal{A}'^{\circlearrowleft} \\ \Big\downarrow \pi & & \Big\downarrow h^{\circlearrowleft} \\ |R| & \xrightarrow{box} & \mathcal{A}^{\circlearrowleft} \end{array}$$

$|R'| =$ 

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We can define in an "easy" but rigorous way operations on proof-nets that are:

- intuitively clear,
- tricky to define precisely.

Examples: cut-elimination, Taylor expansion, identity of proof-nets, correctness . . .

For correctness criterion, identity of proof-nets and the Taylor expansion in MELL:

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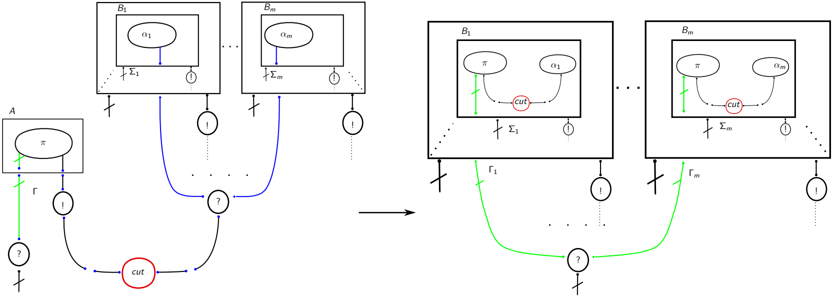

It is tricky (and usually boring) to define this step in a rigorous way!

Sketch of the definition in our framework $\rightsquigarrow$ get rid of most tedious aspects.

1. The exponential cut says which thick subtree to consider (how many copies?).
2. Pullback from that thick subtree, in which we duplicate the box.
3. Local rewriting on the underlying graph (erase !/? nodes, add new cut nodes).
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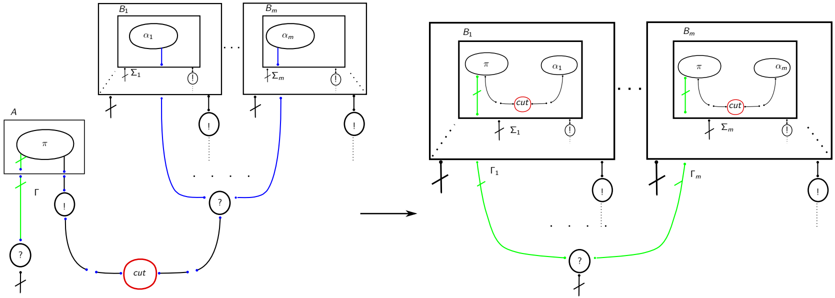

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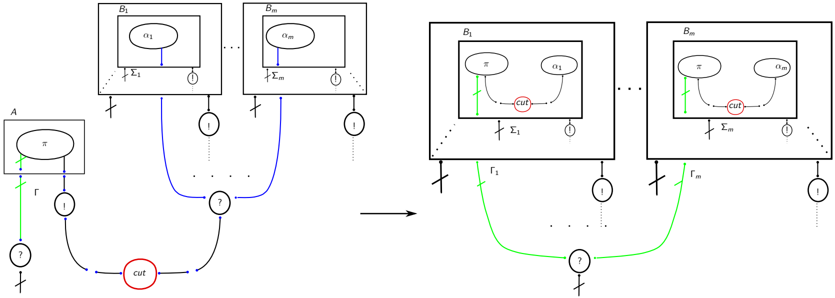

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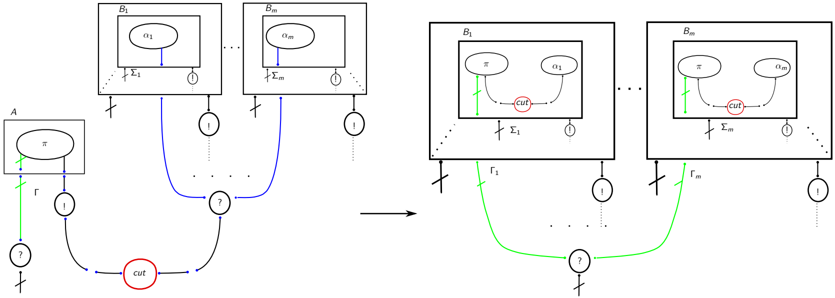

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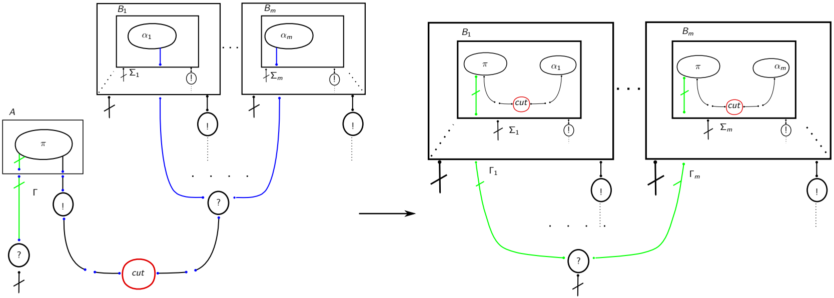

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1. Parallel cut-elimination for MELL proof-nets:
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   - strong normalization.
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3. . . .

Many of these results are already established with different methods.
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