A Parametrised Functional Interpretation of Affine Logic

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Functional interpretations

A functional interpretation $I : S \to T$ is a translation of formulas such that if a formula A is provable in S then there exist terms t such that $A^I(t)$ is provable in T.

Functional interpretations: applications

- Relative consistency of HA (Gödel)
- Independence of Markov's principle (Kreisel)
- Proof mining (Kohlenbach)
- Interpretation Weak König's Lemma (Ferreira, Oliva)
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Different interpretations for different purposes.

We try to capture their common structure.

A little history

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Kohlenbach (monotone functional interpretation) (1996)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
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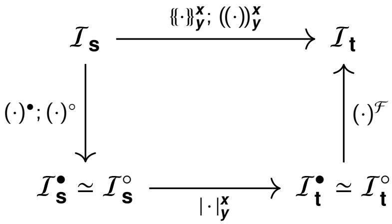
- Compare the various existing functional interpretations.
- Help explain subtle details of the more recent interpretations (BFI, Herbrandized,...)
- Obtain new interpretations

Parametrised interpretations of $\mathcal{I}_{\mathbf{s}}$ into $\mathcal{I}_{\mathbf{t}}$



$\mathcal{I}_{\mathbf{s}}$: (intuitionistic) source theory

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$(\cdot)^{\bullet}$; $(\cdot)^{\circ}$: Girard's translations

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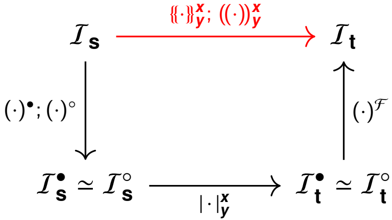

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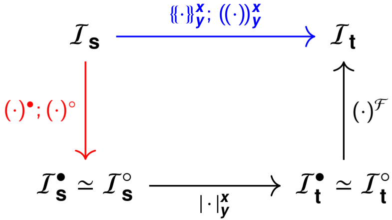

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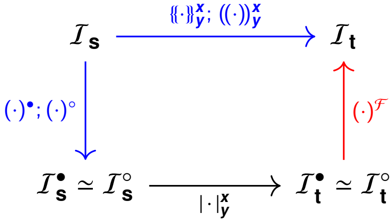

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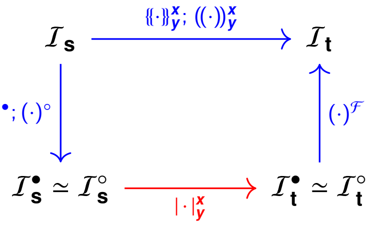

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AL Rules

$$\frac{}{A \vdash A}\ (\mathrm{id}) \qquad \frac{}{\Gamma, \bot \vdash A}\ (\mathrm{efq})$$

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ (\mathrm{cut}) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\ (\mathrm{per})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes \mathrm{R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes \mathrm{L})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap \mathrm{R}) \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap \mathrm{L})$$

AL Rules

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ (\forall \mathrm{R}, x \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}\ (\forall \mathrm{L})$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists \mathrm{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}\ (\exists \mathrm{L}, x \notin \mathrm{FV}(\Gamma, B))$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\mathrm{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ (\mathrm{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\mathrm{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!\mathrm{L})$$

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From $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$

We use Girard's translations of $\mathbf{IL}^{\mathbb{B}}$ into $\mathbf{AL}^{\mathbb{B}}$:

$$
\begin{array}{llll}
(P(\boldsymbol{x}))^\bullet & :\equiv P(\boldsymbol{x}) & (P(\boldsymbol{x}))^\circ & :\equiv\, !P(\boldsymbol{x}), \quad \text{if } P \not\equiv \bot \\
\bot^\bullet & :\equiv \bot & \bot^\circ & :\equiv \bot \\
(A \wedge B)^\bullet & :\equiv A^\bullet \otimes B^\bullet & (A \wedge B)^\circ & :\equiv A^\circ \otimes B^\circ \\
(A \rightarrow B)^\bullet & :\equiv\, !A^\bullet \multimap B^\bullet & (A \rightarrow B)^\circ & :\equiv\, !(A^\circ \multimap B^\circ) \\
(\forall x A)^\bullet & :\equiv \forall x A^\bullet & (\forall x A)^\circ & :\equiv\, !\forall x A^\circ \\
(\exists x A)^\bullet & :\equiv \exists x !A^\bullet & (\exists x A)^\circ & :\equiv \exists x A^\circ
\end{array}
$$

Back into $\mathbf{IL}^{\mathbb{B}}$: the forgetful function

Define a translation of formulas of $\mathbf{AL}^{\mathbb{B}}$ into formulas of $\mathbf{IL}^{\mathbb{B}}$ inductively as follows:

$$
\begin{aligned}
(P(\boldsymbol{x}))^{\mathcal{F}} &:\equiv P(\boldsymbol{x}), \quad \text{for the predicate symbols } P \\
(A \otimes B)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \wedge B^{\mathcal{F}} \\
(A \multimap B)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \rightarrow B^{\mathcal{F}} \\
(!A)^{\mathcal{F}} &:\equiv A^{\mathcal{F}} \\
(\forall x A)^{\mathcal{F}} &:\equiv \forall x A^{\mathcal{F}} \\
(\exists x A)^{\mathcal{F}} &:\equiv \exists x A^{\mathcal{F}}
\end{aligned}
$$

Towards the parametrised interpretation

Our parametrised interpretation of $\mathscr{A}_{\mathbf{s}}$ into $\mathscr{A}_{\mathbf{t}}$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(\boldsymbol{x})$, associate, $\boldsymbol{x} \prec^{P} a$.

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   ($\mathbf{Q}_1$) If $A \vdash_{\mathcal{A}_{\mathbf{t}}} B$ then $!\forall \boldsymbol{x} \sqsubset_{\tau} \boldsymbol{a}\, A \vdash_{\mathcal{A}_{\mathbf{t}}} \forall \boldsymbol{x} \sqsubset_{\tau} \boldsymbol{a}\, B$

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   (**Q**$_2$) $\vdash_{\mathcal{A}_\mathbf{t}} \forall \boldsymbol{x} \sqsubset_\tau \boldsymbol{a}\, \mathrm{W}(\boldsymbol{x})$

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($\mathbf{C}_\eta$) $\rightsquigarrow$ to deal with substitutions.

($\mathbf{C}_\sqcup$) $\rightsquigarrow$ to have a sort of union/maximum of two terms.

($\mathbf{C}_\circ$) $\rightsquigarrow$ to deal with application of terms.

Parametrised **AL**-interpretation

For each formula A of $\mathcal{A}_\mathbf{s}$, let us associate a formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ of $\mathcal{A}_\mathbf{t}$, with two fresh lists of free-variables $\boldsymbol{x}$ and $\boldsymbol{y}$, inductively as follows:

$$|P(\boldsymbol{x})|^a \quad :\equiv \quad \boldsymbol{x} \prec^P a, \qquad (P \text{ computational})$$

$$|P(\boldsymbol{x})| \quad :\equiv \quad P(\boldsymbol{x}), \qquad (P \text{ non-computational})$$

$$|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} \quad :\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{gxw}} \multimap |B|^{\boldsymbol{fx}}_{\boldsymbol{w}}$$

$$|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \quad :\equiv \quad |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|\exists z A|^{\boldsymbol{x}}_{\boldsymbol{y}} \quad :\equiv \quad \exists z |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$|\forall z A|^{\boldsymbol{x}}_{\boldsymbol{y}} \quad :\equiv \quad \forall z |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \quad :\equiv \quad !\forall \boldsymbol{y} \sqsubset_{\tau_A^-} \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Soundness

Theorem (Soundness)

If $\mathcal{A}_{\mathbf{t}}$ is adequate and the axioms of $\mathcal{A}_{\mathbf{s}}$ are witnessable in $\mathcal{A}_{\mathbf{t}}$, then the parametrised **AL***-interpretation is sound.*

Instances

$\forall x \sqsubset_\tau a\, A$	$x \prec^\tau a$	$W_\tau(a)$	**Interpretation**
$A[a/x]$	$x = a$	true	Dialectica interpretation
$\forall x A$	$x = a$	true	Modified realizability
$\forall x \leq^* a\, A$	$x = a$	true	(*combination not sound*)
$\forall x \in a\, A$	$x = a$	true	Diller-Nahm interpretation
$A[a/x]$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	(*combination not sound*)
$\forall x\, A$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	Bounded modified realizability
$\forall x \leq^* a\, A$	$x \leq^* a$	$a \leq^* a$	Bounded functional interpretation
$\forall x \in a\, A$	$x \leq^*_\tau a$	$a \leq^*_\tau a$	**Bounded Diller-Nahm interpretation**
$A[a/x]$	$x \in a$	true	**Herbrand Dialectica** ($\simeq$ Dialectica)
$\forall x\, A$	$x \in a$	$\tau^*(a)$	Herbrand realizability (for IL)
$\forall x \leq^* a\, A$	$x \in a$	$a \leq^*_\tau a$	**Herbrandized bfi**
$\forall x \in a\, A$	$x \in a$	$\tau^*(a)$	Herbrand Diller-Nahm interpretation

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- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
- Composing with Krivine's negative translation does one obtain classical interpretations? Factorization?

Thank you!