A Parametrised Functional Interpretation of Affine Logic

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A functional interpretation $I : S \rightarrow T$ is a translation of formulas such that if a formula $A$ is provable in $S$ then there exist terms $t$ such that $A^I(t)$ is provable in $T$. 
Functional interpretations: applications

- Relative consistency of HA (Gödel)
- Independence of Markov’s principle (Kreisel)
- Proof mining (Kohlenbach)
- Interpretation Weak König’s Lemma (Ferreira, Oliva)
- Interpretation of principles of Nonstandard analysis (Van den Berg, Briseid, Safarik)
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Different interpretations for different purposes.
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Different interpretations for different purposes.

We try to capture their common structure.
A little history

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
- ...
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Goal

Give a parametrised functional interpretation to unify all the well known functional interpretations (including the approximate ones).
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▶ Help explain subtle details of the more recent interpretations (BFI, Herbrandized,...)
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Give a parametrised functional interpretation to unify all the well-known functional interpretations (including the approximate ones).

- Compare the various existing functional interpretations.
- Help explain subtle details of the more recent interpretations (BFI, Herbrandized,...)
- Obtain new interpretations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

$\mathcal{I}_s \xrightarrow{\llbracket \cdot \rrbracket^x_y; \llbracket \cdot \rrbracket^x_y} \mathcal{I}_t$

$\mathcal{I}_s^\bullet \approx \mathcal{I}_s^\circ \quad \llbracket \cdot \rrbracket^x_y \quad \mathcal{I}_t^\bullet \approx \mathcal{I}_t^\circ$

$\mathcal{I}_s$: (intuitionistic) source theory

$\mathcal{I}_t$: (intuitionistic) target theory

$(\cdot)^\bullet; (\cdot)^\circ$: Girard’s translations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

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$\mathcal{I}_s^\circ \approx \mathcal{I}_s^\bullet \xrightarrow{\cdot\cdot^x_y} \mathcal{I}_t^\bullet \approx \mathcal{I}_t^\circ$

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Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

$\mathcal{I}_s \xrightarrow{\{\cdot\}_y^x; (\cdot)_y^x} \mathcal{I}_t$

$\mathcal{I}_s^\bullet \simeq \mathcal{I}_s \quad \mathcal{I}_t^\bullet \simeq \mathcal{I}_t$

$\mathcal{I}_s^\circ \quad \mathcal{I}_t^\circ$

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Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

\[\begin{align*}
\mathcal{I}_s & \quad \xrightarrow{\{\cdot\}_y; \{\cdot\}_{x}^y} \quad \mathcal{I}_t \\
\mathcal{I}_s^\bullet & \simeq \mathcal{I}_s^\circ \quad |\cdot|_y^x \quad \mathcal{I}_t^\bullet & \simeq \mathcal{I}_t^\circ \\
\end{align*}\]

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$\mathcal{I}_s$: (intuitionistic) source theory
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### AL Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(id)</strong></td>
<td>$A \vdash A$</td>
</tr>
</tbody>
</table>
| **(cut)** | $\Gamma \vdash A \Delta, A \vdash B$  
  $\Gamma, \Delta \vdash B$ |
| **(efq)** | $\Gamma, \bot \vdash A$ |
| **(per)** | $\Gamma \vdash A$  
  $\pi\{\Gamma\} \vdash A$ |
| **(⊗R)** | $\Gamma \vdash A \Delta \vdash B$  
  $\Gamma, \Delta \vdash A \otimes B$ |
| **(⊗L)** | $\Gamma, A \otimes B \vdash C$ |
| **(→R)** | $\Gamma \vdash A \rightarrow B$  
  $\Gamma \vdash A \rightarrow B$ |
| **(→L)** | $\Gamma \vdash A \Delta, B \vdash C$  
  $\Gamma, \Delta, A \rightarrow B \vdash C$ |
### AL Rules

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<tr>
<td>( \Gamma \vdash A )</td>
<td>(( \forall R, x \notin \text{FV}(\Gamma) ))</td>
</tr>
<tr>
<td>( \Gamma \vdash \forall x A )</td>
<td>( \Gamma, \forall x A \vdash B ) (( \forall L ))</td>
</tr>
<tr>
<td>( \Gamma \vdash A[t/x] )</td>
<td>( \Gamma, \exists x A \vdash B ) (( \exists R ))</td>
</tr>
<tr>
<td>( \Gamma \vdash \exists x A )</td>
<td>( \Gamma, \exists x A \vdash B ) (( \exists L, x \notin \text{FV}(\Gamma, B) ))</td>
</tr>
<tr>
<td>( \Gamma, !A, !A \vdash B )</td>
<td>( \Gamma \vdash B ) (( \text{con} ))</td>
</tr>
<tr>
<td>( \Gamma, !A \vdash B )</td>
<td>( \Gamma, A \vdash B ) (( \text{wkn} ))</td>
</tr>
<tr>
<td>( !\Gamma \vdash A )</td>
<td>( !\Gamma \vdash !A ) (( \text{!R} ))</td>
</tr>
<tr>
<td>( \Gamma, A \vdash B )</td>
<td>( \Gamma, !A \vdash B ) (( \text{!L} ))</td>
</tr>
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### AL Rules

- **(∀R, x ∉ FV(Γ))**
  - \(\Gamma \vdash A\)
  - \(\Gamma \vdash \forall x A\)
  - \(\Gamma \vdash A[t/x]\)
  - \(\Gamma \vdash \exists x A\)

- **(∀L)**
  - \(\Gamma, A \vdash B\)
  - \(\Gamma, \forall x A \vdash B\)
  - \(\Gamma, A \vdash B\)
  - \(\Gamma, \exists x A \vdash B\)

- **(con)**
  - \(\Gamma, !A, !A \vdash B\)
  - \(\Gamma, !A \vdash B\)

- **(wkn)**
  - \(\Gamma \vdash B\)
  - \(!\Gamma \vdash A\)
  - \(\Gamma \vdash B\)
  - \(!\Gamma \vdash !A\)

- **(!R)**
  - \(\Gamma, A \vdash B\)
  - \(!\Gamma \vdash A\)

- **(!L)**
  - \(\Gamma, !A \vdash B\)
  - \(\Gamma, !A \vdash B\)
From $\text{IL}_B$ into $\text{AL}_B$

We use Girard’s translations of $\text{IL}_B$ into $\text{AL}_B$:

$$(P(x))^* \equiv P(x)$$

$$\bot^* \equiv \bot$$

$$(A \land B)^* \equiv A^* \otimes B^*$$

$$(A \rightarrow B)^* \equiv !A^* \rightarrow B^*$$

$$(\forall xA)^* \equiv \forall xA^*$$

$$(\exists xA)^* \equiv \exists x!A^*$$

$$(P(x))^\circ \equiv !P(x), \quad \text{if } P \neq \bot$$

$$\bot^\circ \equiv \bot$$

$$(A \land B)^\circ \equiv A^\circ \otimes B^\circ$$

$$(A \rightarrow B)^\circ \equiv !(A^\circ \rightarrow B^\circ)$$

$$(\forall xA)^\circ \equiv !\forall xA^\circ$$

$$(\exists xA)^\circ \equiv \exists xA^\circ$$
Back into $\textbf{IL}^B$: the forgetful function

Define a translation of formulas of $\textbf{AL}^B$ into formulas of $\textbf{IL}^B$ inductively as follows:

$$(P(x))^F \equiv P(x), \text{ for the predicate symbols } P$$

$$(A \otimes B)^F \equiv A^F \land B^F$$

$$(A \rightarrow B)^F \equiv A^F \rightarrow B^F$$

$$(\neg A)^F \equiv A^F$$

$$(\forall x A)^F \equiv \forall x A^F$$

$$(\exists x A)^F \equiv \exists x A^F$$
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec^P a$.

2. **Domain of witnesses and counter-witnesses**: For each finite type $\tau$ we associate in $\mathcal{A}_t$ a formula $W_\tau(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses. We assume combinatorial completeness for $W_\tau$.

3. **Interpretation of $!_{\mathcal{A}}$**: A form of bounded quantification $\forall x \in \tau a A$satisfying:
   - ($Q_1$) If $A \vdash A_t B$ then $!\forall x \in \tau a A \vdash A_t \forall x \in \tau a B$,
   - ($Q_2$) $\vdash A_t \forall x \in \tau a W(x)$,
   - $\vdash A_t \forall x \in \tau a \left(\exists y \in \tau b B(y) \land x \prec^P b \right)$.
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x <^P a$.

2. **Domain of witnesses and counter-witnesses**: For each finite type $\tau$ we associate in $\mathcal{A}_t$ a formula $W_\tau(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses.

   We assume combinatorial completeness for $W$
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec ^P a$.

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3. **Interpretation of !A**: A form of bounded quantification $\forall x \sqcup _\tau a A$ satisfying:
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec^P a$.

2. **Domain of witnesses and counter-witnesses**: For each finite type $\tau$ we associate in $\mathcal{A}_t$ a formula $W_\tau(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses.

   We assume combinatorial completeness for $W$

3. **Interpretation of $!A$**: A form of bounded quantification $\forall x \sqsubseteq_\tau a A$ satisfying:

   (Q$_1$) If $A \vdash_{\mathcal{A}_t} B$ then $!\forall x \sqsubseteq_\tau a A \vdash_{\mathcal{A}_t} \forall x \sqsubseteq_\tau a B$
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec_P a$.

2. **Domain of witnesses and counter-witnesses**. For each finite type $\tau$ we associate in $\mathcal{A}_t$ a formula $W_{\tau}(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses.

   We assume combinatorial completeness for $W$

3. **Interpretation of $!A$**: A form of bounded quantification $\forall x \sqsubseteq_{\tau} a A$ satisfying:

   (Q₁) If $A \vdash_{\mathcal{A}_t} B$ then $!\forall x \sqsubseteq_{\tau} a A \vdash_{\mathcal{A}_t} \forall x \sqsubseteq_{\tau} a B$

   (Q₂) $\vdash_{\mathcal{A}_t} \forall x \sqsubseteq_{\tau} a W(x)$
Towards the parametrised interpretation

Finally, for each formula, terms \( \eta(\cdot), (\cdot) \sqcup (\cdot) \) and \( (\cdot) \circ (\cdot) \) satisfying conditions
Towards the parametrised interpretation

Finally, for each formula, terms \( \eta(\cdot) \), \( (\cdot) \sqcup (\cdot) \) and \( (\cdot) \circ (\cdot) \) satisfying conditions

\((C_\eta) \rightsquigarrow\) to deal with substitutions.

\((C_\sqcup) \rightsquigarrow\) to have a sort of union/maximum of two terms.

\((C_\circ) \rightsquigarrow\) to deal with application of terms.
Parametrised \textbf{AL}-interpretation

For each formula $A$ of $\mathcal{A}_s$, let us associate a formula $|A|^x_y$ of $\mathcal{A}_t$, with two fresh lists of free-variables $x$ and $y$, inductively as follows:

$$|P(x)|^a \equiv x <^P a, \quad (P \text{ computational})$$

$$|P(x)| \equiv P(x), \quad (P \text{ non-computational})$$

$$|A \rightarrow B|_{x,w}^{f,g} \equiv |A|^x_{gxyw} \rightarrow |B|_w^{fx}$$

$$|A \otimes B|_{x,v}^y \equiv |A|^x_{gxw} \otimes |B|_w^v$$

$$|\exists z A|^x_y \equiv \exists z |A|^x_y$$

$$|\forall z A|^x_y \equiv \forall z |A|^x_y$$

$$|! A|^a_x \equiv !\forall y \subset_{\tau_A} a |A|^x_y.$$
Soundness

Theorem (Soundness)
If $\mathcal{A}_t$ is adequate and the axioms of $\mathcal{A}_s$ are witnessable in $\mathcal{A}_t$, then the parametrised AL-interpretation is sound.
## Instances

<table>
<thead>
<tr>
<th>$\forall x \sqsubseteq_\tau a A$</th>
<th>$x \prec_\tau a$</th>
<th>$W_\tau(a)$</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A[a/x]$</td>
<td>$x = a$</td>
<td>true</td>
<td>Dialectica interpretation</td>
</tr>
<tr>
<td>$\forall x A$</td>
<td>$x = a$</td>
<td>true</td>
<td>Modified realizability</td>
</tr>
<tr>
<td>$\forall x \leq^* a A$</td>
<td>$x = a$</td>
<td>true</td>
<td><em>(combination not sound)</em></td>
</tr>
<tr>
<td>$\forall x \in a A$</td>
<td>$x = a$</td>
<td>true</td>
<td>Diller-Nahm interpretation</td>
</tr>
<tr>
<td></td>
<td>$x \leq^* a$</td>
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</tr>
<tr>
<td>$\forall x A$</td>
<td>$x \leq^* a$</td>
<td>$a \leq^* a$</td>
<td>Bounded modified realizability</td>
</tr>
<tr>
<td>$\forall x \leq^* a A$</td>
<td>$x \leq^* a$</td>
<td>$a \leq^* a$</td>
<td>Bounded functional interpretation</td>
</tr>
<tr>
<td>$\forall x \in a A$</td>
<td>$x \leq^* a$</td>
<td>$a \leq^* a$</td>
<td><strong>Bounded Diller-Nahm interpretation</strong></td>
</tr>
<tr>
<td>$A[a/x]$</td>
<td>$x \in a$</td>
<td>true</td>
<td><strong>Herbrand Dialectica</strong> ((\simeq) Dialectica)</td>
</tr>
<tr>
<td>$\forall x A$</td>
<td>$x \in a$</td>
<td>$\tau^*(a)$</td>
<td>Herbrand realizability (for IL)</td>
</tr>
<tr>
<td>$\forall x \leq^* a A$</td>
<td>$x \in a$</td>
<td>$a \leq^* a$</td>
<td><strong>Herbrandized bfi</strong></td>
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<tr>
<td>$\forall x \in a A$</td>
<td>$x \in a$</td>
<td>$\tau^*(a)$</td>
<td>Herbrand Diller-Nahm interpretation</td>
</tr>
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Questions and future work

- Other ways to instantiate the parameters?
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- Characterization theorem?
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- Variants with truth?
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$$|!A|^x_a \equiv \forall y \sqsubseteq \tau. a |A|^x_y \otimes A.$$
Questions and future work

- Other ways to instantiate the parameters?
- Characterization theorem?
- Variants with truth?

\[ |!A|_a^x \equiv \forall y \sqsubseteq \tau \ a |A|_y^x \otimes A. \]

- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
Questions and future work

- Other ways to instantiate the parameters?
- Characterization theorem?
- Variants with truth?

\[ \text{\textit{A}}|_a^x \equiv \forall y \subseteq \tau \text{\textit{a}}|_y^x \otimes \text{\textit{A}}. \]

- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
- Composing with Krivine’s negative translation does one obtain classical interpretations? Factorization?
Thank you!