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TLLA – Evening Lecture

June 27, 2021

TLLA Evening Lecture.

Linear Logic:

methodological and philosophical contributions

Just, a first list of contributions of linear logic to methodology and philosophy

I am preparing a book on this subject.

1.
A methodological innovation:

birth of a new logic as a "refinement" of classical logic.

Previously: new logics have been obtained
- by a restriction of classical logic (example: intuitionist logic, or substructural logics)
- by adding to classical logic new operators (modal logic, temporal logic, etc.)

Linear Logic is obtained from classical logic by a "refinement":
- if you delete exponential connectives, and replace
    - symbols for multiplicative and additive conjunction by the symbol for classical conjunction,
    - symbols for multiplicative and additive disjunction by the symbol for classical disjunction,
    - symbol for the linear negation by the symbol for classical negation,
- then you obtain classical logic-

Linear Logic is to read classical logic under a microscope!

$$\vdash A \vee \neg A \qquad\qquad \vdash A \parr A^{\perp}$$

$$A \to (B \to A)$$

$$\neg A \vee (\neg B \vee A)$$

$$\vdash A^{\perp} \parr (?B^{\perp} \parr A)$$

Linear Logic is - as far as I know - the first logic obtained from a refinement of classical logic.

Refinement of the raw is a typical component of philosophical thinking and of modern scientific thinking.

Linear Logic inaugurates a new way of introducing new logics: by refinement.

Non commutative Logic - intropduced by P. Ruet and myself - is a refinement of linear logic.

Linear logic is not born by "eliminating" the structural rules (i.e. it is not a substructural logic) but by "analyzing" them, and discovering that the intuitionist implication between A and B could be "refined" as a linear implication between !A and B.

$$A \to B \;=\; !A \multimap B$$

$$= \; ?A^{\perp} \parr B$$

2.

Considerable contributions to the understanding of the notion of "constructive".

Linear logic follows and develops an idea of "constructive" which is affirmed in the second half of the last century.

- Idea of "constructive": a logic is constructive when its propositions can admit "different" proofs.

- Idea of "equal" proofs: proofs that are reduced to the same proof by means of the transformation rules.

$$\pi = \psi \quad \text{when} \quad \pi \rightsquigarrow \chi \quad \text{and} \quad \psi \rightsquigarrow \chi$$
$$\text{for some } \chi$$

Classical logic is not constructive, since (by structural rules) for every proposition A all the proofs of A must be equal.

$$\pi : \vdash A \qquad \psi : \vdash \vdash A \qquad \Rightarrow \qquad \pi = \psi$$

For many reasons, from the philosophical point of view this idea of constructive is preferable to that according to which a logic is constructive if it has the property of disjunction (prove A or prove B when proving A or B) and existential (prove A [t] for some t, when it proves that there is an x such that A [x])

$\vdash A \vee B \implies \vdash A$ or $\vdash B$

$\vdash \exists x\, A[x] \implies$ there is $\underline{t}$ s.t. $\vdash A[\underline{t}]$

The main contribution of Linear Logic to better understand what is "constructive", is given by the following result:

Linear Logic - i.e. refined classical logic, is "constructive", in the sense that its propositions can admit "different" proofs.

This means that excluded third and involutive negation are compatible with "constructivism", contrary to what was the common opinion before the birth of linear logic!

Excluded third and involutive negation of linear logic express the interaction between a proposition and its negation, and this interaction is constructive!

Hilbert aimed - within his program - to show the constructive character of the excluded third as well.

The result obtained from Linear Logic highlights how the constructive character of the excluded third lies precisely in overcoming the idea that the construction must always be done by a single subject: the excluded third requires the presence of two poles, of two subjects, and it expresses the interaction between a proposition (supported by one of the two) and its negation (supported by the other).

3.

Geometric representation of proofs, using graphs (not only trees): Proof Nets

Throughout the history of logic, the representation of proofs has always been imposed:
- as sequences
- as trees

And these two representations were adopted by Hilbert, and then by Gentzen with the aim that each branch in the tree corresponded to a "natural rule" of reasoning (natural deduction).

In Linear Logic, the proofs are thought to be produced also by the cooperation between several subjects, and therefore we accept rules with more conclusions, and proofs with more conclusions.

Axiom: $\overline{A^{\perp} \quad A}$

$A^{\perp}$ and A interact

$\begin{cases} \text{a refutation of } A^{\perp} \text{ is a proof of } A \\ \text{a refutation of } A \text{ is a proof of } A^{\perp} \end{cases}$

$$\frac{A \otimes B}{A \quad B}$$

Proof-nets: graphs associated to proofs in linear logic. a natural deduction for classical logic.

Explained in the tutorial given by Lionel Vaux,

this morning.

- Some philosophically relevant consequences of the geometric representation of proofs using proof nets

  the simplest proof nets n (axiom only): identity $A \vdash A$

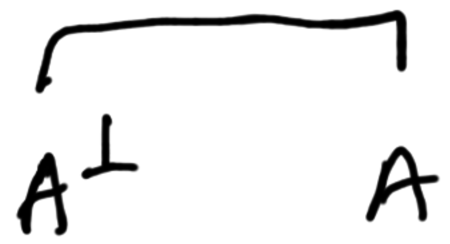
  $A ⅋ B \vdash A ⅋ B$

  Proof nets with two axioms, a tensor and a par: identity $A \otimes B \vdash A \otimes B$

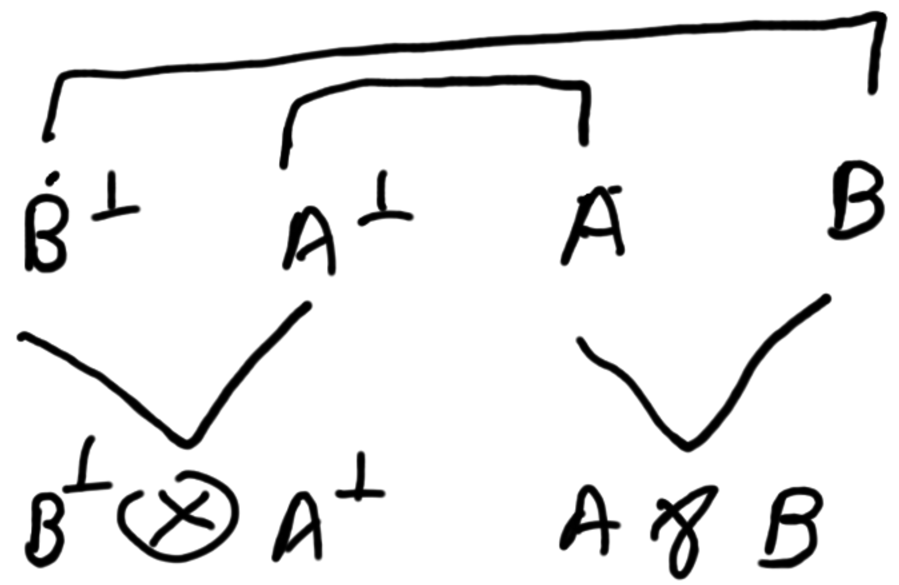
  Proof nets with three axioms, two tensors and a par: syllogisms

  (transitivity)

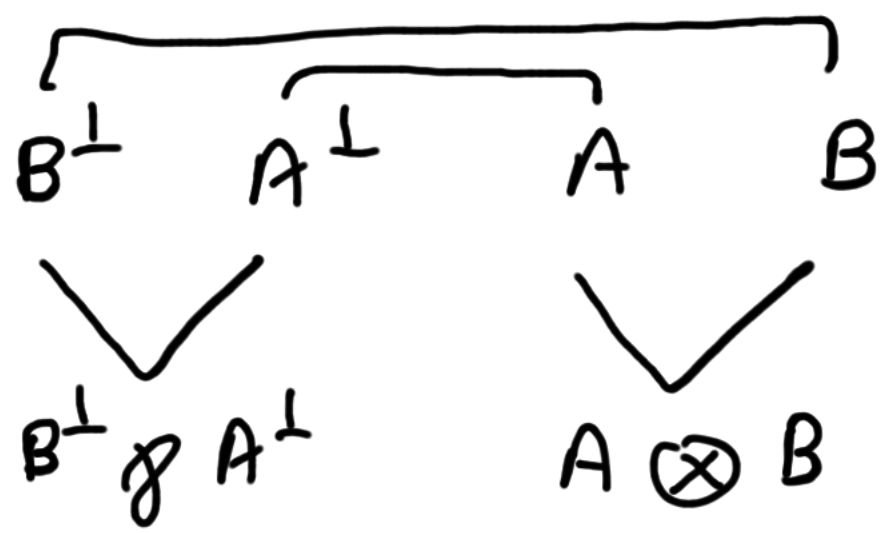
  not depending on bivalence of classical logic!



$A \vdash A$



$A \parr B \vdash A \parr B$

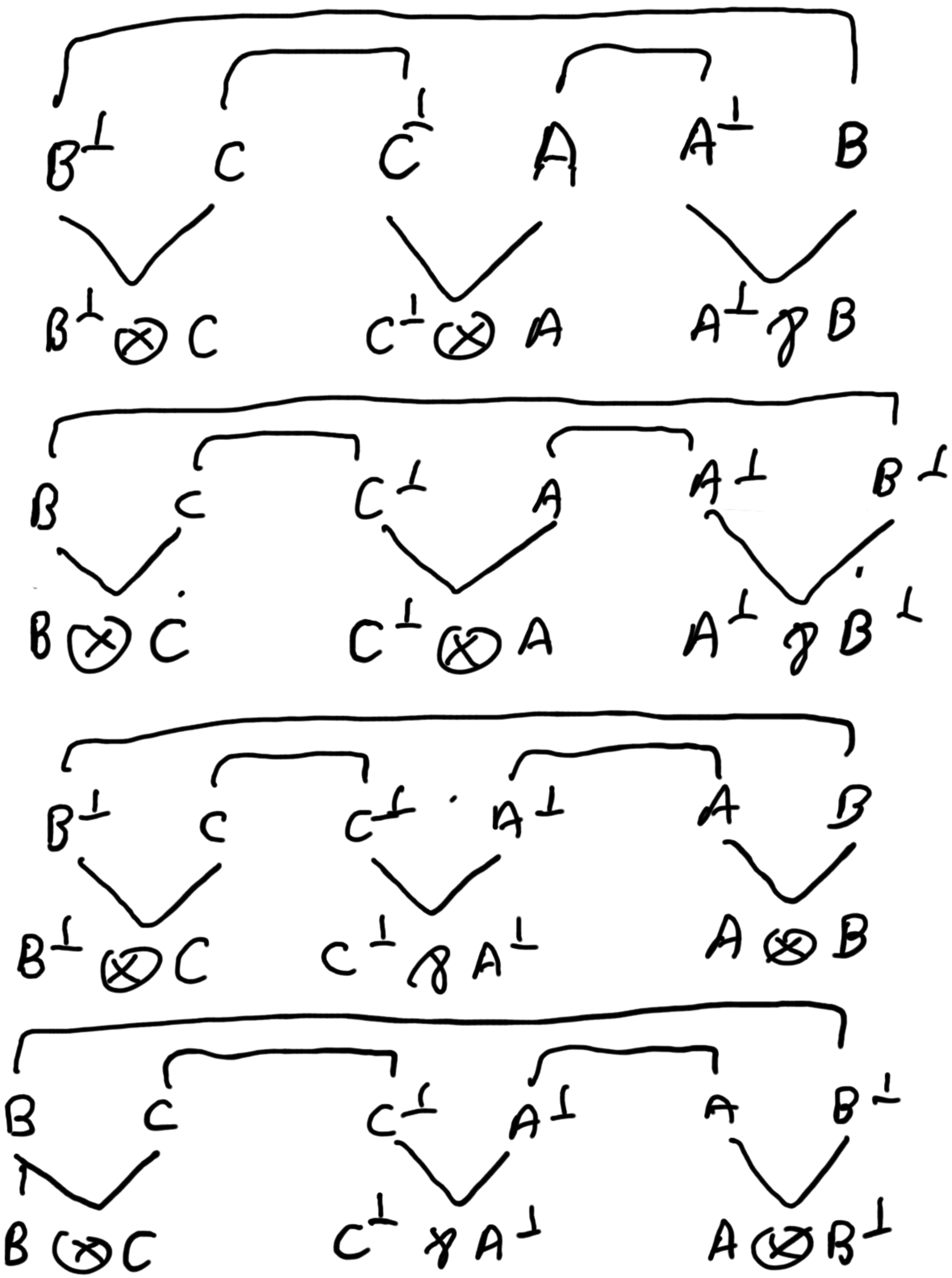


$A \otimes B \vdash A \otimes B$

(planar graphs,

non-commutative logic)

planar graphs



$A \multimap C,\ C \multimap B \vdash A \multimap B$

BARBARA

$A \multimap C,\ C \multimap B^{\perp} \vdash A \multimap B^{\perp}$

CELARENT

$A \otimes C,\ C \multimap B \vdash A \otimes B$

DARII

$A \otimes C,\ C \multimap B^{\perp} \vdash A \otimes B^{\perp}$

FERIO

4.

Conditions of possibility - in the Kantian sense - for the validity of a proof

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Correctness criterion of Proof-nets: an internal property of graphs,

An internal property of graphs - not a property defined in reference to a system of rules, not a semantic concept - which ensures that such a graph is a valid proof, that is, it comes from the sequent calculus, and (seen semantically) passes from the validity of the premises to the validity of the conclusion.

Correctness criterion. explained in the tutorial given by Lionel Vaux

Kant's program: to establish the validity of our knowledge
through internal properties,
without referring to external reality or to the content of our knowdlege ;
these "internal properties" are the "conditions of possibility" for a valid knowledge.

Hilbert's program was heavily inspired by Kant's program

Hilbert's program:
- to establish the validity of mathema9tics through internal properties of mathematical theories, without referring to the content. -
- Internal properties: coherence and completenness ("conditions of possibility" for a mathematical valid theory)
- Establish these properties by means of finitistic proofs.

Hilbert's Program failed , by Incompleteness Theorems:
- Formal systems for arithmetic or analysis cannot be coherent and complete, i.e. if such a system is coherent then it is incomplete.
- Moreover, a typical example of incompleteness of such a system is the aritmentical statement expressing the coherence of the system.

Girard's discovery: it is possible to establish the correctness of a proof-net by means of internal properties (in the case of multiplicative fragment of Linear Logic, a proof-stricture s.t. for every switching becomes a connected and acyclic graph).

These internal properties (i.e. correctness criterion of proof nets) are "conditions of possibility" for a valid proof.

For the first time in the history of logic, the validity of a proof is ensured - in perfect Kantian style - through its exclusive internal properties, without reference to semantics or to a system of rules.

5.
What is truth, what is validity?

An ancient concept, which has been taken up in many ways by linear logic.

5.
What is truth, what is validity?

Being able to defeat any objection, being irrefutable.

An ancient concept, which has been taken up in many ways by linear logic.

For example:

a proof is valid if it defeats any counter-proof (and the switches of the proof-nets correspond to counter-proofs ...)

In this concept of validity and truth, the importance of "incorrect" objects emerges (objections, counter-proofs) that allow us to establish the truth or validity of something.

Moreover,

Goedel's incompleteness theorem establishes that there are truths of logic that cannot be established except through the use of tools that are "incorrect" for logic (tools from other disciplines).

A conception of truth and validity that is in some way close to Hegel's conception, and in any case typical of

- many parts of culture,
- many disciplines (for example, jurisprudence where truth is determined as the result of a debate)
- scientific practice (a paper is accepted when it overcomes all the objections of the referees).

6. Living in incompleteness

The incompleteness concerns precisely logic, if logic is not understood (as was done in the last century) as " first order Logic".

And the closure within the first-order logic was unfortunately the answer of a part of the logical community of the twentieth century, precisely in order not to confront with the fact that the logic is incomplete and cannot have a totally corresponding syntax and semantics.

The research program in linear logic, proposed by Girard, constantly takes into account the incompleteness of logic, and develops essentially in second order logic where the incompleteness theorem holds.

So, linear logic lives in incompleteness.

Living in incompleteness means interacting strongly and strictly with all disciplines, because by interacting with them you can also discover results of a logical nature that logic alone could not discover.

In fact, logic has discovered, thanks to the contribution of computer Science, many new approaches, many new directions.

And the same goes for relations of logic with linguistics.

And I believe that in the future the relations of logic with biology, with the economy, and with the law also, will be interesting and fruitful.

The beautiful multi-year experience (in the first decade of this century) of the LIGC group (Logique et interaction: vers une géometrie de la cognition), promoted by Girard and others, was inspired precisely by the need to make disciplines interact with logic, and was fruitful.

I hope that in the future this or similar experience can be resumed and developed.

7. Questions and answers.

Logic is certainly useful in order to establish whether the answer to a problem is correct, that is, if it is a proposition that follows logically from the hypotheses of the problem by means of a logical proof.

Deductive logic aims precisely at studying logical proofs, and therefore the correctness of the answers to problems.

However, only correct answers to sensible (i.e., meaningful) questions are important for the growth and development of knowledge.

The importance of question-constitution in mathematics was well illustrated by David Hilbert in his 1900 lecture entitled "Mathematische Probleme" and containing a large list of well-chosen and fruitful open problems.

Today, unfortunately dominates - and is becoming obsessive in evaluation - a conception for which only the results obtained and the number of such results count.

Instead, each result is worthy of being accepted if it answers a question worthy of being asked and if it is fruitful in that it leads to new questions worthy of being asked.

Surely, the evaluation of a scientist or a community of scientists must take into account how many sensible and fruitful problems have been proposed and not just how many results have been obtained.

Certainly, well-posed problems are those that question principles or beliefs that are commonly accepted without proof.

The history of logic over the past two centuries is marked by the questioning (and by giving a rigorous refutation) of commonly accepted beliefs:
- "all infinite sets are equipotent"
- "every true logical proposition can be proved logically"
- etc.

Linear logic has questioned and refuted some important commonly accepted beliefs:
- the intuitionist implication is a primitive connective, not definable in terms of other connectives;
- a logic with involutive negation is not constructive
- proofs are defined only on the basis of rules of inferences and axioms;
etc.

(Girard)

The transcendental syntax is also a program in which due attention is paid to the "acceptability of the questions", the "sensibility of the questions", as well as the correct answer to the questions.

And it is the resumption of a Kantian attitude too: investigate what are the questions that can be faced, and why.

An attitude that is found in many parts of our cognitive activity, for example in law (when a question can be posed and can therefore be addressed)

Thank you for your attention!

Thanks for the invitation to give

a "evening lecture"!