WBTS: the new class of WSTS without WQO

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SynCoP + PV
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- Based on joint works with Michael Blondin & Pierre McKenzie, to appear in LMCS.

```
one_task(int k) {
 // some work...
}
main(int P) {
 for i := 1 to P step 1
  fork(one_task(i))
}
```

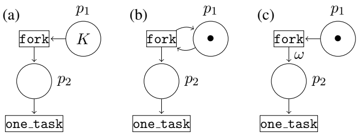


Figure 1. An example of a parametric system with three possible models

From parameterized systems

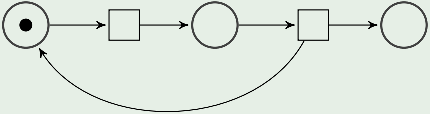
- Parameterized protocols: mutual exclusion, broadcast protocols
- Parametric concurrent systems with dynamic thread creation (ex: Geeraerts & al. FI, 2015)
- Parameterized systems in BIP (ex: Konnov & al. Concur'2016)

To infinite-state models

- Counter (abstraction) machines with reset-transfer-affine extensions
- ω-Petri nets (infinite branching)
- WSTS

Well Structured Transition Systems (WSTS) encompass a large number of infinite state systems (PN and reset-transfer-affine-ω extensions, lossy fifo systems, broadcast protocols, CFG, graph rewriting, depth bounded processes, fragments of the π-calculus,....)

Example of WSTS: Petri nets



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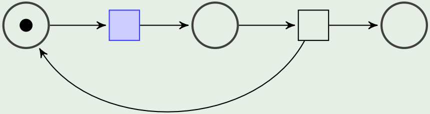

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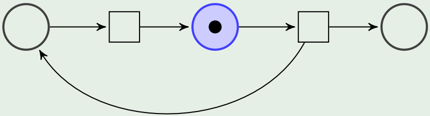

Multiple decidability results are known for (finitely branching) WSTS.

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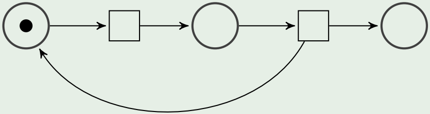

Post(⊙ ○ ○) = ○ ⊙ ○

And also for (infinitely branching) WSTS such as systems with infinitely many initial states and parametric systems

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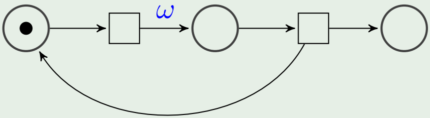

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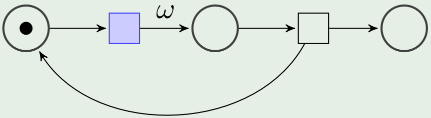

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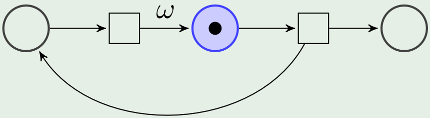

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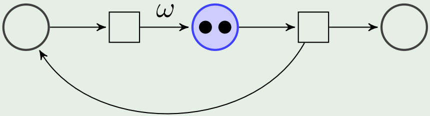

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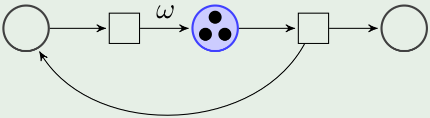

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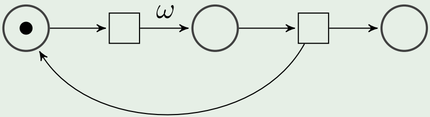
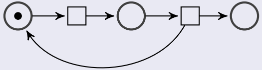


Well structured transition system (F, ICALP'87)

$S = (X, \rightarrow, \leq)$ where

- X set,
- $\rightarrow \subseteq X \times X$,
- monotony,
- well-quasi-ordered.



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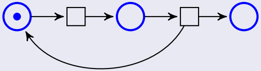

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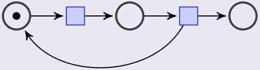

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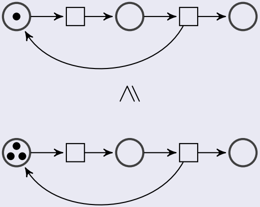

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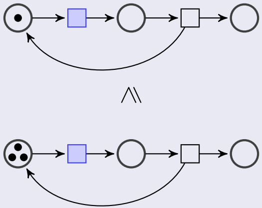

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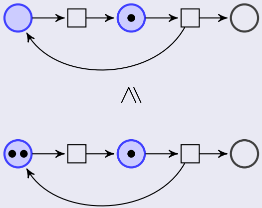

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$$
\begin{array}{cccc}
\forall\, x & \rightarrow & y & \\
\leq & & \leq & \\
x' & \xrightarrow{*} & y' & \exists
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$$

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- transitive monotony,
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$$\begin{array}{cccc} \forall\; x & \rightarrow & y & \\ \leq & & \leq & \\ x' & \boxed{\xrightarrow{+} \;\; y'} & & \exists \end{array}$$

Well structured transition system (F, ICALP'87)

$S = (X, \rightarrow, \leq)$ where

- X set,
- $\rightarrow \subseteq X \times X$,
- strong monotony,
- well-quasi-ordered.

$$\begin{array}{cccc} \forall\; x & \rightarrow & y & \\ \wedge\!\!\backslash & & \wedge\!\!\backslash & \\ x' & \rightarrow & y' & \exists \end{array}$$

Well structured transition system (F, ICALP'87)

$S = (X, \rightarrow, \leq)$ where

- X set,
- $\rightarrow \subseteq X \times X$,
- monotony,
- well-quasi-ordered:
  $\forall x_0, x_1, \ldots \; \exists i < j$ s.t. $x_i \leq x_j$.

WSTS Everywhere! (F, Schnoebelen LATIN'98, TCS'01)

- $T(w)$ = length of a longest computation starting from $w \in \Sigma^*$.
- $T(w) \in \mathbb{N}_\omega$.
- $w \leq_T w'$ if $T(w) \leq T(w')$.
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Theorem

Turing machines are WSTS with strict and strong monotony wrt $\leq_T$.

WSTS Everywhere!

- $\leq_T$ is not decidable.
- Hence TM are non-effective WSTS.
- This also proves that there is no (non-trivial) decidability result for non-effective WSTS (not surprising !).

Effective WSTS

A WSTS $S = (X, \rightarrow, \leq)$, given as a tuple $(M_X, M_{\rightarrow}, M_{\leq})$ of Turing machines, is *effective* if:

(1) M_X decides X

(2) $M_{\rightarrow}$ decides $\rightarrow$.

(3) $M_{\leq}$ decides $\leq$.

(Effective) WSTS Everywhere!

- $S = (\mathbb{N}^k, \leq)$.
  - Petri nets: WSTS with strict and strong monotony.
  - Positive Affine nets, Reset/Transfer Petri nets: WSTS with strong (but not strict) monotony.

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- $S = (Q \times \Sigma^{*k}, = \times \sqsubseteq^k)$.
  - LCS: WSTS with non-strict monotony.

WSTS still verywhere!

- Data nets: $S = (Q \times \mathbb{N}^k)^*$
  - Lazic, Newcomb, Ouaknine, Roscoe, Worrell (PN'07)
  - Hofman, Lasota, Lazić, Leroux, Schmitz, Totzke (FOSSACS'16).
  - Lasota (PN'16)
- ν-Petri nets: $S = (Q \times \mathbb{N}^k)^{\oplus}$.
  - Rosa-Velardo, de Frutos-Escrig (PN'07)
  - Lazić and Schmitz (LICS'16).
- Pi-calculus: Depth-Bounded Processes (trees).
  - Wies, Zufferey, Henzinger (FOSSACS'10, VMCAI'12).
- Timed Petri nets: $Regions = ((Q \times \mathbb{N}^k)^{\oplus})^*$
  - Bonnet, F, Haddad, Rosa-Velardo (FOSSACS'10)
  - Haddad, Schmitz, Schnoebelen (LICS'12).
- Process algebra (BPP,...).

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We want to study the usual reachability problems, e.g.,

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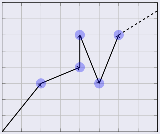

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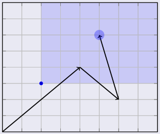
- Coverability (the most used property)



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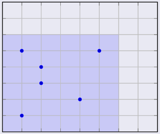

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- Termination
- Coverability (the most used property)
- Boundedness
- And other properties like eventuality, simulation by finite automaton...

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  - WSTS definitions
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- LICS'98 (Emerson, Namjoshi), LICS'99 (Esparza, F, Mayr)
  - broadcast protocols are WSTS
  - model checking of WSTS (with procedures)
- WSTS everywhere, TCS'01 (F, Schnoebelen)

- FSTTCS'04 (Geeraerts, Raskin and Van Begin):
  - The first forward coverability algorithm for WSTS (with ADL).
- STACS'09, ICALP'09 (F, Goubault-Larrecq), ICALP'14 (Blondin, F, McKenzie)
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- 2015-2016: Use of ideals decomposition in:
  - RP'15: The Ideal View on Rackoff's Coverability Technique (Lazić, Schmitz)
  - LICS'15: Demystifying Reachability in Vector Addition Systems (Leroux, Schmitz).
  - FOSSACS'16: Coverability Trees for Petri Nets with Unordered Data (Schmitz and a lot of authors...)
  - LICS'16: ν-Petri nets (Lazić, Schmitz).
  - ...

The survey/story of coverability for WSTS

Year	Authors	Mathematical hyp.	Effectivity hyp.	back/forwar
1978	Arnold & Latteux	reset VAS	reset VAS are effective	backward
1987	F.	(very) WSTS+strong+strict+ω^2-wqo+...	effective very WSTS	forward
1996	Abdulla & CJT	strong monotony	$\text{Pre}_S(\uparrow x)$ comp.	backward
1998	F. Schnoebelen	monotony	$\uparrow \text{Pre}_S(\uparrow x)$ comp.	backward
2004	Geeraerts & RV	strong monotony, ADL	effective ADL	forward
2006	Geeraerts & RV	monotony, ADL	effective ADL	forward
2009	F. & Goubault-Larrecq	strong monotony, weak ADL, flattable	effective WADL	forward
2009	F. & Goubault-Larrecq	strong monotony, flattable	ideally effective	forward
2014	Blondin & FM	monotony,	ideally effective	forward
2016	Blondin & FM	monotony, no wqo but FAC	ideally effective	forward

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Remark

- $\mathrm{Pre}^*(\uparrow y) = \uparrow \mathrm{Pre}^*(\uparrow y)$
- $\downarrow \mathrm{Post}^*(x) = \downarrow \mathrm{Post}^*(\downarrow x)$.

Consequence

- Compute $\uparrow \mathrm{Pre}^*(\uparrow y)$ or $\downarrow \mathrm{Post}^*(x)$.
- Compute $\uparrow$ or $\downarrow$ over-approximations invariants.

The backward coverability algorithm (based on ↑-sets)

- 1978: coverability for reset VAS is decidable (Arnold and Latteux published in French in CALCOLO'78). Their algorithm is an instance of the backward algorithm (LICS'96).
- 1993: decidability of coverability for LCS (Abdulla, Cerans, Jonsson, Tsay, LICS'93)
- 1996: decidability of coverability for strong WSTS assuming Pre($\uparrow x$) is computable (Abdulla, Cerans, Jonsson, Tsay, LICS'96)
- 1998: decidability of coverability for WSTS assuming $\uparrow$Pre($\uparrow x$) is computable (F., Schnoebelen LATIN'98)

Remarks on the backward coverability algorithm

- It computes $\text{Pre}^*(\uparrow y)$ that is more than solving coverability.
- It is often but not always computable, ex: depth-bounded processes (Wies, Zufferey, Henzinger, FOSSACS'10)
- Backward algorithms are often less efficient than forward algorithms.

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Execute two procedures in parallel, one looking for a coverability certificate and one looking for a non coverability certificate.

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  - One enumerates all the finite sets $J \subseteq X$ such that $y \in \uparrow J$ and $Pre(\uparrow J) \subseteq \uparrow J$ and $x \notin \uparrow J$, hence $\uparrow J_m \subseteq Pre^*(\uparrow J) = \uparrow J$. Enumeration of upward closed sets by a finite set of minimal elements is a consequence of $(X, \leq)$ is WQO.

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    Enumeration of upward closed sets by a finite set of minimal elements is a consequence of $(X, \leq)$ is WQO.
  - One finally will find such an invariant J. May be we find a large J_p s.t. $\uparrow J_m = Pre^*(\uparrow y) \subsetneq \uparrow J_p$ but $x \notin \uparrow J_p \implies x \notin \uparrow J_m = Pre^*(\uparrow y)$.

A very interesting characterization of WQO (Finite Basis Property):

Theorem (Higman'52, Nash-Williams' 63-64, Fraïsse'86,...)

$$(X, \leq) \text{ WQO} \iff \text{for all } U = \uparrow U \subseteq X \text{ we have: } U = \bigcup_{\text{finite}} \uparrow x$$

The downward approach for coverability

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- Still simplified and extended with Blondin, McKenzie: WQO is not necessary. (arxiv, august 2016, to appear in LMCS'2017).

In order to decide whether y is coverable from x, Procedure 1 iteratively computes

$$\mathrm{Post}(x), \mathrm{Post}(\mathrm{Post}(x)), \mathrm{Post}^3(x), \ \ldots$$

until it finds y in $\downarrow \mathrm{Post}^n(x)$.

procedure 1: searches for a coverability certificate of y from x

$D \leftarrow x$;
while $y \notin \downarrow D$ **do**
    $D \leftarrow D \cup \mathrm{Post}(D)$
return *true*

y is not coverable from x iff $y \notin \downarrow \text{Post}^*(x)$.

Let $(D_i)_i$ be an enumeration of dc sets, hence $\downarrow \text{Post}^*(x) = D_m$, for some m.

procedure 2: enumerates dcs to find non coverability certificate of y from x

$i \leftarrow 0$;
while $\neg(\downarrow \text{Post}(D_i) \subseteq D_i$ and $x \in D_i$ and $y \notin D_i)$ **do**
  $i \leftarrow i + 1$
return *false*

procedure 1: searches for a coverability certificate of y from x

$D \leftarrow x$;
while $y \notin \downarrow D$ **do**
    $D \leftarrow D \cup \mathrm{Post}(D)$
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Effective hypotheses

- dcs are recursive.
- Union of dcs is computable
- $\downarrow \mathrm{Post}(D)$ is computable.
- Inclusion between dcs is decidable.

Theorem

Let $S = (X, \rightarrow, \leq)$ be a monotone transition system + there exists an enumeration of downward closed sets of X, and let $x, y \in X$.

1. y is coverable from x iff Procedure 1 terminates.
2. y is not coverable from x iff Procedure 2 terminates.

This theorem does not provide an algorithm.

Remark

WSTS, hence WQO implies possible enumeration of downward closed sets (by minimal elements of upward closed sets) but the converse is false: $(\mathbb{Z}, \leq)$ is not WQO but one may enumerate the D_i as follows: $D_i = \downarrow x_i$ for $x_i \in \mathbb{Z}$ or $D_i = \mathbb{Z}$.

Question

How to enumerate downward closed sets ?

Answer

By enumerating ideals ! (Erdös & Tarski)

Definition

- Now $(X, \leq)$ is a qo (shortly written X or $\leq$)

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- The converse is false since $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are FAC but not WF: $13/7, \pi, 3, 2, 1, 0, -1/2, -1, -2, -3, -41, -78695/12, \ldots$

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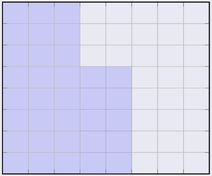
- $\mathbb{Z}^2$ contains infinite antichains, $A = \{(n, -n) \mid n \in \mathbb{N}\}$, hence the cartesian product of two FAC's is not necessary a FAC.

Ideals

$I \subseteq X$ is an *ideal* if

- downward closed: $I = \downarrow I$,



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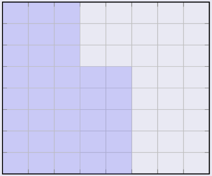

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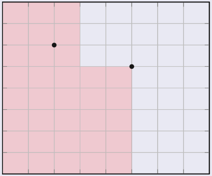

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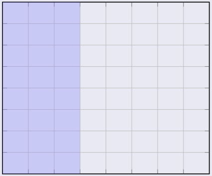


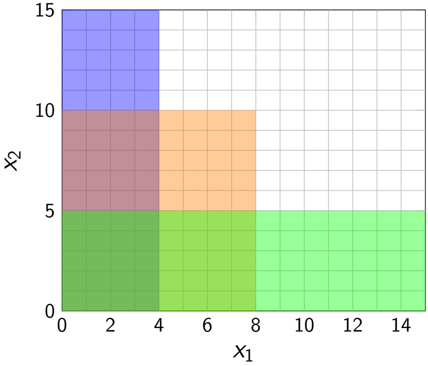
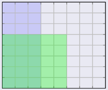


Figure: Decomposition of $X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}$ into finitely many ideals. The three ideals $\downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10$ and $\mathbb{N} \times \downarrow 5$ appear respectively in blue, orange and green.

A very interesting but unknown theorem in the verification community like was the Higman theorem before WSTS'87 - *front* lossy channel systems'89 - lossy channel systems'93.

Theorem (Erdös & Tarski'43, Bonnet'75, Fraïsse'86,...)

$$(X,\leq) \text{ FAC} \iff \text{for all } D = \downarrow D \subseteq X \text{ we have: } D = \bigcup_{\text{finite}} \text{Ideals}$$



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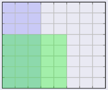

Corollary

Every downward closed set decomposes canonically as the union of its $\subseteq$-maximal ideals.

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ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdös and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.

Theorem

Let $D = \downarrow D \subseteq X$ and X a WQO. Then $D = I_1 \cup I_2 \cup \cdots \cup I_m$ for some $I_1, I_2, \ldots, I_m \in Ideals(X)$.

Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

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Assume that a bad D (bad = dc set that does not admit a finite (may be empty) decomposition in ideals) exists.

$\exists D$ bad and minimal for inclusion among bad subsets (strictly decreasing subsequences of dc subsets are finite in a WQO).

- $D \neq \emptyset$ since $\emptyset$ is equal to an empty union ($\emptyset$ is not an ideal).
- $D \neq \{d\}$ since $\{d\}$ is an ideal.

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- Let $x_1 \neq x_2 \in D$.

Since $D \setminus \uparrow x_1$ and $D \setminus \uparrow x_2$ are dc and strictly included in D, they are not bad (by minimality of D).

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Since $D \setminus \uparrow x_1$ and $D \setminus \uparrow x_2$ are dc and strictly included in D, they are not bad (by minimality of D).
Thus, $D \setminus \uparrow x_1 = \bigcup_{j=1}^{n} I_j$ and $D \setminus \uparrow x_2 = \bigcup_{j=n+1}^{m} I_j$ for some ideals $I_1, I_2, \ldots, I_m \subseteq X$.

Hence

$$D' = (D \setminus \uparrow x_1) \cup (D \setminus \uparrow x_2) = \bigcup_{j=1}^{m} I_j$$

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$$D' = D \setminus (\uparrow x_1 \cap \uparrow x_2)$$

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Hence D is directed and therefore D is an ideal, contradicting our assumption. Thus, D is equal to a finite union of ideals. □

Definition (new)

A *Well Behaved Transition System (WBTS)* is a monotone transition system $S = (X, \rightarrow, \leq)$ such that $(X, \leq)$ is FAC.

WSTS and WBTS

- WSTS are WBTS but the converse is false: $\mathbb{Z}$-VASS are WBTS but are not WSTS.
- Weighted VASS are WBTS for $\leq$ (but are not WSTS).
- Multi-weighted VASS are WBTS for $\leq_{lex}$ (but are not WSTS).

Corollary (new)

Coverability is decidable for any ideally effective class of WBTS.

Remark

The backward coverability procedure does not terminate on $\mathbb{Z}$-VASS, weighted VASS and multi-weighted VASS.

The survey/story of coverability for WSTS

Year	**Authors**	**Mathematical hyp.**	**Effectivity hyp.**	**back/forward**
1978	Arnold & Latteux	reset VAS	YES	backward
1987	F.	very WSTS (strong+strict, ω^2-wqo,...)	effective very WSTS	forward
1996	Abdulla & CJT	strong monotony	$\mathrm{Pre}_S(\uparrow x)$ comp.	backward
1998	F. Schnoebelen	monotony	$\uparrow \mathrm{Pre}_S(\uparrow x)$ comp.	backward
2004	Geeraerts & RV	strong monotony, ADL	effective ADL	forward
2006	Geeraerts & RV	monotony, ADL	effective ADL	forward
2009	F. & Goubault-Larrecq	strong monotony, weak ADL, flattable	effective WADL	forward
2009	F. & Goubault-Larrecq	strong monotony, flattable	ideally effective	forward
2014	Blondin & FM	monotony,	ideally effective	forward
2016	Blondin & FM	monotony, no wqo but FAC	ideally effective	forward
2017	Trivial	no monotony, wqo (Minsky machines)	ideally effective	Undec.
2017	New question	monotony, no wqo but WF	ideally effective	Undec ?

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- Different levels: Bachelor, Master, PhD, post-PhD.

Thank you!