Fault-tolerant matrix factorisation: a formal model and proof

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Content

Motivation

Matrix operations

- Addition, transposition, matrix multiplication
- Row operations, submatrix
- Diagonal matrix, triangular matrix, identity matrix, orthogonal matrix
- Determinant, eigenvalues, eigenvectors

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Decompositions

- QR, LU, Cholesky
  - TSQR: iterative methods use it
    - Linear systems with multiple right-hand sides
    - Block iterative eigensolvers
    - s-step Krylov methods

Motivation

Fault tolerance in HPC

- System-level
  - Transparent for the application
  - Specific middleware to ensure coherent state of the application
- Application-level
  - The application itself handles the failures and adapt to them
  - The middleware must be robust enough to provide primitives

High Performance Systems

Platforms at large scale

- Have their own technical challenges
  - The total number of hardware and software components grows exponentially
  - Platforms needed to manage and handle complex computational problems
  - Hardware or software failures may occur anytime during the execution of high parallel applications
- System reliability, availability and scalability are factors to deal with
  - Failures may result in a high execution times and high cost

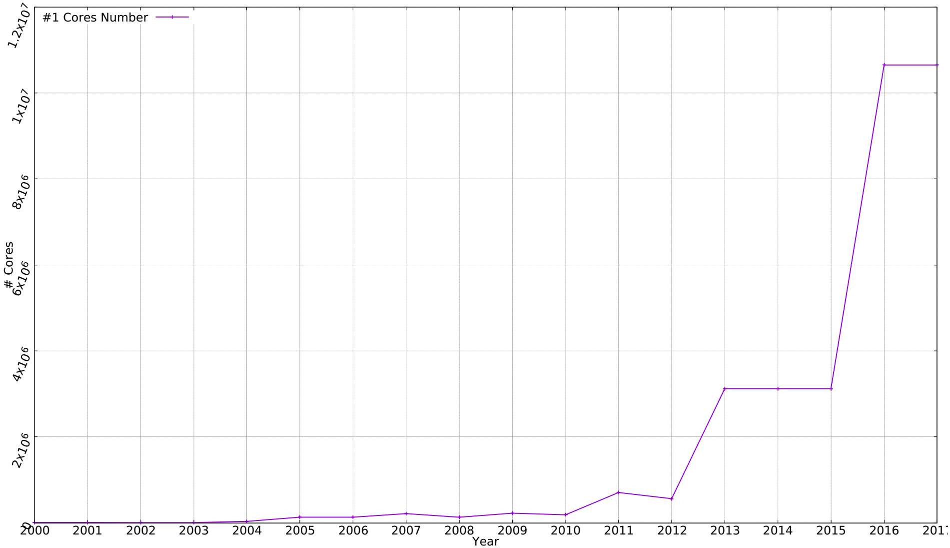
Top500.org

Top500

- A statistical list with ranks and details of the 500 world's most powerful supercomputers
- It shows that performance has almost doubled each year

Rank	System	**Cores**	Rmax (TFlop/s)	Rpeak (TFlop/s)	Power (kW)
1	Summit - IBM Power System AC922, IBM POWER9 22C 3.07GHz, NVIDIA Volta GV100, Dual-rail Mellanox EDR Infiniband , IBM DOE/SC/Oak Ridge National Laboratory United States	**2,397,824**	143,500.0	200,794.9	9,783
2	Sierra - IBM Power System S922LC, IBM POWER9 22C 3.1GHz, NVIDIA Volta GV100, Dual-rail Mellanox EDR Infiniband , IBM / NVIDIA / Mellanox DOE/NNSA/LLNL United States	**1,572,480**	94,640.0	125,712.0	7,438
3	Sunway TaihuLight - Sunway MPP, Sunway SW26010 260C 1.45GHz, Sunway , NRCPC National Supercomputing Center in Wuxi China	**10,649,600**	93,014.6	125,435.9	15,371
4	Tianhe-2A - TH-IVB-FEP Cluster, Intel Xeon E5-2692v2 12C 2.2GHz, TH Express-2, Matrix-2000 , NUDT National Super Computer Center in Guangzhou China	**4,981,760**	61,444.5	100,678.7	18,482
5	Piz Daint - Cray XC50, Xeon E5-2690v3 12C 2.6GHz, Aries interconnect , NVIDIA Tesla P100 , Cray Inc. Swiss National Supercomputing Centre (CSCS) Switzerland	**387,872**	21,230.0	27,154.3	2,384

#1 Cores Number



Failures

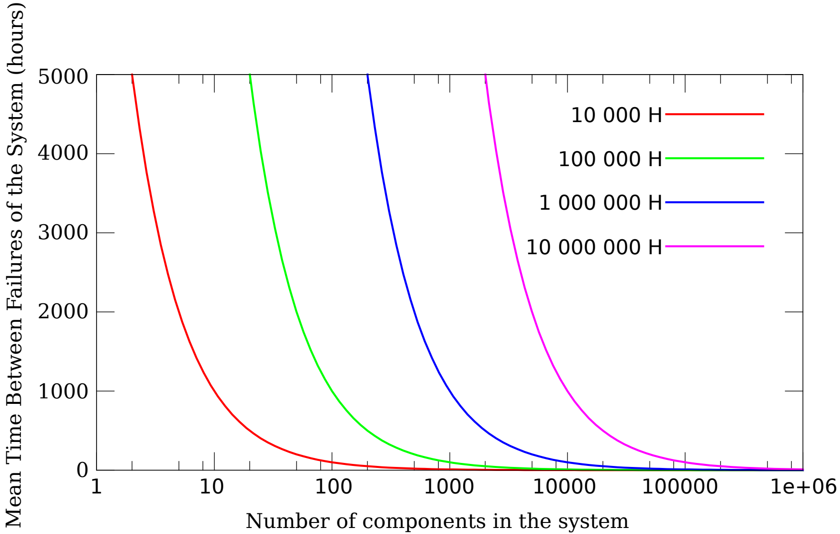
Failures in High Performance Systems

- Node increase in HPC $\Rightarrow$ platforms more subject to failures
  - Mean Time Between Failures (MTBF): measure of system reliability
  - Defined as the probability that the system performs without deviations from agreed-upon behavior for a specific period of time

$$MTBF_{\mathcal{T}} = \left(\sum_{i=0}^{n-1} \frac{1}{MTBF_i}\right)^{-1}$$

- Failures arise anytime
  - Stops partially or totally the execution (crash-type failures)
  - Provides incorrect results (bit errors)
- With an increase in the number of components, the system will experience a component failure every few hours or even minutes

MTBF


Mean Time Between Failures of the System (hours)
5000
4000
3000
2000
1000
0
1
10
100
1000
10000
100000
1e+06
Number of components in the system
10 000 H
100 000 H
1 000 000 H
10 000 000 H

Fault tolerance

Challenges in HPC

- HPC algorithms should be designed to:
  - expect failures: very difficult to predict all possible failures
  - take suitable actions: ensure that intensive applications run smoothly with reduced overhead
- Fault tolerant solutions are being incorporated
  - Have the ability to contain failures
  - Minimize the impact of failures
- Provide a fault tolerant environment
  - Enhance the utilization of the system at high scale
  - Ensure the failure-free execution of critical algorithms
- Hard to describe and verify the system's properties: how to simplify it?

Formal models

Coloured Petri Nets (CPN)

- Better understanding of the system
- The system ensures mathematically that it is correct
- Modelling, validating properties and synchronizing communications of parallel and distributed algorithms
- Allow for better readability and understandability

FT-TSQR Formal Model

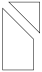
Formal model and associated verifications

- Proves it tolerates the failures
- Guarantees that the final results are correct
  - Data flow is correct
  - Each process calculates and shares its results
  - At the end, all the process have the same result

Los Tres Amigos

QR factorization: $A = QR$

- R upper triangular
- Q orthogonal



LU decomposition: $A = LU$

- L lower triangular
- U upper triangular

Cholesky factorization: $A = LL^T$

- A symmetric, positive definite
- L lower triangular

Tall and Skinny QR

Tall and Skinny QR (TSQR) Factorisation

- It calculates the QR factorisation of a tall and skinny matrix A, i.e. a matrix with m rows and n columns, $m \gg n$
- Linear algebra applications depend on the algorithm:
  - $Ax = b$
  - numerically stable: eigenvalues computation is sensitive to the accuracy of the orthogonalization

Fault-Tolerant TSQR

t processes

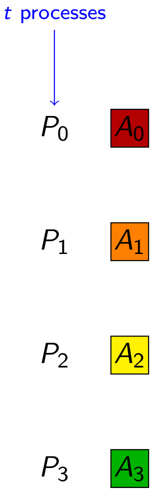
P_0

P_1

P_2

P_3

Fault-Tolerant TSQR



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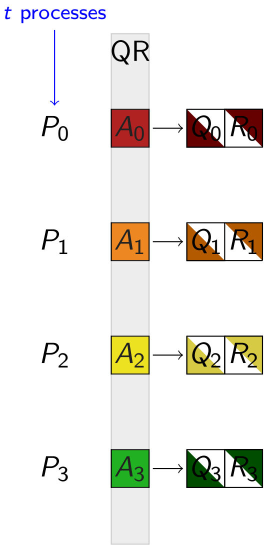

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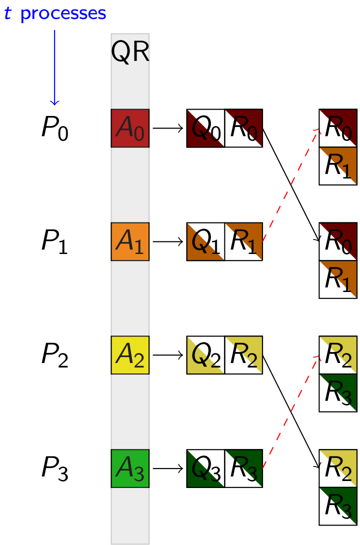

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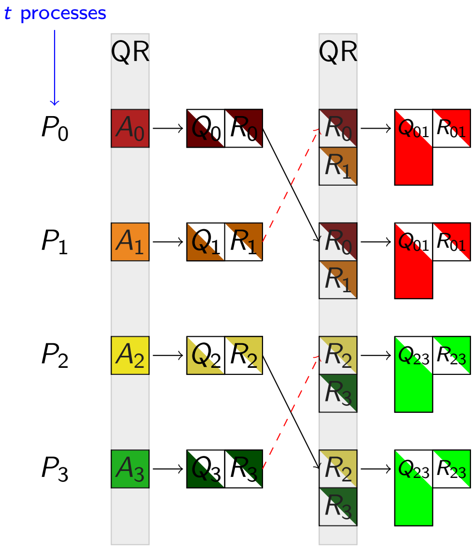

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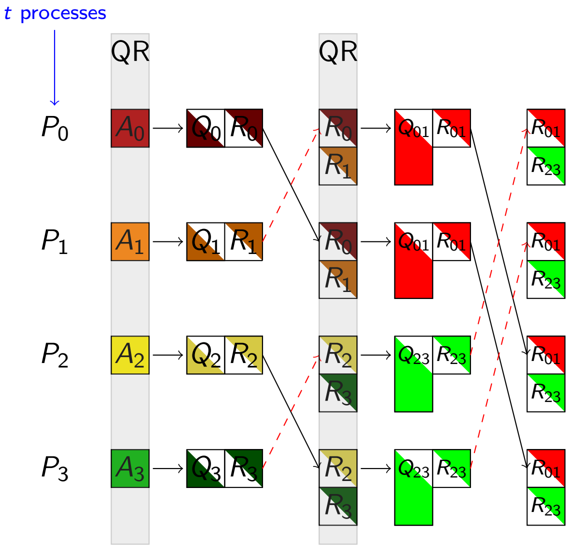

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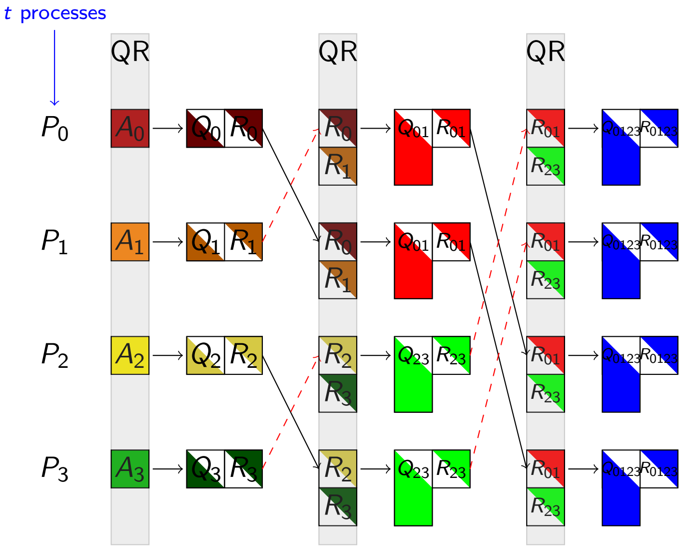

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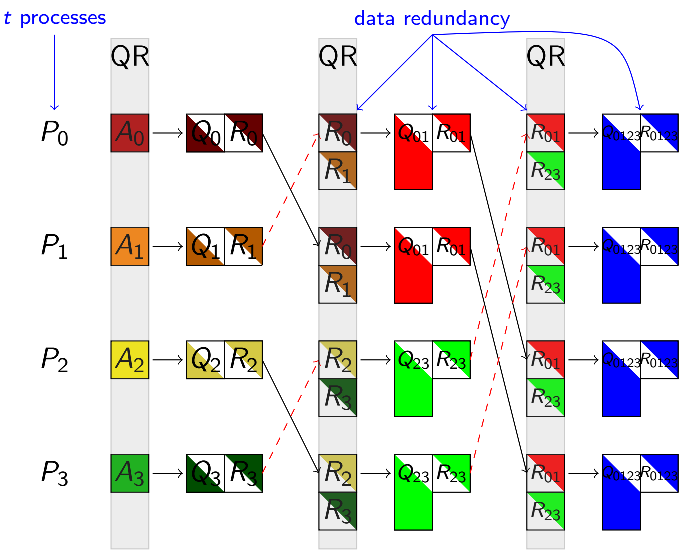

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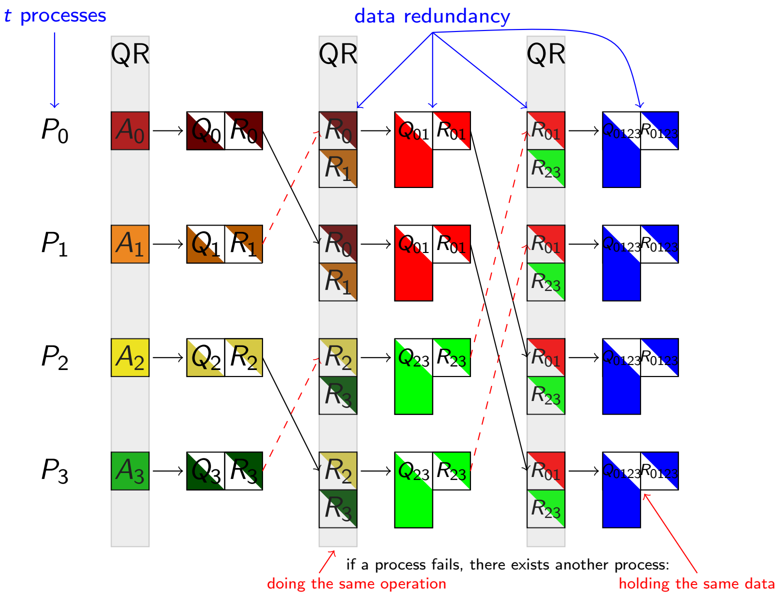
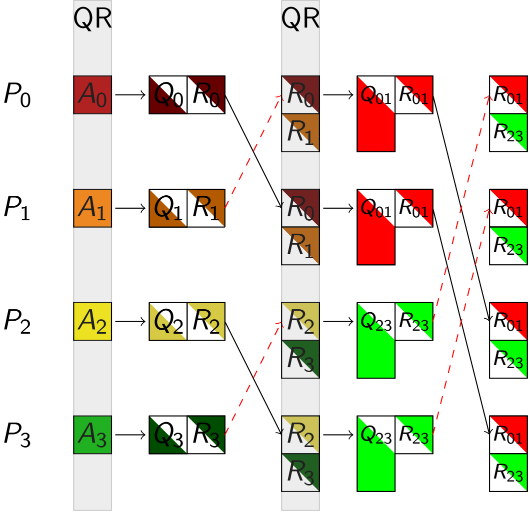


Fault-Tolerant TSQR Failure



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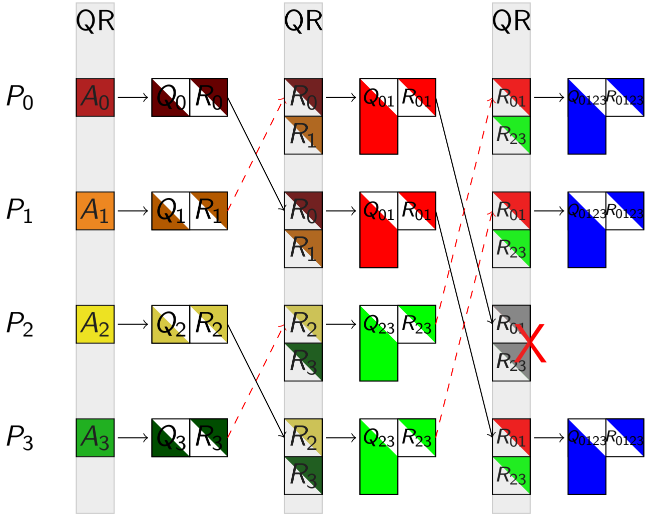

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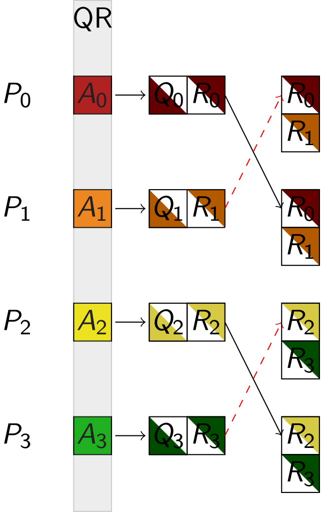

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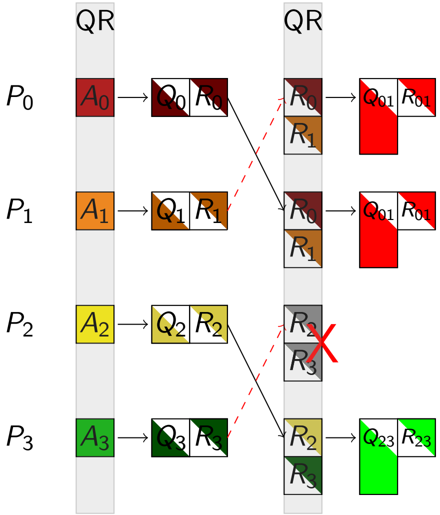

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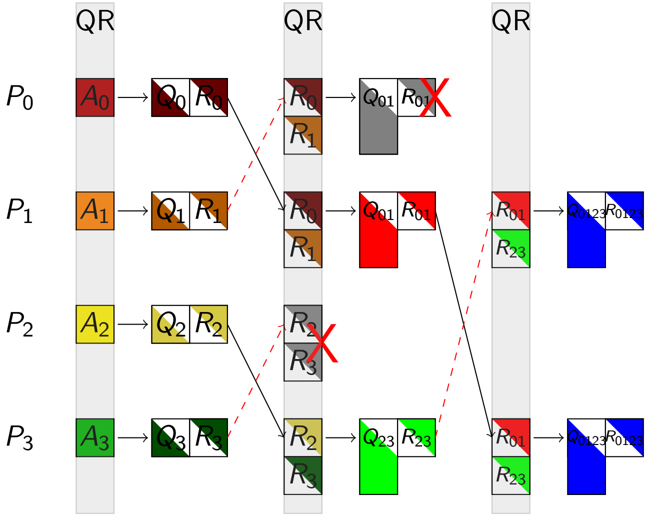

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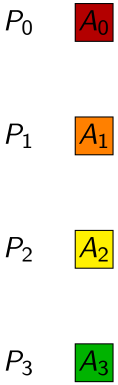

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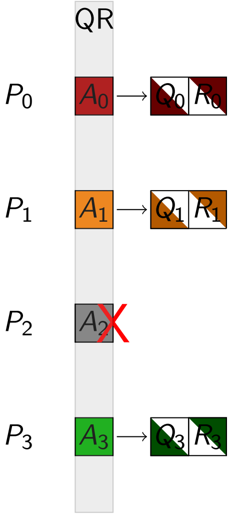

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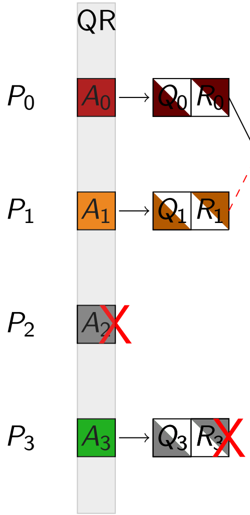

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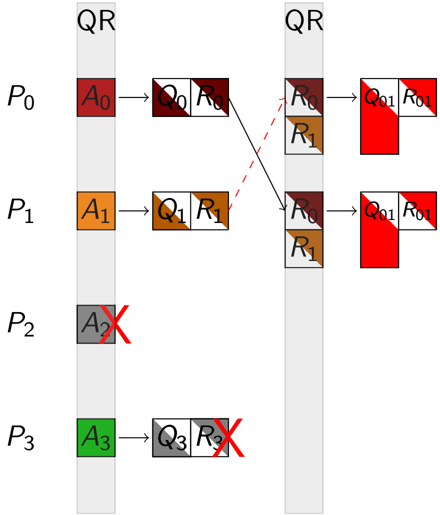

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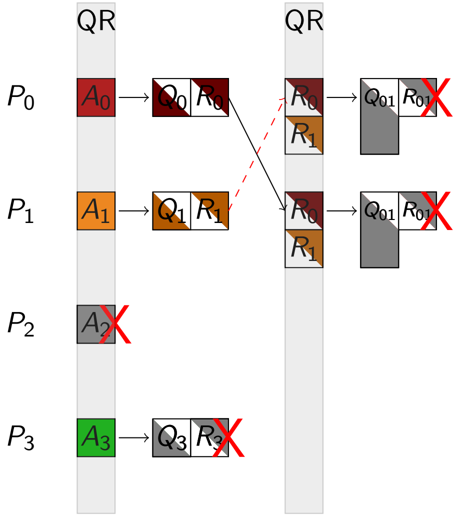

The algorithm

Algorithm 1: FT-TSQR

Data: Submatrix A

$\tilde{Q}, \tilde{R} = \texttt{QR}(A)$;
s = 0 ;
while ! done() **do**
    $p_i = \texttt{myPartner}(s)$;
    $f = \texttt{sendRecv}(\tilde{R}, \tilde{R}', p_i)$;
    if FAIL == f **then**
        return;
    A = $\texttt{concatenate}(\tilde{R}, \tilde{R}')$;
    $\tilde{Q}, \tilde{R} = \texttt{QR}(A)$;
    $s = s + 1$;
/* All the surviving processes reach this point and own the final $\tilde{R}$ */
return $\tilde{R}$;

The model

$\sum_{0 \leq q < t} (q, 0, q)$

Processes

$PROC \times INT \times PROC$

contains triples (q, s, k)

q: a process number

s: the current step

k: index of the $\tilde{R}_i$ matrix

The model

finds a partner process q'
executes a step of the algorithm

compute
$\square[q + 2^s - q \mod 2^{s-1} \leq q' < q + 2^{s+1} - q \mod 2^{s-1}$
$\wedge k'' = \min(k, k')]$

$\sum_{0 \leq q < t}(q, 0, q)$ ◯
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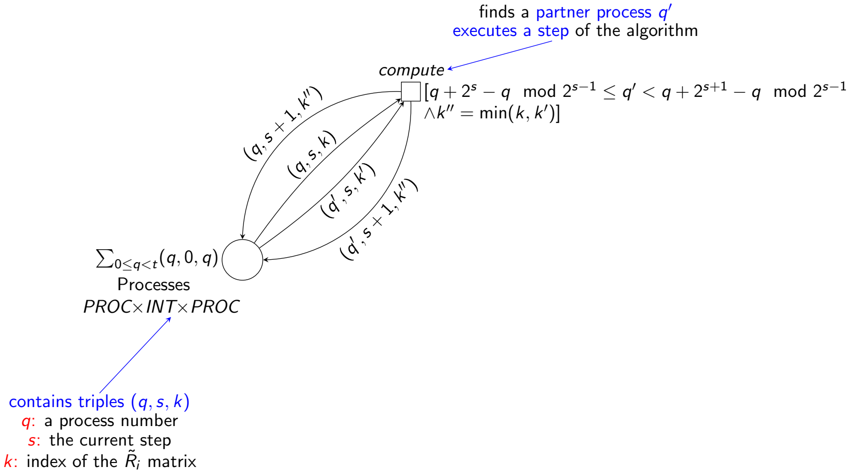
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$[q + 2^s - q \mod 2^{s-1} \leq q' < q + 2^{s+1} - q \mod 2^{s-1}$
$\wedge k'' = \min(k, k')]$
$(q, s+1, k'')$
(q, s, k)
(q', s, k')
$(q', s+1, k'')$
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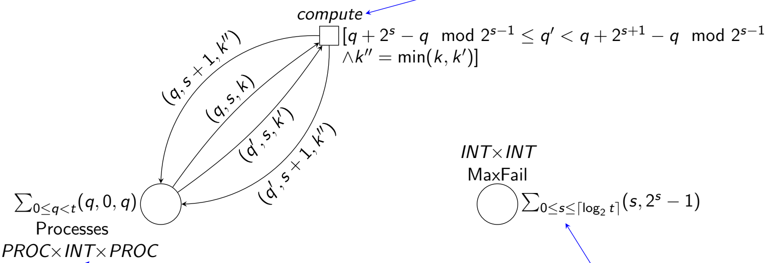
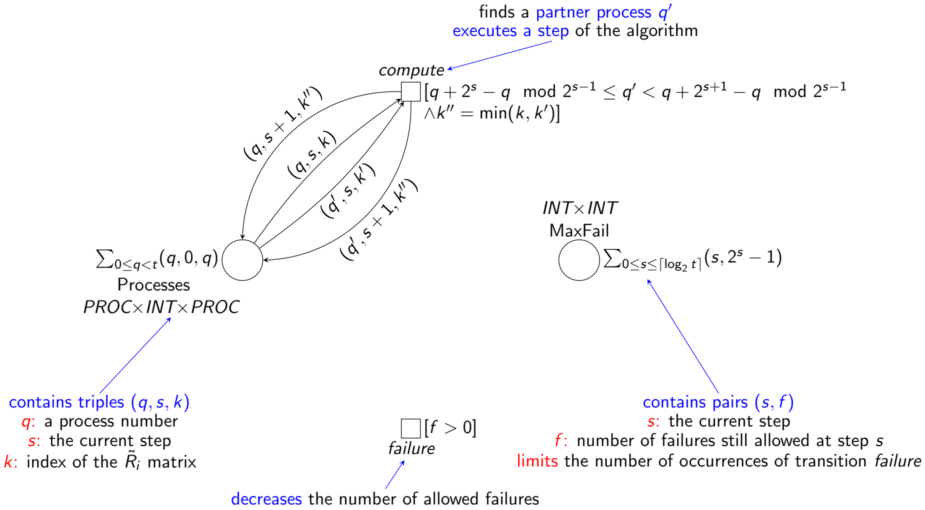


contains triples (q, s, k)
q: a process number
s: the current step
k: index of the $\tilde{R}_i$ matrix

contains pairs (s, f)
s: the current step
f: number of failures still allowed at step s
limits the number of occurrences of transition *failure*

The model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$\sum_{0 \leq q < t} (q, 0, q)$
Processes
$PROC \times INT \times PROC$
$INT \times INT$
MaxFail
$\sum_{0 \leq s \leq \lceil \log_2 t \rceil} (s, 2^s - 1)$
contains triples (q, s, k)
q: a process number
s: the current step
k: index of the $\tilde{R}_i$ matrix
$[f > 0]$
failure
decreases the number of allowed failures
contains pairs (s, f)
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The model

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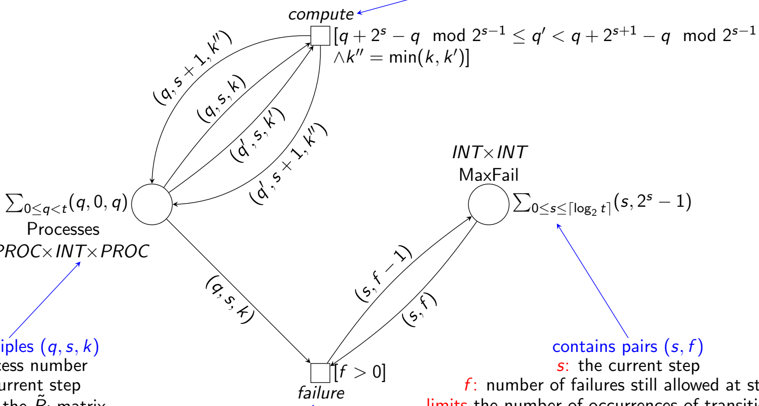

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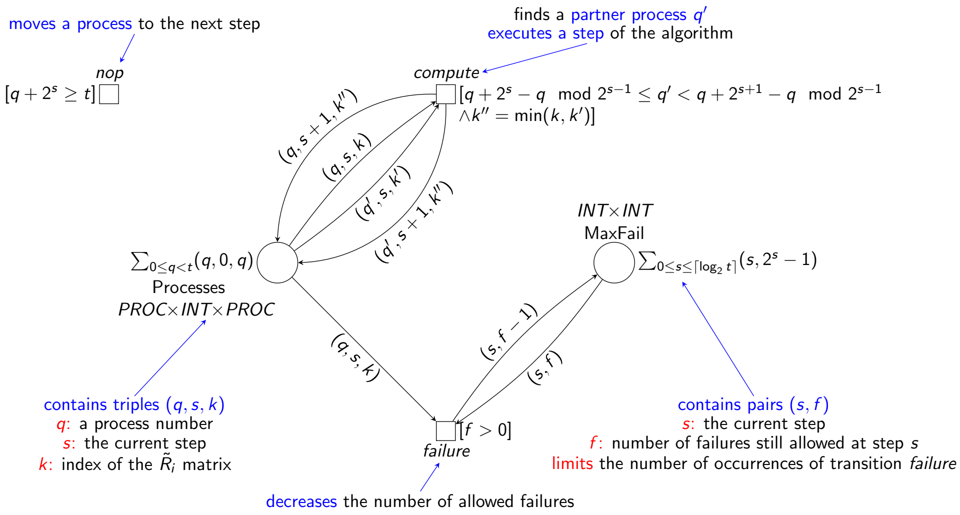
moves a process to the next step
nop
$[q + 2^s \geq t]$
finds a partner process q'
executes a step of the algorithm
compute
$[q + 2^s - q \mod 2^{s-1} \leq q' < q + 2^{s+1} - q \mod 2^{s-1}$
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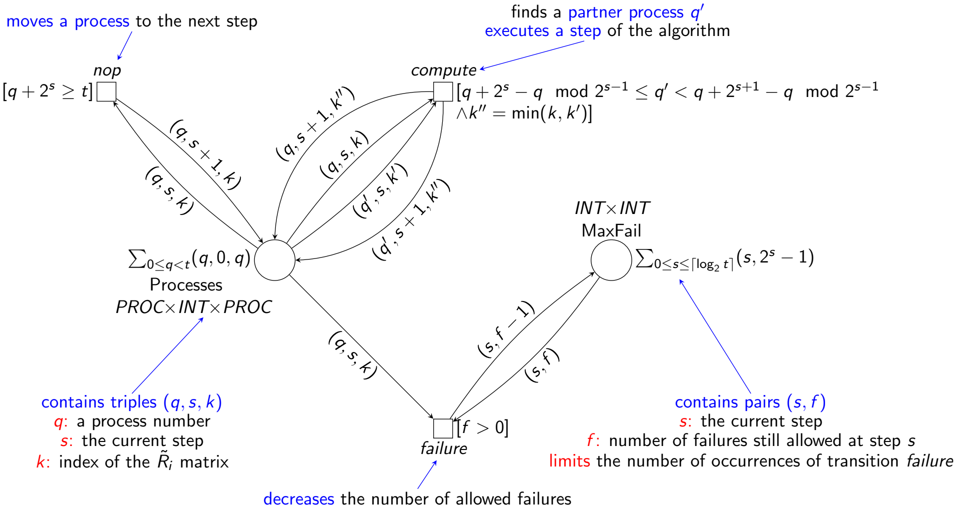
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Properties

- The system can reach the end of the computation (prop. 1)
- The final result is unique and is the expected one (prop. 2)

- Projection functions
  - Π_x: select the xth element of a token which has a tuple value
  - $\Pi_{x,y}$: select the xth and yth elements to form a pair
- Π_x^s denotes the value of Π_x when the step number is s

Properties

Property 1

At every step $s > 0$, the system can tolerate at most $2^s - 1$ failures

Proof.

- When $s = 0$ each process performs a local computation
- When $s > 0$ transition *compute* takes the $\tilde{R}$ and $\tilde{R}'$ from two processes q and q' and produces $\tilde{R}''$ on both q and q' or transition *failure* consumes a process
  - By recursion, at each step $s > 0$, $\forall M \in [M_0 >$, it holds that:

$$|\Pi_3^s(M(Processes))| + \Pi_2^s(M(MaxFail)) = 2^s \leftarrow \text{invariant}$$

    At each step, the number of process with the same information increases by 2
  - The guard on transition *failure* ensures:

$$0 \leq \Pi_2^s(M(MaxFail)) \leq 2^s - 1$$

  - $1 \leq |\Pi_3^s(M(Processes))| \leq 2^s$: at least one process holds each intermediate $\tilde{R}$

Properties

Property 2

At the end of the computation, if the system did not suffer too many failures, at least one process holds the final R

Proof.

- From property 1, when $s > 0$: $|\Pi_3^s(M(Processes))| \geq 1$
  - For each $\tilde{R}$: $|\Pi_3^s(M(Processes))| + \Pi_2^s(M(MaxFail)) = 2^s$
  - $0 \leq \Pi_2^s(M(MaxFail)) \leq 2^s - 1$
- When $s = \log_2 t$: the algorithm reaches the final step
  - $|\Pi_3^s(M(Processes))| + \Pi_2^s(M(MaxFail)) = t$
  - Then, all non-failed processes hold the same $\tilde{R} \Rightarrow \tilde{R}$ is unique and is the final R

Conclusion and perspectives

Conclusions

- A formal model for a fault tolerant algorithm
  - How the failures are modelled
  - Design of proofs of fault tolerance properties
- Number of processes, size of the matrix: parameters of the model
  - The proof holds for any value of the parameters

Perspectives

- How to derive a general modelling and verification approach for:
  - fault tolerant algorithms?
  - square matrices?
  - the Trailing Matrix Update?
- Maximum number of errors allowed?
- Improvements
  - choosing among more partners?
  - recovery after failure: handled by a spawned node, an inactive one?

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