Population Protocols and Predicates

Pierre Ganty
IMDEA Software Institute
The computer science debate: Journals vs Conferences

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<th>Tired</th>
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The computer science debate: Journals vs Conferences

![Diagram showing the debate between Journals and Conferences]

- **Journals**
  - Tired
  - Rested

- **Conferences**
  - Tired
  - Rested
The computer science debate: Journals vs Conferences

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Journals

Conferences

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The computer science debate: Journals vs Conferences

- If two thinking the same debate then they remain the same
- Rested scientists convince tired ones of anything
- If two feeling the same debate then the one for conferences convince the one for journals, and both feel tired afterwards
The computer science debate: Journals vs Conferences

- If two thinking the same debate then they remain the same
- Rested scientists convince tired ones of anything
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The computer science debate: Journals vs Conferences

Tired  Rested

Journals

Conferences

• (■, ■) ↦→ (■, ■)
• (■, ■) ↦→ (■, ■)
• If two thinking the same debate then they remain the same

• Rested scientists convince tired ones of anything

• If two feeling the same debate then the one for conferences convince the one for journals, and both feel tired afterwards
The computer science debate: Journals vs Conferences

- \((\text{Journals}, \text{Tired})\) \rightarrow (\text{Conferences}, \text{Rested})
- \((\text{Journals}, \text{Rested})\) \rightarrow (\text{Conferences}, \text{Tired})
- \((\text{Journals}, \text{Tired})\) \rightarrow (\text{Conferences}, \text{Tired})
- \((\text{Journals}, \text{Rested})\) \rightarrow (\text{Conferences}, \text{Rested})
The computer science debate: Journals vs Conferences

What is the result of endless debates?
The computer science debate: Journals vs Conferences

What is the result of endless debates?

¿initial distribution?
¿scheduler policy?

What is the result of endless debates?
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Take a population of 3 rested computer scientists

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<tr>
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<tr>
<td>J</td>
<td>0</td>
<td>0</td>
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<tr>
<td>C</td>
<td>0</td>
<td>3</td>
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Initial distributions & reachability graphs

Take a population of 3 rested computer scientists
Initial distributions & reachability graphs

Take a population of 3 rested computer scientists
PP define infinitely many graphs

\[
\begin{array}{c|c}
T & R \\
\hline
J & 0 \quad x \\
C & 0 \quad y \\
\end{array}
\]

\[
\begin{align*}
x + y &= 3 \\
x + y &= 4 \\
x + y &= 5 \\
\ldots 
\end{align*}
\]
What is the result of endless debates?

✓ initial distributions

¿scheduler policy?

What is the result of endless debates?
Scheduler policy: fair

“if $C$ appears infinitely often in a fair execution, then every step enabled at $C$ is taken infinitely often in the execution”
“if \( C \) appears infinitely often in a fair execution, then every step enabled at \( C \) is taken infinitely often in the execution”
Scheduler policy: fair

“if $C$ appears infinitely often in a fair execution, then every step enabled at $C$ is taken infinitely often in the execution”
Scheduler policy: fair

“if $C$ appears infinitely often in a fair execution, then every step enabled at $C$ is taken infinitely often in the execution”
Scheduler policy: fair

“if $C$ appears infinitely often in a fair execution, then every step enabled at $C$ is taken infinitely often in the execution”

- infinite executions end up in Strongly Connected Components
- **fair** executions end up in a **bottom SCC** visiting **every** state
Under a fair scheduler

If for each initial distribution

1. the agents reach a stable consensus

2. the outcome only depends on the initial distribution

then

PP maps initial distributions onto consensus values
Under a fair scheduler

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Under a fair scheduler

If for each initial distribution

1. the agents reach a stable consensus

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then

PP maps initial distributions onto consensus values

\[ \varphi(x, y) \mapsto \{\text{True, False}\} \]
Does my PP compute a predicate?

Does my PP compute this predicate?

What predicate does my PP compute?
Does my PP compute a predicate?

1. the agents reach a stable consensus under a fair scheduler
   • broken when, in a reachable bottom SCC,

2. the outcome only depends on the initial distribution
   • broken when, two bottom SCC are such that

PP does not compute a predicate iff either condition holds:

1.a

1.b

Equivalently, there exist configurations $c_0, c_1, c_2$ such that

- $c_0$ is initial
- $c_1, c_2$ belongs to bottom SCC
- $c_1(\text{blue}) > 0$ and $c_2(\text{red}) > 0$
- $c_1, c_2$ are reachable from $c_0$
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land$ $c_1(\text{blue}) > 0 \land c_2(\text{red}) > 0$

$c_1, c_2$ are reachable from $c_0$
Decision procedure

$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land c_1(\text{blue}) > 0 \land c_2(\text{red}) > 0$

$c_1, c_2$ are reachable from $c_0$
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land c_1(\text{blue}) > 0 \land c_2(\text{red}) > 0$

$c_1, c_2$ are reachable from $c_0$
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land \ c_1(\ ) > 0 \land c_2(\ ) > 0$

$c_1, c_2$ are reachable from $c_0$
<table>
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<th>Population Protocols</th>
<th>Petri nets</th>
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<td><strong>State</strong></td>
<td><strong>Place</strong></td>
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<tr>
<td><strong>Configuration</strong></td>
<td><strong>Marking</strong></td>
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<tr>
<td>(([\bullet],[\bullet])) (\mapsto) (([\bullet],[\bullet]))</td>
<td>[\text{PN with infinitely many initial markings}]</td>
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\[\text{PP}\]
Petrification of the debating scientists
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land c_1(\text{Blue}) > 0 \land c_2(\text{Red}) > 0$

$c_1, c_2$ are reachable from $c_0$ in $N$, the Petrified PP
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land c_1(\text{blue}) > 0 \land c_2(\text{red}) > 0$

$c_1, c_2$ are reachable from $c_0$ in $N$, the Petrified PP
$c_0$ is initial

$c_1, c_2$ belongs to bottom SCC $\land c_1(\square) > 0 \land c_2(\blacksquare) > 0$

$c_1, c_2$ are reachable from $c_0$ in $N$, the Petrified PP

$A(C_0, C_0') \overset{\text{def}}{=} \text{Init}_N(C_0) \land \text{Init}_N(C_0') \land C_0 = C_0'$

$B(C_1, C_2) \overset{\text{def}}{=} \Omega_N(C_1) \land \Omega_N(C_2) \land C_1(\square) > 0 \land C_2(\blacksquare) > 0$

Check reachability from $A$ to $B$ in $NN$
How to compute $\Omega$?

The reachability relation is not Presburger definable.
How to compute $\Omega$?

The reachability relation is not Presburger definable.

But, the mutual reachability relation is effectively Presburger.

That is, a Presburger formula $MR(m, m')$ is computable.
Finally

\[
A(C_0, C'_0) \overset{\text{def}}{=} \text{Init}_N(C_0) \land \text{Init}_N(C'_0) \land C_0 = C'_0
\]

\[
B(C_1, C_2) \overset{\text{def}}{=} \Omega_N(C_1) \land \Omega_N(C_2) \land C_1(\text{blue}) > 0 \land C_2(\text{red}) > 0
\]

Check reachability from \(A\) to \(B\) in \(NN\)

\[
\Omega_N(C) \overset{\text{def}}{=} \forall C', C'': (MR_N(C, C') \land C' \rightarrow C'') \Rightarrow MR_N(C, C'')
\]
Finally

\[
A(C_0, C'_0) \overset{\text{def}}{=} \text{Init}_N(C_0) \land \text{Init}_N(C'_0) \land C_0 = C'_0
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Check reachability from \(A\) to \(B\) in \(NN\)

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Finally

\[ A(C_0, C'_0) \overset{\text{def}}{=} Init_N(C_0) \land Init_N(C'_0) \land C_0 = C'_0 \]

\[ B(C_1, C_2) \overset{\text{def}}{=} \Omega_N(C_1) \land \Omega_N(C_2) \land C_1(\ ) > 0 \land C_2(\ ) > 0 \]

Check reachability from \( A \) to \( B \) in \( NN \)

\[ \Omega_N(C) \overset{\text{def}}{=} \forall C', C'': (MR_N(C, C') \land C' \rightarrow C'') \Rightarrow MR_N(C, C'') \]
Finally

\[ A(C_0, C'_0) \overset{\text{def}}{=} \text{Init}_N(C_0) \land \text{Init}_N(C'_0) \land C_0 = C'_0 \]

\[ B(C_1, C_2) \overset{\text{def}}{=} \Omega_N(C_1) \land \Omega_N(C_2) \land C_1(\ ) > 0 \land C_2(\ ) > 0 \]

Check reachability from \( A \) to \( B \) in \( NN \)

\[ \Omega_N(C) \overset{\text{def}}{=} \forall C', C'' : (MR_N(C, C') \land C' \rightarrow C'') \Rightarrow MR_N(C, C'') \]
Does my PP compute this predicate?

What predicate does my PP compute?
What predicate can PPs compute?

PP compute exactly the Presburger predicates [Angluin et al.,'04]
What predicate can PPs compute?

PP compute exactly the Presburger predicates [Angluin et al.,’04]

Not less than Presburger is “easy”: threshold, $\equiv_m, \land, \lor, \neg$

Not more Presburger is “involved”
Does my PP compute this (Presburger) predicate?

What (Presburger) predicate does my PP compute?
Does my PP compute this (Presburger) predicate?

What (Presburger) predicate does my PP compute?

decidable

effectively computable
Does my PP compute (Presburger) predicate Π?
Does my PP compute (Presburger) predicate Π?

It does not compute Π when:

• for a valuation $v_1, \ldots, v_n$ such that $\Pi(v_1, \ldots, v_n) = \text{Init}_N(C_0)$ reaches $C_1(\text{Init}_N(C_0)) > 0 \land \Omega_N(C_1)$ where

$$C_0(x_i) = \begin{cases} v_i & \text{if } 1 \leq i \leq n \\ 0 & \text{elsewhere} \end{cases}$$
Does my PP compute (Presburger) predicate \( \Pi \)?

It does **not** compute \( \Pi \) when:

- for a valuation \( v_1, \ldots, v_n \) such that \( \Pi(v_1, \ldots, v_n) = \)

\[
\text{Init}_N(C_0) \text{ reaches } C_1() > 0 \land \Omega_N(C_1) \text{ where } \]

\[
C_0(x_i) = \begin{cases} 
  v_i & \text{if } 1 \leq i \leq n \\
  0 & \text{elsewhere} 
\end{cases}
\]

**OR**

- for a valuation \( v_1, \ldots, v_n \) such that \( \Pi(v_1, \ldots, v_n) = \)

\[
\text{Init}_N(C_0) \text{ reaches } C_1() > 0 \land \Omega_N(C_1) \text{ where } \]

\[
C_0(x_i) = \begin{cases} 
  v_i & \text{if } 1 \leq i \leq n \\
  0 & \text{elsewhere} 
\end{cases}
\]
“Does my PP compute a predicate?” is PN-reachability hard
“Does my PP compute a predicate?” is PN-reachability hard

Does $p_0 \in \text{Reach}(N, M)$ for some $M$ s.t. $M(p_0 \cup P) = 0$?

$N$ “looks like” a PP: transitions have 1 or 2 input/output arcs

$p_0$ is “write only”: no transition consumes from $p_0$
PP states = \{ \text{places of } N \} \cup \{ \text{Budget} \} \cup \{ \text{zombie} \}
PP states = \{\text{places of } N\} \cup \{\text{Budget}\} \cup \{\text{zombie}\}

PP transitions:

\[(p_1, p_2) \mapsto (p'_1, p'_2)\]

\[(p, \text{Budget}) \mapsto (p'_1, p'_2)\]

\[(p_1, p_2) \mapsto (p', \text{Budget})\]

\[(p, \text{Budget}) \mapsto (p', \text{Budget})\]

PN-reachability hardness (con’td)
From PN reachability to “does my PP compute a predicate?”

• $p_0 \in \text{Reach}(N, M)$ for some marking $M$
From PN reachability to “does my PP compute a predicate?”

• \( p_0 \in \text{Reach}(N, M) \) for some marking \( M \)
From PN reachability to “does my PP compute a predicate?”

- $p_0 \in \text{Reach}(N, M)$ for some marking $M$
From PN reachability to “does my PP compute a predicate?”

- \( p_0 \in \text{Reach}(N, M) \) for \textit{some} marking \( M \)

- \( p_0 \in \text{Reach}(N, M) \) for \textit{no} marking \( M \)
From PN reachability to “does my PP compute a predicate?”

- $p_0 \in \text{Reach}(N, M)$ for some marking $M$
- $p_0 \in \text{Reach}(N, M)$ for no marking $M$
From PN reachability to “does my PP compute a predicate?”

- \( p_0 \in \text{Reach}(N, M) \) for some marking \( M \)

- \( p_0 \in \text{Reach}(N, M) \) for no marking \( M \)

\[(p_0, X) \mapsto (p_0, \text{zombie}) \quad \forall X \in \text{places of } N\]

\[(\text{zombie}, Y) \mapsto (\text{zombie}, \text{zombie}) \quad \forall Y\]
From general PN-reachability to "our" PN-reachability

Does $M \in \text{Reach}(N, M_0)$?
From general PN-reachability to "our" PN-reachability

Does $M \in \text{Reach}(N, M_0)$?

Does $M \in \text{Reach}(N_0, M_0)$ from some $M$ with $M(\hat{p}) = 0$?
From general PN-reachability to "our" PN-reachability

Does $M \in \text{Reach}(N, M_0)$?

Does $M \in \text{Reach}(N_0, M_0)$ from some $M$ with $M(\hat{p}) = 0$?

Does $M \in \text{Reach}(N_1, M_0 + \ell)$ for some $M$ s.t. $M(\hat{p}) = 0$ and $M(P_{aux}) = 0$ and where $N_1$ is PP like?
From general PN-reachability to "our" PN-reachability

Does $M \in \text{Reach}(N, M_0)$?

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Does $M \in \text{Reach}(N_2, p_0)$ for a $M$ s.t. $M(\{\hat{p}, p_0\} \cup P_{aux}) = 0$?
From general PN-reachability to "our" PN-reachability

Does $M \in \text{Reach}(N, M_0)$?

Does $M \in \text{Reach}(N_0, M_0)$ from some $M$ with $M(\hat{p}) = 0$?

Does $M \in \text{Reach}(N_1, M_0 + \ell)$ for some $M$ s.t. $M(\hat{p}) = 0$ and $M(P_{aux}) = 0$ and where $N_1$ is PP like?

Does $M \in \text{Reach}(N_2, p_0)$ for a $M$ s.t. $M(\{\hat{p}, p_0\} \cup P_{aux}) = 0$?

Does $p_0 \in \text{Reach}(N_3, M)$ for a $M$ s.t. $M(\{\hat{p}, p_0\} \cup P_{aux}) = 0$?
Population Protocols

Theoretical model with reasonable motivations

- distributed computations among sensor nodes
- “soups” of chemical molecules
- people in social networks
- ...

Does my PP compute a predicate?

Does my PP compute this predicate?

What predicate does my PP compute?

Credits: Javier Esparza, Rupak Majumdar, Jerome Leroux