Inapproximability Ratios for Crossing Number

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A drawing of a graph $G$ is a function $D : G \rightarrow \mathbb{R}^2$ that maps each vertex to a distinct point and each edge $uv \in E(G)$ to an arc connecting $D(u)$ to $D(v)$ that does not contain the image of any other vertex.
A crossing occurs whenever the image of two edges coincide somewhere other than their extremes.
The crossing number \( cr(D) \) of the drawing \( D \) is the sum, for all pairs \( e, e' \) of edges, of all crossings between \( D(e) \) and \( D(e') \).
The crossing number $\text{cr}(D)$ of the drawing $D$ is the sum, for all pairs $e, e'$ of edges, of all crossings between $D(e)$ and $D(e')$.

When the edges $e$ and $e'$ have weights $w(e)$ and $w(e')$, each crossing counts $w(e)w(e')$ to the sum.
Crossing Number

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- When the edges \( e \) and \( e' \) have weights \( w(e) \) and \( w(e') \), each crossing counts \( w(e)w(e') \) to the sum.
- The minimum crossing number of all drawings of \( G \) is denoted by \( \text{cr}(G) \).
The crossing number $cr(D)$ of the drawing $D$ is the sum, for all pairs $e, e'$ of edges, of all crossings between $D(e)$ and $D(e')$.

When the edges $e$ and $e'$ have weights $w(e)$ and $w(e')$, each crossing counts $w(e)w(e')$ to the sum.

The minimum crossing number of all drawings of $G$ is denoted by $cr(G)$.

### Crossing Number Problem

**Input:** A graph $G$.

**Output:** A drawing of $G$ whose crossing number is $cr(G)$.
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• Cabello (2013) showed there is a constant $c > 1$ such that it is NP-hard to approximate Crossing Number by a constant ratio less than $c$.
• Here, some precise values for such a $c$ are presented.
For simplicity, weighted graphs will be used in the constructions.

Despite of this, the results also apply for unweighted graphs.

For translating a weighted result to the unweighted case, it is enough to substitute each weighted edge \( e = uv \), whose weight \( w(e) \) is a positive integer, by \( w(e) \) parallel paths connecting \( u \) to \( v \).
Weighted Edges

\[ w = 4 \]
Multiway Cut

**Input:** A graph $G$ and a set $T \subset V(G)$ of terminals.

**Output:** The minimum set $C \subset E(G)$ such that the vertices in $T$ are disconnected in $G \setminus C$. 
Maximum Cut

**Input:** A graph $G$.

**Output:** The set $X \subset V(G)$ with largest $\delta(X)$. 
Given a graph $G$ with $n$ vertices and $m$ edges, we construct a new graph $G'$ as follows.

- For each vertex in $G$, create a gadget composed of five vertices and heavy edges (weighting $m^3$):
Seat the gadgets on a circular frame, also made by heavy edges:
For every edge $uv \in E(G)$, link the opposite sides of the corresponding gadgets with light edges (with unitary weight):
New Reduction

- Note that solving Crossing Number for the constructed graph is the same as deciding, for each vertex, on which side to put the blue and the red ends.
- Moreover, deciding the side of the blue and red ends is equivalent to solving Maximum Cut on the original graph.
Let:

- $A_{cr}$ be a polynomial-time $c$-approximation algorithm for Crossing Number.
- $A_{mc}$ be a polynomial-time approximation for Maximum Cut derived from $A_{cr}$.
- $\overline{mc}(G)$ be the size of the returned by $A_{mc}$ applied to $G$.
- $\overline{cr}(G)$ be the number of crossings in the drawing returned by $A_{cr}$ applied to $G'$. 
Let:

- \( A_{cr} \) be a polynomial-time \( c \)-approximation algorithm for Crossing Number.
- \( A_{mc} \) be a polynomial-time approximation for Maximum Cut derived from \( A_{cr} \).
- \( \overline{mc}(G) \) be the size of the returned by \( A_{mc} \) applied to \( G \).
- \( \overline{cr}(G') \) be the number of crossings in the drawing returned by \( A_{cr} \) applied to \( G' \).

**Lemma 1**

\[
2m^3(m - \overline{mc}(G)) \leq \overline{cr}(G')
\]

**Lemma 2**

\[
\overline{cr}(G') \leq 2m^3(m - mc(G)) + 4m^2
\]
Lemma 1

\[ 2m^3(m - mc(G)) \leq \text{cr}(G') \]

Lemma 2

\[ \text{cr}(G') \leq 2m^3(m - mc(G)) + 4m^2 \]
Applying Lemmas 1 and 2:

\[ 2m^3 (m - \overline{mc}(G')) \leq \overline{cr}(G') \]

\[ \leq c \cdot \text{cr}(G') \]

\[ \leq 2cm^3 (m - mc(G')) + c4m^2. \]
Approximation Ratio of $A_{mc}$

- Applying Lemmas 1 and 2:

$$2m^3(m - \overline{mc}(G)) \leq \overline{cr}(G') \leq c \overline{cr}(G') \leq 2cm^3(m - mc(G)) + c4m^2.$$ 

- Assuming $G$ is not bipartite:

$$m - \overline{mc}(G) \leq c(m - mc(G)) + \frac{2c}{m} \leq c(m - mc(G)) + \frac{2c(m - mc(G))}{m} \leq c \left(1 + \frac{2}{m}\right) \left(m - mc(G)\right) \leq c \left(1 + \frac{2}{m}\right) m - c \left(1 + \frac{2}{m}\right) mc(G).$$
The last inequality gives:

\[ \overline{mc}(G') \geq c \left( 1 + \frac{2}{m} \right) mc(G) + \left( 1 - c \left( 1 + \frac{2}{m} \right) \right) m. \]
The last inequality gives:

$$\overline{mc}(G) \geq c \left( 1 + \frac{2}{m} \right) mc(G) + \left( 1 - c \left( 1 + \frac{2}{m} \right) \right) m.$$ 

$c \geq 1$ implies:

$$1 - c \left( 1 + \frac{2}{m} \right) \leq 0.$$
The last inequality gives:

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From $mc(G') \geq \frac{m}{2}$ it follows that:

$$\overline{mc}(G') \geq c \left( 1 + \frac{2}{m} \right) mc(G) + \left( 1 - c \left( 1 + \frac{2}{m} \right) \right) 2mc(G)$$

$$= \left( 2 - c \left( 1 + \frac{2}{m} \right) \right) mc(G).$$
Results

**Theorem**

If $A_{cr}$ is a constant-factor polynomial $c$-approximation for Crossing Number, then $c \geq 2 - \frac{16}{17} \approx 1.058824$, assuming $P \neq NP$.

- It is NP-hard to approximate Maximum Cut polynomially by a constant ratio greater than $\frac{16}{17}$ (Håstad, 2001).
- $A_{mc}$ satisfies
  $$2 - c \left(1 + \frac{2}{m}\right) \leq \frac{16}{17}.$$  
- The inequality must hold for every $m$.  

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Khot et al. (2007) proved that, if $P \neq NP$ and the Unique Games Conjecture is true, then the approximation ratio $\alpha \approx 0.878567$ obtained by the algorithm by Goemans and Williamson is the best possible.
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**Theorem**

If $A_{cr}$ is a constant-factor polynomial $c$-approximation for Crossing Number, then $c \geq 2 - \alpha \approx 1.121433$, assuming $P \neq NP$ and the UGC.
Thank You!
Hardness of approximation for crossing number. 

Adding one edge to planar graphs makes crossing number and 1-planarity hard. 

An algorithm for the graph crossing number problem. 


Optimal inapproximability results for MAX-CUT and other 2-variable CSPs.