The Solitaire Clobber game and correducibility

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September 12, 2017
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$k$-reducible graph $G$: For any non-monochromatic initial configuration of stones, there exists a succession of moves that leaves $k$ stones on the board.
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Reducibility of a graph $G$:

$$r(G) := \min \{ k \geq 1 \mid G \text{ is } k\text{-reducible.} \}.$$
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Correducibility

For every non-monochromatic initial configuration of stones and for every subset $S$ of cardinality at most $k$, there exists a Solitaire Clobber game on $G$ that empties $S$.

Example: $K_3$. 
Correducibility

$k$-correducible graph $G$: For every non-monochromatic initial configuration of stones and for every subset $S \subseteq V(G)$ of cardinality at most $k$, there exists a Solitaire Clobber game on $G$ that empties $S$. 
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Diagram of $K_3$: A triangle with three vertices.
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Correducibility of a graph \( G \):

\[
\text{cr}(G) := \max \{ k \in \mathbb{N} \mid G \text{ is } k\text{-correducible} \}.
\]
Correducibility of complete graphs

Proposition
If $n \geq 3$, then $cr(K_n) = n^2$.

Corollary
If $|H| = n$, then $cr(H) \geq n^2$. 
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Connectivity and correducibility

A graph $G$ is $k$-connected if and only if every pair of vertices are joined by $k$ pairwise internally disjoint paths.
Connectivity and coreducibility

$k$-connected graph $G$: $|G| > k$ and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. 

Theorem (Menger' 27)

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For all pair $(v,w)$
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Proposition
A graph $G$ is 1-correducible if and only if it is 1-connected.
Note that $K_{k+1}$ is the only $k$-connected graph of order $k + 1$, and this graph is not $k$-correducible. Actually, it is $(k - 1)$-correducible.
Our main theorem states that every other $k$-connected graph is $k$-correducible.
Theorem
Let $k \geq 1$, and let $G$ be a graph with $|G| \geq k + 2$. If $G$ is $k$-connected, then it is $k$-correducible.
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Idea of the proof

Lemma
Let \( k \geq 1 \), and let \( G \) be a \( k \)-connected graph such that \(|G| \geq k + 2\). Given any initial configuration \( \Phi_1 : V(G) \to \{0, 1\} \) with \(|\Phi_1^{-1}(0)| = 1\) and a subset \( S \) of \( V(G) \) with \(|S| = k\), there exists a Solitaire Clobber game on \( G \) that empties \( S \).
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Now, we may always assume that $|\Phi^{-1}(0)| > 1$ and $|\Phi^{-1}(1)| > 1$. 
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If there exists $v \in S$ and $u$ adjacent to $v$ such that $\Phi_1(u) \neq \Phi_1(v)$:

- $|G_2| \geq k + 1 = (k - 1) + 2$;
- $k(G) = k$.
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- \(k(G_2) \geq k - 1;\)
- \(|S_2| = k - 1.\)
- \(\Phi_2\) is non-monochromatic.
Now assume that for each vertex $v \in S$ and any vertex $u$ adjacent to $v$, $\Phi_1(u) = \Phi_1(v)$. 

Theorem (Dirac '60)

If $G$ is a $k$-connected graph (with $k \geq 2$), and $S$ is a set of $k$ vertices in $G$, then $G$ has a cycle $C$ including $S$ in its vertex set.

Theorem (Fan Lemma, Dirac '60)

Let $G$ be a $k$-connected graph, let $x$ be a vertex of $G$, and let $Y \subseteq V(G) \setminus \{x\}$ be a set of at least $k$ vertices of $G$. Then there exists a $k$-fan in $G$ from $x$ to $Y$ (that is, a family of $k$ internally disjoint $(x; Y)$-paths whose terminal vertices are distinct).

We may assume that the restriction of $\Phi_1$ to $V(C)$ is non-monochromatic.

If $\Phi_1(x) = \Phi_1(y)$ for all $x, y \in S$, then there is an obvious Solitaire Clobber game on the cycle $C$ that empties $S$. 

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If both $S$ and $C$ are non-monochromatic, we use the following lemma which extends Dirac’s Theorem.
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**Lemma**

Let $k \geq 2$, $G$ be a $k$-connected graph, $S \subseteq V(G)$ with $|S| = k$, and $T_i = \{v_{i,1}, \ldots, v_{i,s_i}\}$, $1 \leq i \leq m$, $m$ pairwise disjoint subsets of $S$. Suppose that $G$ contains a cycle $C$ that satisfies the following condition: ($\ast$) For each $1 \leq i \leq m$, $(v_{i,1}, \ldots, v_{i,s_i})$ is a path in $C$. Then $G$ contains a cycle that includes $S$ in its vertex set and satisfies ($\ast$).
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Let $G$ be a graph with $|G| \geq 4$. Then $G$ is 2-connected if and only if it is 2-correducible.
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Proposition
For all $n \geq 2$ and $k \leq n - 1$, $k(G(n, k)) = k$. In contrast, for a fixed $k$,$\lim_{n \to \infty} cr(G(n, k)) = \infty$. 
NEXT STEPS:
° Find graphs with \( k(G) = \text{cr}(G) \);
° Study other types of connectivity;
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Thank you for your attention!

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This research was partially supported by CAPES, CNPq and FAPERJ.