Transversals of longest paths

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September 14, 2017
Outline

1. Gallai’s question
2. Longest path transversal
3. Proof idea for chordal graphs
Gallai’s question

- In a connected graph, every pair of longest paths have a common vertex
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- Gallai’s question: *In any connected graph, is there a common vertex to all longest paths?*
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- Gallai’s question: *In any connected graph, is there a common vertex to all longest paths?*
- Negative answer: [Walther ’69] and [Zamfirescu ’76]
Gallai’s question

Figure: The classical 12-vertex graph with a negative answer to Gallai’s question
Gallai’s question
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Transversals of longest paths

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- Dually chordal graphs [Jobson et al. ’16]
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Longest path transversal

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- **Longest path transversal**: set of vertices that intersects all longest paths
- $lpt(G)$: minimum size of a longest path transversal in $G$
- How big can $lpt(G)$ be?
Known upper bounds for $\text{lpt}(G)$

- Connected graph $G$
**Known upper bounds for \( lpt(G) \)**

- Connected graph \( G \)
- \( lpt(G) \leq \left\lfloor \frac{n}{4} - \frac{n^{2/3}}{90} \right\rfloor \) [Rautenbach and Sereni '14]
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- $\text{lpt}(G) \leq \text{tw}(G) + 1$ where $\text{tw}(G)$ is the tree-width of $G$ [Rautenbach and Sereni ’14]
Our results on $lpt(G)$

- Connected graph $G$

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- If $G$ is chordal, then $\text{lpt}(G) \leq \max\{1, \omega(G) - 2\}$, where $\omega(G)$ is the size of a maximum clique of $G$
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- $lpt(G) \leq tw(G)$, where $tw(G)$ is the tree-width of $G$
- Corollary: $lpt(G) \leq 3.182\sqrt{n}$ when $G$ is planar
Our results on $\text{lpt}(G)$

<table>
<thead>
<tr>
<th>Graph class</th>
<th>Previous</th>
<th>Ours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded tree-width</td>
<td>$\text{tw} + 1$</td>
<td>$\text{tw}$</td>
</tr>
<tr>
<td>Chordal</td>
<td>$\omega$</td>
<td>$\omega - 2$</td>
</tr>
<tr>
<td>Planar</td>
<td>$9\sqrt{n \log n}$</td>
<td>$3.182\sqrt{n}$</td>
</tr>
<tr>
<td>Bipartite permutation</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>Full substar</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>
Graph classes

Figure: Results for lpt

- Yellow: our results
- Green: another people results
- White: no results
Proof idea for chordal graphs

- Use tree-decomposition
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- For every node of the tree, find a longest path that does not “fit into” it
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- For every node of the tree, find a longest path that does not "fit into" it
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Proof idea for chordal graphs

- Use tree-decomposition
- For every node of the tree, find a longest path that does not “fit into” it
- Direct some edges of this tree
- Take the last arc of a maximal directed path
- Find a contradiction
Proof idea for chordal graphs
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Tree-decomposition

**Clique tree**: Special tree-decomposition for chordal graphs

**Proposition**

*Any chordal graph has a tree-decomposition \((T, \mathcal{V})\) such that any \(V_t \in \mathcal{V}\) is a maximal clique* [Gavril ’74]
Tree-decomposition

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Let $S$ be a set of vertices in $G$
Crossing paths and fenced paths

- Let $S$ be a set of vertices in $G$
- Let $P$ a path in $G$
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- Let $S$ be a set of vertices in $G$
- Let $P$ a path in $G$
- Assume that $P - S \not= \emptyset$ and $S - P \not= \emptyset$
Let $S$ be a set of vertices in $G$
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Assume that $P - S \neq \emptyset$ and $S - P \neq \emptyset$
$S$ fences $P$ if all vertices of $P - S$ are in one component of $G - S$
Crossing paths and fenced paths

- Let $S$ be a set of vertices in $G$
- Let $P$ a path in $G$
- Assume that $P - S \neq \emptyset$ and $S - P \neq \emptyset$
- $S$ fences $P$ if all vertices of $P - S$ are in one component of $G - S$
- Otherwise, we say that $P$ crosses $S$
Crossing paths and fenced paths

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- \( P \) \textit{t-touches} \( S \) if \( |P \cap S| = t \)
Crossing paths and fenced paths

- $P$ \textit{t-touches} $S$ if $|P \cap S| = t$
- $\omega(G)$: size of a maximum clique in $G$
Proof idea

Lemma

Let $G$ be a connected chordal graph with a clique $K$. One of the following is true:

(a) $\text{lpt}(G) \leq \max\{1, \omega(G) - 2\}$.

(b) There exists a longest path that does not touch $K$.

(c) There exists a vertex $v$ of $K$ such that there is a longest path that is fenced by $K$ and 1-touches $K$ at $v$. Moreover, no longest path that 1-touches $K$ at $v$ crosses $K$.

(d) There exists an edge $e$ of $K$ such that there is a longest path that is fenced by $K$ and 2-touches $K$ at the end vertices of $e$. Moreover, no longest path that 2-touches $K$ at the end vertices of $e$ crosses $K$. 
Proof idea
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Proof idea
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Main result

**Theorem**

*For every connected chordal graph* $G$, $\text{lpt}(G) \leq \max\{1, \omega(G) - 2\}$
Corollaries

**Corollary**

*If G is a tree or a 2-tree, then \( lpt(G) = 1 \)*

**Corollary**

*If G is a 3-tree, then \( lpt(G) \leq 2 \)*

**Corollary**

*If G is a connected chordal planar graph, then \( lpt(G) \leq 2 \)*
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Thank you