Constant Threshold Intersection Graphs of Orthodox Paths in Trees

Claudson F. Bornstein\textsuperscript{1}  \hspace{1em} José W. C. Pinto\textsuperscript{1,2}
Dieter Rautenbach\textsuperscript{3}  \hspace{1em} Jayme Szwarcfiter\textsuperscript{1,4}

\textsuperscript{1}Federal University of Rio de Janeiro, Brazil
\textsuperscript{2}Technological Education Faculty of Rio de Janeiro, Brazil
\textsuperscript{3}University of Ulm, Germany
\textsuperscript{4}Rio de Janeiro State University, Brazil

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Purpose

- Present a solution to a problem posed by Golumbic, Lipshteyn and Stern (2008)

- Show that the graphs in $\text{ORTH}[3, 2, 3]$ are line graphs of planar graphs
Intersection Graphs of Subtrees of a Tree

A graph $G$ is an Intersection Graph of Subtrees of a Tree if:

- $\exists$ a family of subtrees $\{S_v\}_{v \in V(G)}$ of a host tree $T$
- $uv \in E(G) \iff S_u \cap S_v \neq \emptyset$
(h,s,t)-representation

A graph $G$ has an $(h, s, t)$-representation if $\exists$ a family of subtrees $\{S_v\}_{v \in V(G)}$ of a host tree $T$, such that:

- $\Delta(T) \leq h$
- $\Delta(S_v) \leq s$
- $uv \in E(G) \iff |S_u \cap S_v| \geq t$

We denote the class of graphs having an $(h, s, t)$-representation by $[h, s, t]$. 
(h,s,t)-representation

(3,3,1)-representation

G
(h,s,t)-representation

(3, 3, 3)-representation
orthodox \((h, s, t)\)-representation

- The leaves of each subtree \(S_v\) must be leaves of \(T\)
- The vertices \(u, v \in G\) are adjacent iff:
  \(S_u, S_v\) have at least \(t\) vertices in common, iff
  \(S_u, S_v\) share a leaf of \(T\).
orthodox \((h, s, t)\)-representation

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\textbf{orthodox \((3,3,3)\)-representation}

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orthodox \((h, s, t)\)-representation

We denote the class of graphs having an orthodox \((h, s, t)\)-representation by \(\text{ORTH}[h, s, t]\).
ORTH\([h, 2, t]\) and line graphs

**Theorem**

Let \(G\) be a connected, twin-free graph with \(|V(G)| \geq 4\). If \(G\) is \(ORTH[h,2,t]\) with \(h \geq 3\) then \(G\) is the line graph of a connected graph \(H\).
ORTH\[h, 2, t\] and line graphs

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\[\text{G} \quad \text{T} \]

- Diagram of graph \(G\) and \(T\).
- Edge coloring in \(T\).

There is a bijection \(\phi : V(H) \rightarrow L(T)\), and two distinct vertices \(x\) and \(y\) are adjacent in \(H\) if and only if there exists a path in \(T\) between \(\phi(x)\) and \(\phi(y)\).
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![Graph H and T](image)

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Tree layout

Let $H$ be a graph.
If $\exists$ a tree $T$ and two integers $h \geq 3$ and $t \geq 1$, s.t.

- $\Delta(T) \leq h$,
- $V(H) = \mathcal{L}(T)$, and
- for every two independent edges $xy$ and $x'y'$ of $H$, the two paths $x - y$ and $x' - y'$ in $T$ share at most $t - 1$ vertices.

Then, $T$ is an $(h, t)$-tree layout of $H$. 
ORTH\([h, 2, t]\) and \((h, t)\)-tree layout

\[\text{Theorem}\]

Let \(G\) be a connected twin-free line graph of order at least 4, \(H\) a connected graph with \(L(H) = G\), and \(h \geq 3, t \geq 1\) integers. Then

\[G \in \text{ORTH}[h, 2, t] \iff H \text{ has an } (h, t)\text{-tree layout.}\]
The Question

Golumbic, Lipshteyn and Stern asked if \( \text{ORTH}[\infty, 2, t] \) and \( \text{ORTH}[3, 2, t] \) coincide or there is a separating example between these families.

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We find some of these examples.

For each pair $h, t \geq 3$, we exhibit a graph $G \in (\text{ORTH}[h + 1, 2, t] \setminus \text{ORTH}[h, 2, t])$. 

Maximum number of leaves of $T$

Let $H$ be a complete graph $K_n$, and $T$ an $(h, t)$-tree layout of $H$. Since $H$ is complete, there is a path between each pair of leaves. Let $x - y$ be the longest path between two internal vertices. Then, $x - y$ has at most $(t - 1)$ vertices. This implies that the diameter of the tree $T$ is at most $t$. 
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Maximum number of leaves of $T$

Let $T$ be a tree of maximum degree $\leq h$, such that every two leaves are at distance $\leq t$. Then

$$|L(T)| \leq \begin{cases} 2(h - 1)(t - 1/2), & \text{if } t \text{ is odd,} \\ h(h - 1)(t/2 - 1), & \text{if } t \text{ is even.} \end{cases}$$
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Lemma

Let $T$ be a tree of maximum degree $\leq h$, such that every two leaves are at distance $\leq t$. Then

$$|\mathcal{L}(T)| \leq \begin{cases} 2(h - 1)^\left(\frac{t-1}{2}\right), & \text{if } t \text{ is odd}, \\ h(h - 1)^\left(\frac{t}{2}-1\right), & \text{if } t \text{ is even}. \end{cases}$$
ORTH\([h + 1, 2, t]\) and ORTH\([h, 2, t]\)

We describe some graphs \(G \in (ORTH\([h + 1, 2, t]\) \setminus ORTH\([h, 2, t]\))\)

Which are line graphs of complete graphs, whose orders depend on \(h\) and \(t\).
ORTH\([h + 1, 2, t]\] and ORTH\([h, 2, t]\]

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Which are line graphs of complete graphs, whose orders depend on \(h\) and \(t\).

**Theorem**

*Let \(h\) and \(t\) be integers with \(h \geq 3\) and \(t \geq 3\).*

*If \(N = \left\{ n \in \mathbb{N}^* : L(K_n) \in ORTH[h + 1, 2, t] \setminus ORTH[h, 2, t] \right\},* then

\[
N = \begin{cases} 
2(h - 1)\left(\frac{t - 1}{2}\right) + 1, & \text{if } t \text{ is odd}, \\
h(h - 1)\left(\frac{t}{2} - 1\right) + 1, & \text{if } t \text{ is even}.
\end{cases}
\]
The largest value of $n$ such that $L(K_n) \in \text{ORTH}[h, 2, t]$

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Example: $L(K_5) \in \text{ORTH}[4, 2, 3] \setminus \text{ORTH}[3, 2, 3]$
The largest value of $n$ such that $L(K_n) \in \text{ORTH}[h, 2, t]$

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ORTH[3, 2, 3] and planar graphs

Lemma

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If a connected twin-free graph $G$ of order $\geq 4$ is in ORTH[3, 2, 3], then $G$ is the line graph of a planar graph.

Such necessary condition is not sufficient.

Example: $K_{5} - e$ is planar, but $L(K_{5} - e) \notin$ ORTH[3, 2, 3].
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Open questions

• Recognition of the graphs $G \in \text{ORTH}[3, 2, 3]$

• Characterize and determine the complexity of recognizing graphs $G \in \text{ORTH}[3, 3, 3]$
Merci beaucoup !!!