THE MINIMUM CHROMATIC VIOLATION PROBLEM:
A POLYHEDRAL APPROACH

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• Chromatic violation problem in a graph.
Summary

- Chromatic violation problem in a graph.

- $P_{CV}(G, E, F)$ chromatic violation polytope.
Chromatic violation problem in a graph.

\( P_{CV}(G,E,F) \) chromatic violation polytope.

Limit cases: Coloring polytope \( P_{col}(G,E) \) and \( k \)-partition polytope \( P_k(G) \).
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- Limit cases: Coloring polytope $P_{col}(G, E)$ and $k$-partition polytope $P_k(G)$.

- Polyhedral study of $P_{CV}(G)$. 
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- Chromatic violation problem in a graph.
- \( P_{CV}(G, E, F) \) chromatic violation polytope.
- Limit cases: Coloring polytope \( P_{col}(G, E) \) and \( k \)-partition polytope \( P_k(G) \).
- Polyhedral study of \( P_{CV}(G) \).
- General Lifting Procedure for generating valid inequalities
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- Polyhedral study of \( P_{CV}(G) \).

- General Lifting Procedure for generating valid inequalities

- Families of new facets without using Lifting Procedure
Vertex coloring

- **k-coloring** of $G = (V, E)$: partition of $V$ into $k$ stable sets.
- **vertex coloring problem (VCP):** smallest $k$ needed to color the nodes of $G$
**Definitions**

Vertex coloring

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**Definitions**

**Vertex coloring**
- **$k$-coloring** of $G = (V, E)$: partition of $V$ into $k$ stable sets.
- **vertex coloring problem (VCP)**: smallest $k$ needed to color the nodes of $G$

**$k$-partition**
- **$k$-partition** of $G = (V, E)$: partition of $V$ into at most $k$ nonempty sets.
- **$k$-partition problem ($k$-P)**: $G$ edge weighted. Minimum weight $r$-partition, $r \leq k$. 

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The minimum chromatic violation problem: a polyhedral approach
**Definitions**

**Vertex coloring**
- *k-coloring* of $G = (V, E)$: partition of $V$ into $k$ stable sets.
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**k-partition**
- *k-partition* of $G = (V, E)$: partition of $V$ into at most $k$ nonempty sets.
- *k-partition problem (k-P)*: $G$ edge weighted. Minimum weight $r$-partition, $r \leq k$. 

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![Graph with vertex coloring and k-partition examples](image)
Our problem:

Given $G = (V, E)$, $c$ colors, $F \subseteq E$ weak edges

Minimum chromatic violation problem (MCVP)

Find $c$-coloring of $G' = (V, E \setminus F)$ minimizing the weak edges with both endpoints at the same color.
Our problem:

Given $G = (V, E)$, $\mathcal{C}$ colors, $F \subset E$ weak edges

Minimum chromatic violation problem (MCVP)

Find $\mathcal{C}$-coloring of $G' = (V, E \setminus F)$ minimizing the weak edges with both endpoints at the same color.

$F = \{23, 36, 46, 16\}$
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**Minimum chromatic violation problem (MCVP)**

Find $\mathcal{C}$-coloring of $G' = (V, E \setminus F)$ minimizing the weak edges with both endpoints at the same color.

$F = \emptyset$ (VCP)

$F = E$ ($k$-P)
For $i \in V$ and $c \in \mathcal{C}$ let

$$x_{ic} = \begin{cases} 1 & \text{if } i \text{ colored by } c \\ 0 & \text{otherwise} \end{cases}$$

For $ij \in F$ let

$$z_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ have the same color} \\ 0 & \text{otherwise} \end{cases}$$
**Integer Programming Formulation**

For $i \in V$ and $c \in \mathcal{C}$ let

$$x_{ic} = \begin{cases} 
1 & \text{if } i \text{ colored by } c \\
0 & \text{otherwise}
\end{cases}$$

For $ij \in F$ let

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1 & \text{if } i \text{ and } j \text{ have the same color} \\
0 & \text{otherwise}
\end{cases}$$

The MCVP is

$$\min \sum_{ij \in F} z_{ij}$$

subject to

$$\sum_{c \in \mathcal{C}} x_{ic} = 1 \quad i \in V$$

$$x_{ic} + x_{jc} \leq 1 \quad ij \in E \setminus F, c \in \mathcal{C}$$

$$x_{ic} + x_{jc} \leq 1 + z_{ij} \quad ij \in F, c \in \mathcal{C}$$

$$x_{ic}, x_{jc}, z_{ij} \in \{0, 1\} \quad i \in V, j \in V, ij \in F, c \in \mathcal{C}.$$
For $i \in V$ and $c \in C$ let
\[
   x_{ic} = \begin{cases} 
   1 & \text{if } i \text{ colored by } c \\
   0 & \text{otherwise}
   \end{cases}
\]

For $ij \in F$ let
\[
   z_{ij} = \begin{cases} 
   1 & \text{if } i \text{ and } j \text{ have the same color} \\
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   x_{ic} + x_{jc} \leq 1 \quad ij \in E \setminus F, c \in C
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   x_{ic}, x_{jc}, z_{ij} \in \{0, 1\} \quad i \in V, j \in V, ij \in F, c \in C.
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The minimum chromatic violation problem: a polyhedral approach
**INTEGER PROGRAMMING FORMULATION**

For \( i \in V \) and \( c \in C \) let

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The **MCVP** is

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\min \sum_{ij \in F} z_{ij}
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subject to

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x_{ic} + x_{jc} \leq 1 \quad ij \in E \setminus F, c \in C
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\[
x_{ic}, x_{jc}, z_{ij} \in \{0, 1\} \quad i \in V, j \in V, ij \in F, c \in C.
\]
Chromatic violation polytope

\[ P_{CV}(G, F, C) = \text{conv} \left\{ (x, z) \in \{0, 1\}^s : \begin{array}{c} \sum_{c \in C} x_{ic} = 1 \quad i \in V \\ x_{ic} + x_{jc} \leq 1 \quad ij \in E \setminus F, c \in C \\ x_{ic} + x_{jc} \leq 1 + z_{ij} \quad ij \in F, c \in C \end{array} \right\} \]

where \( s = |V||C| + |F| \).
Chromatic violation polytope

\[ P_{CV}(G, F, C) = \text{conv} \left\{ (x, z) \in \{0, 1\}^s : \begin{align*}
\sum_{c \in C} x_{ic} &= 1 & i \in V \\
x_{ic} + x_{jc} &\leq 1 & ij \in E \setminus F, c \in C \\
x_{ic} + x_{jc} &\leq 1 + z_{ij} & ij \in F, c \in C
\end{align*} \right\} \]

where \( s = |V||C| + |F| \).

Observe that

- \( P_{col}(G, C) = P_{CV}(G, \emptyset, C) \) where

\[ P_{col}(G, C) = \text{conv} \left\{ x \in \{0, 1\}^s : \begin{align*}
\sum_{c \in C} x_{ic} &= 1 & i \in V \\
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\[ P_{CV}(G, F, C) = \text{conv} \left\{ (x, z) \in \{0, 1\}^s : \begin{array}{l}
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Observe that

- \( P_{col}(G, C) = P_{CV}(G, \emptyset, C) \)
- \( P_k(G) \subset P_{CV}(G, E, C) \) where

\[ P_k(G) = \text{conv} \left\{ (x, z) \in \{0, 1\}^s : \begin{array}{l}
\sum_{c \in C} x_{ic} = 1 \quad i \in V \\
x_{ic} + x_{jc} \leq 1 + z_{ij} \quad ij \in E, c \in C \\
-x_{ic} + x_{jc} \leq 1 - z_{ij} \quad ij \in E, c \in C \\
x_{ic} - x_{jc} \leq 1 - z_{ij} \quad ij \in E, c \in C
\end{array} \right\} \]
Chromatic violation polytope

\[
P_{CV}(G, F, \mathcal{C}) = \text{conv}\left\{ (x, z) \in \{0, 1\}^s : \begin{aligned}
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Observe that

- \( P_{col}(G, \mathcal{C}) = P_{CV}(G, \emptyset, \mathcal{C}) \)
- \( P_k(G) \subset P_{CV}(G, E, \mathcal{C}) \)

**Lemma**

If \( |\mathcal{C}| > \chi(G - F) \) then

- \( \sum_{c \in \mathcal{C}} x_{ic} = 1, i \in V \) minimal equation system for \( P_{CV}(G) \)
Chromatic violation polytope

\[ P_{CV}(G, F, \mathcal{C}) = \text{conv} \left\{ (x, z) \in \{0, 1\}^s : \begin{align*}
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- \( P_k(G) \subset P_{CV}(G, E, \mathcal{C}) \)

Lemma

If \(|\mathcal{C}| > \chi(G - F)\) then

- \( \sum_{c \in \mathcal{C}} x_{ic} = 1, i \in V \) minimal equation system for \( P_{CV}(G) \)
- \( \dim(P_{CV}(G)) = |V|(|\mathcal{C}| - 1) + |F| \).
PROPOSITION

If $|\mathcal{C}| > \chi(G - F)$ then

- $x_{ic} \geq 0$, $i \in V, c \in \mathcal{C}$
- $z_{ij} \leq 1$, $ij \in F$
- $z_{ij} \geq 0$, $ij \in F$ such that $|\mathcal{C}| > \chi(G - (F \setminus \{ij\}))$, $x_{ic} + x_{jc} \leq 1 + z_{ij}$, $ij \in F$ maximal clique in $G - (F \setminus \{ij\})$
- $x_{ic} + x_{jc} \leq 1$, $ij \in E \setminus F$ maximal clique in $G - F$

are facet defining inequalities for $P_{CV}(G)$. 
**PROPOSITION**

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$G$ $F = \{23, 36, 46, 16\}$
PROPOSITION

If $|\mathcal{C}| > \chi(G - F)$ then

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- $x_{ic} + x_{jc} \leq 1 + z_{ij}, \quad ij \in F$ maximal clique in $G - (F \{ij\})$
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$G - (F \{16\})$

\[
x_{1c} + x_{6c} \leq 1 + z_{16} \text{ facet } \forall c
\]
### Proposition

If $|\mathcal{C}| > \chi(G - F)$ then

- $x_{ic} \geq 0, \quad i \in V, c \in \mathcal{C}$
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$G - F$
Lemma

If \( \lambda x + \mu z \leq \lambda_0 \) non-boolean facet of \( P_{CV}(G) \) then \( \mu \leq 0 \).
**Lemma**

If $\lambda x + \mu z \leq \lambda_0$ non-boolean facet of $P_{CV}(G)$ then $\mu \leq 0$.

An instance $(G_1, F_1, C_1)$ of MCVP is stronger than $(G_2, F_2, C_2)$ if $G_1 = G_2$, $C_1 = C_2$ and $F_1 \subset F_2$. 

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The minimum chromatic violation problem: a polyhedral approach
**Lemma**

If $\lambda x + \mu z \leq \lambda_0$ non-boolean facet of $P_{CV}(G)$ then $\mu \leq 0$.

**Theorem**

Let $H \subset F$.

$\lambda x + \mu_H z_H \leq \lambda_0$ facet of $P_{CV}(G, F) \iff \lambda x + \mu_H z_H \leq \lambda_0$ facet of $P_{CV}(G', H)$ where $G' = G - (F \setminus H)$
The minimum chromatic violation problem: a polyhedral approach

**Lemma**

If \( \lambda x + \mu z \leq \lambda_0 \) non-boolean facet of \( P_{CV}(G) \) then \( \mu \leq 0 \).

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\lambda x + \mu_H z_H \leq \lambda_0 \text{ facet of } P_{CV}(G, F) \iff \lambda x + \mu_H z_H \leq \lambda_0 \text{ facet of } P_{CV}(G', H)
\]

where \( G' = G - (F \setminus H) \)

Note that: \( \lambda x \leq \lambda_0 \) facet of \( P_{col}(G') \) \( \iff \) facet of \( P_{CV}(G, F) \) where \( G' = (V, E \setminus F) \).
**Lemma**

If $\lambda x + \mu z \leq \lambda_0$ non-boolean facet of $P_{CV}(G)$ then $\mu \leq 0$.

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Relationship with the $k$-partition problem.
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where $G' = G - (F \setminus H)$

Relationship with the $k$-partition problem.

Recall that $P_k(G) \subset P_{CV}(G, E)$.
**Lemma**

If \( \lambda x + \mu z \leq \lambda_0 \) non-boolean facet of \( P_{CV}(G) \) then \( \mu \leq 0 \).

**Theorem**

Let \( H \subset F \).

\[
\lambda x + \mu H z_H \leq \lambda_0 \quad \text{facet of} \quad P_{CV}(G, F) \iff \lambda x + \mu H z_H \leq \lambda_0 \quad \text{facet of} \quad P_{CV}(G', H)
\]

where \( G' = G - (F \setminus H) \)

Relationship with the \( k \)-partition problem.

Recall that \( P_k(G) \subset P_{CV}(G, E) \).

**Lemma**

Let \( \lambda x + \mu z \leq \lambda_0 \) valid for \( P_k(G) \).

- Facet for \( P_k(G) \) and valid for \( P_{CV}(G, E) \) ⇒ facet for \( P_{CV}(G, E) \).
**Lemma**

If \( \lambda x + \mu z \leq \lambda_0 \) non-boolean facet of \( P_{CV}(G) \) then \( \mu \leq 0 \).

**Theorem**

Let \( H \subset F \).

\[
\lambda x + \mu_H z_H \leq \lambda_0 \text{ facet of } P_{CV}(G, F) \iff \lambda x + \mu_H z_H \leq \lambda_0 \text{ facet of } P_{CV}(G', H)
\]

where \( G' = G - (F \setminus H) \)

Relationship with the \( k \)-partition problem.

Recall that \( P_k(G) \subset P_{CV}(G, E) \).

**Lemma**

Let \( \lambda x + \mu z \leq \lambda_0 \) valid for \( P_k(G) \).

- Facet for \( P_k(G) \) and valid for \( P_{CV}(G, E) \) \( \Rightarrow \) facet for \( P_{CV}(G, E) \).
- \( \mu \leq 0 \) \( \Rightarrow \) valid for \( P_{CV}(G, E) \).
Lemma

\[ \mathcal{F} = \{(x, z) \in P_{CV}(G, F \setminus \{ij\}) : \lambda x + \mu z = \lambda_0 \} \] non-empty face. Then

\[ \lambda x + \mu z \leq \lambda_0 + \lambda^* z_{ij}, \]

(1)

with \( \lambda^* = \max \{ |\lambda_{vc_1} - \lambda_{vc_2}| : v \in \{i, j\} \text{ and } c_1, c_2 \in \mathcal{C} \} \) valid for \( P_{CV}(G, F) \).
LIFTING PROCEDURE AND ITS CONSEQUENCES

LEMMA

\( \mathcal{F} = \{ (x, z) \in P_{CV}(G, F \setminus \{ij\}) : \lambda x + \mu z = \lambda_0 \} \) non-empty face. Then

\[
\lambda x + \mu z \leq \lambda_0 + \lambda^* z_{ij},
\]

with \( \lambda^* = \max\{ |\lambda_{vc_1} - \lambda_{vc_2}| : v \in \{i, j\} \text{ and } c_1, c_2 \in \mathcal{C} \} \) valid for \( P_{CV}(G, F) \).

If \( \mathcal{F} \) facet and \( \exists (x, z) \in \mathcal{F}, v \in V \) and \( c_1, c_2 \in \mathcal{C} \) such that

- \( x_{vc_1} = 1 \) and \( \lambda_{vc_2} - \lambda_{vc_1} = \lambda^* \),
- \( x_{uc_2} = 0, \ \forall u \in \Gamma_s(v) \),
- \( x_{uc_2} = 0 \) or \( \mu_{vu} = 0 \) or \( z_{vu} = 1 \), \( \forall u \in \Gamma_w(v) \)

then (1) defines facet of \( P_{CV}(G) \).
**Corollary**

Let $K \subseteq V$ clique in $G$. For $c \in \mathcal{C}$, the semi-clique inequality

$$\sum_{v \in K} x_{vc} \leq 1 + \sum_{e \in F(K)} z_e$$

is valid for $P_{CV}(G)$.  

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Corollary

Let $K \subseteq V$ clique in $G$. For $c \in \mathcal{C}$, the semi-clique inequality

$$\sum_{v \in K} x_{vc} \leq 1 + \sum_{e \in F(K)} z_e$$

is valid for $P_{CV}(G)$.

If $K$ maximal clique in $G - (F \setminus F(K))$ and $|\mathcal{C}| > \chi(G - (F \setminus F(K)))$ then it defines a facet of $P_{CV}(G)$.
COROLLARY

Let $K \subseteq V$ clique in $G$. For $c \in \mathcal{C}$, the semi-clique inequality

$$
\sum_{v \in K} x_{vc} \leq 1 + \sum_{e \in F(K)} z_e
$$

is valid for $P_{CV}(G)$. If $K$ maximal clique in $G - (F \setminus F(K))$ and $|\mathcal{C}| > \chi(G - (F \setminus F(K)))$ then it defines a facet of $P_{CV}(G)$.

$K = \{3, 4, 6\}$
**Corollary**

Let $K \subseteq V$ clique in $G$. For $c \in \mathcal{C}$, the semi-clique inequality

$$
\sum_{v \in K} x_{vc} \leq 1 + \sum_{e \in F(K)} z_e
$$

is valid for $P_{CV}(G)$.

If $K$ maximal clique in $G - (F \setminus F(K))$ and $|\mathcal{C}| > \chi(G - (F \setminus F(K)))$ then it defines a facet of $P_{CV}(G)$.

$K = \{3, 4, 6\}$

$G - (F \setminus F(K))$

$x_{3c} + x_{4c} + x_{6c} \leq 1 + z_{34} + z_{36} + z_{46}$,

$|\mathcal{C}| > 3$

facet of $P_{CV}(G)$.
Recursively applying the Lifting Lemma
Corollary

$G' \subset SG G$ and $F(G')$ weak edges in $G'$. For $T \subset C$, the multirank inequality

$$\sum_{t \in T} \sum_{i \in V'} x_{it} \leq \alpha(G')|T| + \sum_{e \in F(G')} z_e$$

is valid for $P_{CV}(G)$.
**Corollary**

$G' \subset_{SG} G$ and $F(G')$ weak edges in $G'$. For $T \subset C$, the multirank inequality

$$\sum_{t \in T} \sum_{i \in V'} x_{it} \leq \alpha(G')|T| + \sum_{e \in F(G')} z_e$$

is valid for $P_{CV}(G)$.

For general $G'$ is not easy to analyze facetness. Two particular structures: **cliques** and **odd holes**.
**Corollary**

$G' \subset_{SG} G$ and $F(G')$ weak edges in $G'$. For $T \subset \mathcal{C}$, the multirank inequality

$$\sum_{t \in T} \sum_{i \in V'} x_{it} \leq \alpha(G')|T| + \sum_{e \in F(G')} z_e$$

is valid for $P_{CV}(G)$.

**Proposition**

Let $K \subset V$ clique, $T \subset \mathcal{C}$. For $|\mathcal{C}| > \chi(G - (F \setminus F(K))) + 1$ and $1 \leq |T| \leq |K| \leq |\mathcal{C}| + |T|$, the multicolor clique inequality (MKI)

$$\sum_{t \in T} \sum_{i \in V'} x_{it} \leq |T| + \sum_{e \in F(K)} z_e$$

valid for $P_{CV}(G)$. 

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**Corollary**

\( G' \subset_{SG} G \) and \( F(G') \) weak edges in \( G' \). For \( T \subset \mathcal{C} \), the multirank inequality

\[
\sum_{t \in T} \sum_{i \in V'} x_{it} \leq \alpha(G')|T| + \sum_{e \in F(G')} z_e
\]

is valid for \( P_{CV}(G) \).

**Proposition**

Let \( K \subset V \) clique, \( T \subset \mathcal{C} \). For \(|\mathcal{C}| > \chi(G - (F \setminus F(K))) + 1 \) and

\( 1 \leq |T| \leq |K| \leq |\mathcal{C}| + |T| \), the multicolor clique inequality (MKI)

\[
\sum_{t \in T} \sum_{i \in V'} x_{it} \leq |T| + \sum_{e \in F(K)} z_e
\]

valid for \( P_{CV}(G) \).

**Facet** of \( P_{CV}(G) \) ⇔

- \( 1 \leq |T| < |K| < |\mathcal{C}| + |T| \)
- \( \nexists w \in V \setminus K \) with \( K \subseteq \Gamma_s(w) \).
**Proposition**

Let $H \subset V$ odd hole, $T \subset C$. For $|C| > \chi(G - (F \setminus F(H)))$ the multicolor odd hole inequality

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**Facet of $P_{CV}(G)$ $\iff$**

- $|C| > 2$,
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M. Escalante

The minimum chromatic violation problem: a polyhedral approach
PROPOSITION

Let $K \subseteq V$ clique with $F(K) = E(K)$, $T \subseteq C$ and $q_t \in \mathbb{N}$ for each $t \in T$, the multicolor combinatorial clique inequality

$$\sum_{t \in T} \sum_{i \in K} q_t x_{it} \leq \sum_{t \in T} \frac{q_t(q_t + 1)}{2} + \sum_{ij \in F(K)} z_{ij}$$

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If $|\mathcal{C}| > \chi(G - (F \setminus F(K))) + 1$, it is facet of $P_{CV}(G) \iff$

- $\exists w \in V \setminus K$ with $K \subseteq \Gamma_{s}(w)$
- $1 \leq q_{\Sigma} < |K| < |\mathcal{C}| + q_{\Sigma}$, where $q_{\Sigma} = \sum_{t \in T} q_t$. 

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Not arising from the Lifting Lemma

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In this paper:
- Polyhedral study of the minimum chromatic violation problem
Conclusions and Future Work

In this paper:

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- Analyzed its relationship with the two limit cases: coloring and $k$-partition

To do:

- "Projecting procedure" starting from the $k$-partition facets?
- Implement Branch-and-cut algorithm for some of the inequalities defining facets?
- Families of graphs for which the MCVP can be polynomial time solvable?
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Thanks for your attention!
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